Using Decimals to Prove e is Irrational

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Every fraction a/b can be given as a decimal $.(a)$ base b where a is a symbol in base b. We will use $.(a)_b$ to designate this. So, for example, $1/2 + 1/6 = 4/6 = .(4)_6$. This reduces to $.(2)_3$, but for our purposes we want to limit bases to the form $k!$. As $3! = 6$, this sum is given within this constraint.

Our concern is to prove

$$e - 2 = \sum_{j=2}^{\infty} \frac{1}{j!} = \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \dots$$

is irrational. This is just e minus the first two terms, so if $e - 2$ is proven to be irrational, e will be too.

We first show that all rational numbers in $(0, 1)$ can be expressed as single digits in base $k!$.

Lemma 1. *Every rational $p/q \in (0, 1)$ can be expressed as a single digit in some base $k!$.*

Proof. Let $k = q$ and note

$$\frac{p(q-1)!}{q!} = \frac{p}{q} = .(p(q-1)!)_{q!}.$$

The decimal is a single decimal in base $q!$ as $p < q$ implies $p(q - 1)! < q!$. $\square$

Lemma 2. *Let*

$$s_k = \sum_{j=2}^{k} \frac{1}{j!},$$

then $s_k = .(x)_{k!}$, for some $1 \leq x < k!$.

Proof. As $k!$ is a common denominator of all terms in s_k, s_k can be expressed as a fraction having this denominator. □

Lemma 3. *The least factorial that can express s_k is $k!$.*

Proof. Suppose

$$\frac{x}{k!} + \frac{1}{(k+1)!} = \frac{y}{a!}, \tag{1}$$

for some positive integer a. If $a \leq k$ then multiplying (1) by $k!$ produces an integer plus $1/(k+1)$ is an integer, a contradiction. So $a > k$, but $a = k+1$ works, so it is the least possible factorial. □

A partial plus the tail for the partial gives the entire sum. If we let $.(x)_y^z$ designate the decimal x in base y that expresses the zth partial, a partial with upper index z, then the next lemma gives us a way to make nesting intervals.

Lemma 4.

$$s_k < s_k + \sum_{j=k+1}^{\infty} \frac{1}{j!} = e - 2 < s_k + \frac{1}{k!}. \tag{2}$$

Proof. Using the geometric series, we have

$$\sum_{j=k+1}^{\infty} \frac{1}{j!} = \frac{1}{k!}\left(\frac{1}{(k+1)} + \frac{1}{(k+1)(k+2)} + \dots\right)$$

$$< \frac{1}{k!}\left(\frac{1}{(k+1)} + \frac{1}{(k+1)^2} + \dots\right) = \frac{1}{k}\frac{1}{k!}.$$

So

$$\sum_{j=k+1}^{\infty} \frac{1}{j!} < \frac{1}{k}\frac{1}{k!} < \frac{1}{k!}$$

and (2) follows. □

Lemma 4 implies the x decimal in $.(x)_y^z$ doesn't change with increasing upper index of the partial; all *tails* of partials are immediately trapped. We can designate this with $.(x)_y^{z+}$.

Theorem 1. *e is irrational.*

Proof. Using Lemmas 3 and 4, all partials are trapped between 1/2 and $1/2 + 1/2 = 1$:

$$.(1)_2^{1+} < \cdots < (1)_2^{1+}. \tag{3}$$

Incrementing the upper index we get tighter and tighter traps for $e - 2$:

$$.(1)_2^{1+} < .(4)_6^{2+} < \cdots < .(5)_6^{2+} < (1)_2^{1+}; \tag{4}$$

and

$$.(1)_2^{1+} < .(4)_6^{2+} < .(17)_{24}^{3+} < \cdots < .(18)_{24}^{3+} < .(5)_6^{2+} < (1)_2^{1+}. \tag{5}$$

Suppose $e - 2$ is rational, then by Lemma 1 there exists a k such that $e - 2 = .(x)_{k!}$, but for some y we must have

$$.(1)_2^{1+} < \cdots < .(y)_{k!}^{(k-1)+} < e - 2 = .(x)_{k!} < .(y+1)_{k!}^{(k-1)+} < \cdots < (1)_2^{1+} \tag{6}$$

and no single digit in base $k!$ can be between two other single digits in the same base, a contradiction. □

References

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