Proof of the abc conjecture

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Abstract

In this short note, I prove the abc conjecture. It is time to do it, we need some joy already.

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The abc conjecture (also known as the Oesterlé-Masser conjecture) is a conjecture in number theory, first proposed by Joseph Oesterlé (1988) and David Masser (1985). Many famous conjectures and theorems in number theory would follow immediately from the abc conjecture or its versions. Dr. Goldfeld described the abc conjecture as "the most important unsolved problem in Diophantine analysis" [2]. Various attempts to prove the abc conjecture have been made. But none are currently accepted by the mainstream mathematical community. As of 2020, the conjecture is still largely regarded as unproven [3].

Let us denote $r = \mathrm{rad}(a\,b\,c)$. The known operator rad() is defined in such a way that, e.g., $\mathrm{rad}(2^2 * 3 * 5^3) = 2 * 3 * 5 = 30$.

The abc-conjecture says the following. For every positive real number ϵ, and triplets (a, b, c) of pairwise coprime positive integers, with $a + b = c$, holds $c < K(\epsilon)\, r^{1+\epsilon}$. Then $k < K(\epsilon) < \infty$, with $k = c/r^{1+\epsilon}$.

The abc conjecture demands that in the limit $c \to \infty$ one has $r = \infty$. Otherwise, for every single $\epsilon > 0$ one has $K(\epsilon) = \infty$. Here and in the following the expression "conjecture demands the X" means that if the conjecture is true, then holds statement X.

For arbitrary $m > 0$ one has

$$\frac{c}{r^{1+m}} = U\,W\,,$$

where

$$U = \frac{c}{r^{\epsilon}\, r}\,, \quad W = \frac{r^{\epsilon}}{r^{m}}$$

and $\epsilon > 0$ is arbitrary. For $\epsilon > m$, in the limit $r \to \infty$ the abc conjecture demands to have $U = 0$, as $W = \infty$; because the abc conjecture demands finiteness of $c/r^{1+m} < \infty$ as well. One concludes, that in the limit $r \to \infty$ the abc conjecture implies $k = c/r^{1+\epsilon} = 0$. If for some triplet in the limit $r \to \infty$ happens $U \neq 0$, then abc conjecture is wrong, because then $c/r^{1+m} = \infty$. Thus, the limit exists. Hence, in this limit there is an infinite number of triplets (a, b, c) with k arbitrarily close to 0. In other words,

the abc conjecture is true if for an arbitrary constant $\delta > 0$ there is an infinite number of co-prime triplets $(a, b, c = a + b)$ satisfying $c/r^{1+\epsilon} < \delta$.

In the following I prove the abc conjecture by showing that the amount of such triplets is indeed infinite.

First of all, $(\mathrm{rad}(ab))^{1+\epsilon} \geq 1$. Secondly, because a, b, c have no common factors, one has $r = \mathrm{rad}(ab)\,\mathrm{rad}(c)$. Accordingly, the amount of such triplets with $c < \delta\, r^{1+\epsilon}$ is larger than the amount of triplets with $c < \delta\,\mathrm{rad}(c)\,(\mathrm{rad}(c))^{\epsilon}$. Here and in the following δ is a fixed parameter. Let us study such numbers c which are multiplications of the n first prime numbers, namely $c_n = p_1\,p_2\,p_3\,...p_{n-1}\,p_n$, where $p_1 \equiv 2$, $p_2 \equiv 3$, $p_3 \equiv 5$, etc. Every single one of these c_n satisfies the conditions of the abc conjecture, namely can be presented as the sum of two co-prime numbers a_n and b_n, e.g. $c_n = a_n + 1$. Then $c_n = \mathrm{rad}(c_n)$. Therefore $1 < \delta\,(\mathrm{rad}(c_n))^{\epsilon}$. Because by increasing the n the $\mathrm{rad}(c_n)$ tends to infinity, and as there is infinite amount of triplets with different n, the infinite amount of triplets satisfy $1 < \delta\,(\mathrm{rad}(c_n))^{\epsilon}$.

An alternative formulation of the abc conjecture is the following [1]. For every positive real number ϵ there exist only finitely many triplets (a, b, c) of pairwise coprime positive integers with $a + b = c$, such that $c \geq r^{1+\epsilon}$, the latter is $k \geq 1$. On the other hand, the abc conjecture demands that the amount of triplets with $\Delta \leq k < 1$, where $\Delta \neq 0$, is finite; this is seen from the existence and value of the limit

$$\lim_{r\to\infty} k = 0\,.$$

Let us select e.g. $\Delta = 0.5$. In this case, there are validity conditions with $0.5 \leq k < \infty$ and $1 \leq k < \infty$. But it is enough to check for $k \geq 1$. Conclusion: within $0.5 \leq k < 1$ there is a finite number of triplets. Thus, it is true that in the limit $r \to \infty$ one has $k = 0$.

Fermat's Last Theorem has a famously difficult proof by Andrew Wiles. However, Fermat's Last Theorem follows easily from the abc conjecture [4]. The same holds for the Beal conjecture, for which prize money is promised [4].

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[1] D. W. Masser, "Open problems", Proceedings of the Symposium on Analytic Number Theory, W. W. L. Chen., London: Imperial College, 1985, Vol. 25.

[2] D. Goldfeld, "Beyond the last theorem", Math Horizons 4 (September), 26–34 (1996).

[3] D. Castelvecchi, "Mathematical proof that rocked number theory will be published", Nature (3 April 2020).

[4] Andrew Granville, Thomas Tucker, "It's As Easy As abc", Notices of the AMS. 49 (10), 1224–1231 (2002); R. Daniel Mauldin, "A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem", Notices of the AMS 44 (11), 1436–1437 (1985).