Isomorphisms between dual spaces of a vector space

Eduardo Magalhães

December 20, 2020

Abstract

In this small paper, it's deduced that for every finite-dimensional vector space V, the i-th and the j-th dual spaces of V are isomorphic. Two other minor lemmas are also proven: 1) Every vector space V with dimension n over a field $\mathbb{K}$ is isomorphic to $\mathbb{K}^n$, and 2) The i-th dual space of a finite-dimensional vector space V is isomorphic to the $i+1$-th dual space of V.

1 Introduction

Notation:

Firstly I'll introduce the notation that will be used in this paper: $\mathbb{N}$ will be denoting the set of all natural numbers, with the number 0 included. We'll denote the dual space of a vector space V, over a field $\mathbb{K}$ as: $\mathrm{Hom}(V, \mathbb{K})$. The n-th dual space of V will be denoted as: $\mathrm{Hom}_n(V, \mathbb{K})$ and it can be formally defined as the following:

$$\begin{cases} \mathrm{Hom}_0(V, \mathbb{K}) = V \\ \mathrm{Hom}_{n+1}(V, \mathbb{K}) = \mathrm{Hom}(\mathrm{Hom}_n(V, \mathbb{K}), \mathbb{K}) \end{cases}, \forall n \in \mathbb{N}$$

If two vector spaces, V and W are isomorphic, that will be denoted as: $V \simeq W$.

If V is a set of vectors, then the linear span of V will me denoted as: $\mathcal{L}(V)$.

2 Content

2.1 Lemmas

To prove the main result we will make use of the following lemmas:

Lemma 1: Let V be a finite-dimensional vector space over a field $\mathbb{K}$. If $\dim(V) = n$, then:

$$V \simeq \mathbb{K}^n$$

Proof: Let $(v_1, ..., v_n)$ be an ordered basis of V. Then we can define a function $f : V \to \mathbb{K}^n$ such that, if $u \in V$ and u has coordinates $(\alpha_1, ..., \alpha_n)$, then $f(u) = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$.

First let's verify that this function is indeed linear:

- Let $v = (\alpha_1, ..., \alpha_n)$ and $u = (\beta_1, ..., \beta_n)$ be vectors of V. Then: $f(v+u) = (\alpha_1 + \beta_1, ..., \alpha_n + \beta_n) = (\alpha_1, ..., \alpha_n) + (\beta_1, ..., \beta_n) = f(v) + f(u)$.
- Let $\lambda \in \mathbb{K}$ and $v = (\alpha_1, ..., \alpha_n) \in V$. Then: $f(\lambda v) = (\lambda\alpha_1, ..., \lambda\alpha_n) = \lambda(\alpha_1, ..., \alpha_n) = \lambda f(v)$

Now we need to prove that f is bijective. Let $v, u \in V$ if $v \neq u$ then they must have different coordinates, this implies that $f(v) \neq f(u)$ so we conclude that the f is injective. It's also trivial to verify that f is surjective, because for every $(\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$,there exists always a vector $u \in V$ with coordinates $(\alpha_1, ..., \alpha_n)$ such that $f(u) = (\alpha_1, ..., \alpha_n) \in \mathbb{K}^n$. This makes f a bijective linear map, meaning that $V \simeq \mathbb{K}^n$. Q.E.D.

Lemma 2: Let V be a finite-dimensional vector space over a field $\mathbb{K}$. Then it's true that:

$$\forall \alpha \in \mathbb{N},\ \mathrm{Hom}_\alpha(V, \mathbb{K}) \simeq \mathrm{Hom}_{\alpha+1}(V, \mathbb{K})$$

Proof: Let $\{v_1, ..., v_n\}$ be a basis of $\mathrm{Hom}_\alpha(V, \mathbb{K})$. Now, let $\{v'_1, ..., v'_n\} \subseteq \mathrm{Hom}_{\alpha+1}(V, \mathbb{K})$ such that:

$$\begin{array}{r} v'_i : \mathrm{Hom}_\alpha(V, \mathbb{K}) \to \mathbb{K} \\ v_j \rightsquigarrow \delta_{ij} \end{array} ,\forall i, j \in \{1, ..., n\} \tag{1}$$

where δ_{ij} is the Kronecker delta. Then, $\{v'_1, ..., v'_n\}$ is a basis for $\mathrm{Hom}_{\alpha+1}(V, \mathbb{K})$.

- First let's prove that it's spans the space:

We want to show that: $\mathcal{L}(\{v'_1, ..., v'_n\}) = \mathrm{Hom}_{\alpha+1}(V, \mathbb{K})$. The fact that $\mathcal{L}(\{v'_1, ..., v'_n\}) \subseteq \mathrm{Hom}_{\alpha+1}(V, \mathbb{K})$ follows trivially from the fact that, because $\mathrm{Hom}_{\alpha+1}(V, \mathbb{K})$ is a vector space, it's closed under addition and multiplication with a scalar.

Now, let $w \in \mathrm{Hom}_{\alpha+1}(V, \mathbb{K})$. Let's check what w does to a generic element of $\mathrm{Hom}_\alpha(V, \mathbb{K})$. Let $u \in \mathrm{Hom}_\alpha(V, \mathbb{K})$,then:

$$u = \sum_{k=1}^{n} \lambda_k v_k$$

with $\lambda_k \in \mathbb{K}$. So we have the following:

$$w(u) = w\left(\sum_{k=1}^{n} \lambda_k v_k\right)$$

Using the fact that w is linear we can simplify this even further:

$$w\left(\sum_{k=1}^{n} \lambda_k v_k\right) = \sum_{k=1}^{n} w\left(\lambda_k v_k\right) = \sum_{k=1}^{n} \lambda_k w(v_k)$$

We want to prove that w is a linear combination of $\{v_1', ..., v_n'\}$, so we have to get those somewhere into the sum. We'll accomplish this by doing the following: it's trivial that: $\lambda_k = \lambda_k \delta_{kk}$, because $\delta_{kk} = 1$. But $\delta_{kk} = \sum_{m=1}^{n} \delta_{km}$, because, $\forall m \neq k, \delta_{km} = 0$, so we are basically just adding a bunch of 0's, not changing the value of δ_{kk}. So we conclude that: $\lambda_k = \lambda_k \sum_{m=1}^{n} \delta_{km} = \sum_{m=1}^{n} \lambda_m \delta_{km}$. But, because of (1), $\delta_{km} = v_k'(v_m)$. So we end up with:

$\lambda_k = \sum_{m=1}^{n} \lambda_m v_k'(v_m)$. Continuing where we left:

$$\sum_{k=1}^{n} \lambda_k w(v_k) = \sum_{k=1}^{n} w(v_k)\left(\sum_{m=1}^{n} \lambda_m v_k'(v_m)\right)$$

Because v_k' is linear, we have that: $\sum_{m=1}^{n} \lambda_m v_k'(v_m) = \sum_{m=1}^{n} v_k'(\lambda_m v_m) = v_k'\left(\sum_{m=1}^{n} \lambda_m v_m\right)$. But note that $\sum_{m=1}^{n} \lambda_m v_m$ is the vector u that we started with, so: $v_k'\left(\sum_{m=1}^{n} \lambda_m v_m\right) = v_k'(u)$. So we conclude that: $\sum_{m=1}^{n} \lambda_m v_k'(v_m) = v_k'(u)$. Making this substitution we get:

$$\sum_{k=1}^{n} w(v_k)\left(\sum_{m=1}^{n} \lambda_m v_k'(v_m)\right) = \sum_{k=1}^{n} w(v_k) v_k'(u) = \underbrace{\left(\sum_{k=1}^{n} w(v_k) v_k'\right)}_{\in \mathrm{Hom}_{\alpha+1}(V,\mathbb{K})}(u)$$

this allows us the conclude that:

$$\forall u \in \mathrm{Hom}_{\alpha}(V,\mathbb{K}),\ w(u) = \left(\sum_{k=1}^{n} w(v_k) v_k'\right)(u)$$

So now we know that:

$$w = \sum_{k=1}^{n} w(v_k) v_k'$$

With this we can conclude that every $w \in \mathrm{Hom}_{\alpha+1}(V,\mathbb{K})$ is a linear combination of $\{v_1', ..., v_n'\}$, so we infer that: $\mathcal{L}(\{v_1', ..., v_n'\}) \supseteq \mathrm{Hom}_{\alpha+1}(V,\mathbb{K})$. If we pair this conclusion with $\mathcal{L}(\{v_1', ..., v_n'\}) \subseteq \mathrm{Hom}_{\alpha+1}(V,\mathbb{K})$ that we've deduced above, we are finally done proving that: $\mathcal{L}(\{v'_1, ..., v'_n\}) = \text{Hom}_{\alpha+1}(V, \mathbb{K})$, this meaning that the set $\{v'_1, ..., v'_n\}$ spans $\text{Hom}_{\alpha+1}(V, \mathbb{K})$.

- Now, in order to conclude the proof of the second lemma, let's prove that the set $\{v'_1, ..., v'_n\}$ is linearly independent:

Let $\lambda_1, ..., \lambda_n \in \mathbb{K}$ such that:

$$\sum_{k=1}^{n} \lambda_k v'_k = 0$$

This means that, it doesn't matter which input we give to $\sum_{k=1}^{n} \lambda_k v'_k$, the output is always 0. So let's see what happens when we plug the elements of the basis $\{v_1, ..., v_n\}$ as input: let $i \in \{1, ..., n\}$

$$\left(\sum_{k=1}^{n} \lambda_k v'_k\right)(v_i) = 0$$

$$\sum_{k=1}^{n} \lambda_k v'_k(v_i) = 0$$

$$\sum_{k=1}^{n} \lambda_k \delta_{ik} = 0$$

$$\lambda_i = 0$$

So we conclude that, $\forall i \in \{1, ..., n\}, \lambda_i = 0$, proving that $\{v'_1, ..., v'_n\}$ is linearly independent.

- Now we know that $\{v'_1, ..., v'_n\}$ spans $\text{Hom}_{\alpha+1}(V, \mathbb{K})$ and that the set is linearly independent. This means that $\{v'_1, ..., v'_n\}$ is a basis of $\text{Hom}_{\alpha+1}(V, \mathbb{K})$ and, because $\{v_1, ..., v_n\}$ is a basis of $\text{Hom}_{\alpha}(V, \mathbb{K})$ it's clear that $\dim \ \text{Hom}_{\alpha}(V, \mathbb{K}) = \dim \ \text{Hom}_{\alpha+1}(V, \mathbb{K}) = n$. Using lemma 1 we conclude that $\text{Hom}_{\alpha}(V, \mathbb{K}) \simeq \mathbb{K}^n \simeq \text{Hom}_{\alpha+1}(V, \mathbb{K})$, so, using the fact that $\simeq$ is transitive, we end up with: $\text{Hom}_{\alpha}(V, \mathbb{K}) \simeq \text{Hom}_{\alpha+1}(V, \mathbb{K})$. Q.E.D.

2.2 The proposition

Proposition 1: Let V be a finite-dimensional vector space over a field $\mathbb{K}$, then:

$$\forall i, j \in \mathbb{N}, \ \text{Hom}_i(V, \mathbb{K}) \simeq \text{Hom}_j(V, \mathbb{K})$$

Proof: With lemmas 1 and 2 this proof is almost done, we just need to put everything together now. Using lemma 2 and mathematical induction it's trivial to conclude that:

$$\forall \alpha \in \mathbb{N}, V \simeq \mathrm{Hom}_\alpha(V, \mathbb{K}) \tag{2}$$

- For $\alpha = 0$, then we have that $\mathrm{Hom}_\alpha(V, \mathbb{K}) = \mathrm{Hom}_0(V, \mathbb{K}) = V \simeq V$.
- Let's assume that $V \simeq \mathrm{Hom}_\alpha(V, \mathbb{K})$, with $\alpha \in \mathbb{N}$, then, because of lemma 2 we have: $V \simeq \mathrm{Hom}_\alpha(V, \mathbb{K}) \simeq \mathrm{Hom}_{\alpha+1}(V, \mathbb{K})$, so: $V \simeq \mathrm{Hom}_{\alpha+1}(V, \mathbb{K})$.

With this two steps proven, mathematical induction tells us that this statement is true for all $\alpha \in \mathbb{N}$.

So now: Let $i, j \in \mathbb{N}$. Because of (2) we have:

$$\mathrm{Hom}_i(V, \mathbb{K}) \simeq V \simeq \mathrm{Hom}_j(V, \mathbb{K})$$

Using that fact that $\simeq$ is transitive we arrive at the conclusion that:

$$\mathrm{Hom}_i(V, \mathbb{K}) \simeq \mathrm{Hom}_j(V, \mathbb{K})$$

And this concludes the proof. Q.E.D.