THE 2N CONJECTURE ON SPECTRALLY ARBITRARY SIGN PATTERNS IS FALSE

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Abstract. A *sign pattern* is a matrix with entries in $\{+,-,0\}$. An $n\times n$ sign pattern S is *spectrally arbitrary* if, for any monic polynomial f of degree n with real coefficients, one can replace the $+$ and $-$ signs in S with real numbers of the corresponding signs so that the resulting matrix has characteristic polynomial f. This paper refutes a long-standing conjecture with a construction of an $n \times n$ spectrally arbitrary sign pattern with less than $2n$ entries nonzero.

1. Introduction

Qualitative problems in linear algebra intend to extract some meaningful information from the signs of the numbers in a given data set [45]. A classical result in this line of study is the Perron–Frobenius theorem [23, 51], which guarantees that the largest eigenvalue of a real $n\times n$ matrix A is real and positive, provided that the entries of A are all positive. Another notable example is the study of a slightly more general class of nonnegative matrices, which provides important tools in different contexts of pure and applied mathematics [3, 29, 42, 60]. Other areas that benefit from linear algebraic qualitative methods include ecology [6, 44], optimization [34], economics [41, 45, 52], graph theory [58], dynamical systems [59].

A particularly important problem is, for a given property of spectra of $n\times n$ real matrices, to determine the sign patterns that represent matrices with this property. This includes the study of sign patterns which are sign stable [39, 44], potentially stable [38, 40], potentially nilpotent [22, 61], spectrally arbitrary [5, 17], inertially arbitrary [10, 24], and many others presented in the comprehensive survey [7]. We are going to focus on one problem, which was posed in 2004 by Britz, McDonald, Olesky, van den Driessche [5] and later appeared as the main topic of the workshop held in 2006 in the *American Institute of Mathematics* in Palo Alto [56, 57]. We proceed with the statement of this problem, which is referred to as *the $2n$ conjecture* in many published papers, monographs, degree theses and conference talks.

Conjecture 1.1 (The $2n$ conjecture [2, 7, 8, 15, 16, 30, 35, 47, 53, 54, 57, 62])**.** *Every $n\times n$ spectrally arbitrary sign pattern has at least $2n$ nonzero entries.*

In one of the earliest studies of spectrally arbitrary sign patterns, Drew, Johnson, Olesky, van den Driessche [17] constructed a 2×2 spectrally arbitrary sign pattern T_2 with four nonzero entries and a 3×3 spectrally arbitrary sign pattern T_3 with six nonzero entries. Since every real polynomial can be written as the product of several real polynomials of degrees 1 and 2, one can use the direct sums of T_2 and T_3 to check that, if Conjecture 1.1 is true, then the bound of $2n$ is best possible

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for all nontrivial cases, that is, for $n \neq 1$. Many other sign patterns attaining the bound of $2n$ in Conjecture 1.1 are known [4, 5, 27, 28, 53], and we mention an earlier conjecture in [17, 21] proved by Garnett, Shader [27], who presented a tridiagonal spectrally arbitrary $n \times n$ sign pattern with $2n$ nonzero entries.

As said above, the $n = 1$ case of Conjecture 1.1 is trivial, and the $n = 2$ version is implied by the above mentioned paper [17], which gives a more general result that the spectrally arbitrary *tree* sign patterns satisfy the bound in Conjecture 1.1. The $n = 3$ case of Conjecture 1.1 follows from a description of the 3×3 spectrally arbitrary sign patterns as given by Britz, McDonald, Olesky, van den Driessche [5] and also by Cavers, Vander Meulen [10]. The paper [5] gives an explicit statement of Conjecture 1.1 and proves that every *irreducible* spectrally arbitrary $n \times n$ sign pattern has at least $2n - 1$ nonzero entries, where an *irreducible* pattern is the one that cannot be put to the block triangular form by a permutation similarity.

A *zero-nonzero pattern* is a matrix with entries in $\{0, *\}$, where the $*$ element represents an arbitrary nonzero number. We remark that Conjecture 1.1 is implied by its variation that appeals to the zero-nonzero patterns instead of the sign patterns. It makes sense to further replace $\mathbb{R}$ with an arbitrary ground field $\mathbb{F}$, and we refer to the resulting statement as the *$2n$-conjecture over* $\mathbb{F}$. This generalized statement was proved for $n = 4$ by Corpuz, McDonald [12], and, as explained above, this implies the $n = 4$ version of Conjecture 1.1. Further progress came from the paper of DeAlba, Hentzel, Hogben, McDonald, Mikkelson, Pryporova, Shader, Vander Meulen [15], who showed that a $k \times k$ irreducible component of a spectrally arbitrary sign pattern should have at least $2k$ nonzero entries if $k \leqslant 5$, and this result confirms the $n = 5$ version of Conjecture 1.1. The validity of the $n = 6$ case was reported by Shader [53], and the confirmation of the $n = 7$ version was announced by Deaett, Garnett [14]. It seems that the results with $n = 6$ and $n = 7$ are yet to go through the peer review or independent verification, and several recent sources [9, 62] mention $n \leqslant 5$ as the known range of the validity of Conjecture 1.1.

As explained by Shader [53], the *Nilpotent Jacobian* method [5, 17, 28] cannot be sufficient to confirm that a given potential counterexample to the $2n$ conjecture is, in fact, spectrally arbitrary. The complex number version of this conjecture is true for $n \leqslant 4$ as shown by McDonald, Yielding [49], but, nevertheless, it is false over $\mathbb{C}$ for large n [55]. The $2n$ conjecture is true over any finite field as shown by Shader [47], and the bound of $2n$ is optimal for all sufficiently large finite fields by the work of Bodine, McDonald [4]. The lower bound of $2n - 1$ for the number of the $*$ entries in an irreducible $n \times n$ spectrally arbitrary zero-nonzero pattern is valid over arbitrary fields [54]. Kim, McDonald, Olesky, van den Driessche [37] constructed a family of inertially arbitrary $n \times n$ patterns with less than $2n$ nonzero entries and invalidated a possible stronger version of the $2n$ conjecture. Cavers, Vander Meulen, Vanderspek [11] considered the restriction of Conjecture 1.1 to irreducible patterns, and they improved on the above mentioned result in [37] with a construction of irreducible inertially arbitrary $n \times n$ sign patterns with less than $2n$ nonzero entries. Several authors considered the analogue of Conjecture 1.1 for complex sign patterns in which the signs of the entries are taken separately for the real and imaginary parts [26, 43] and also for ray patterns [25, 48], and one of these problems was solved by Mei, Gao, Shao, Wang [46] with a proof that the minimum number of nonzeros in an $n \times n$ irreducible spectrally arbitrary ray pattern is $3n - 1$.

The goal of this paper is to refute Conjecture 1.1.

2. Building blocks

The idea of our construction develops an earlier approach in [55]. Before we explain it in detail, we reproduce one standard result for ease of reference.

Theorem 2.1 (See [5, 17, 27])**.** *For any $n \geqslant 2$, there exists an $n \times n$ spectrally arbitrary sign pattern which has exactly $2n$ nonzero entries.*

We are going to look for a counterexample to Conjecture 1.1 in the form $\mathcal{S} \oplus P$, where, for some k and sufficiently large n, the sign pattern $\mathcal{S}$ has $2k-1$ nonzero entries and size $k \times k$, and P is a pattern satisfying the assumptions of Theorem 2.1. Here and in what follows, the *direct sum* $A \oplus B$ is the block diagonal matrix with the corresponding diagonal blocks equal to A and B, respectively.

Definition 2.2. Let $\mathbb{F}$ be a field. For any positive integer k, we write $\mathbb{F}_k$ to denote the set of all monic polynomials of degree k in $\mathbb{F}[t]$.

Definition 2.3. If P is a $k \times k$ sign pattern, then we define $\chi(P)$ as the set of all polynomials *realized* by P, which means that, for every such polynomial f, there exists a matrix with sign pattern P and characteristic polynomial f.

Remark 2.4. The inclusion $\chi(P) \subseteq \mathbb{R}_n$ holds for any $n \times n$ sign pattern P. Conjecture 1.1 states that this inclusion is strict if P has less than $2n$ nonzero entries.

We are ready for the first step towards a counterexample. In the following auxiliary definition, the term 'sap' corresponds to a commonly used shorthand for the collocation 'spectrally arbitrary pattern' [8, 15, 17, 27].

Definition 2.5. A $k \times k$ sign pattern $\mathcal{S}$ is *almost sap* if there is an integer $q \geqslant k+2$ such that every polynomial in $\mathbb{R}_q$ is divisible by some polynomial in $\chi(\mathcal{S})$.

Observation 2.6. *If there exists an almost sap $k \times k$ pattern with less than $2k$ nonzero entries, then Conjecture 1.1 is false.*

Proof. We take a pattern $\mathcal{S}$ as in Definition 2.5 with at most $2k-1$ entries nonzero. By Theorem 2.1, there exists a $(q-k) \times (q-k)$ spectrally arbitrary sign pattern P with exactly $2(q-k)$ nonzero entries. Then $\mathcal{S} \oplus P$ is a spectrally arbitrary pattern of the order $2q$, and it has at most $2q-1 < 2q$ nonzero entries. □

In the earlier paper [55], we disproved the complex number analogue of the $2n$ conjecture with a similar idea. We showed, implicitly, that the complex number analogue of the property in Definition 2.5 applies to a zero-nonzero pattern $\mathcal{S}$ if

- the polynomials t^k and $(t+1)^k$ are realized by $\mathcal{S}$ over $\mathbb{C}$, and
- a generic monic polynomial of degree k is realized by $\mathcal{S}$ over $\mathbb{C}$.

These conditions are still necessary in the real number setting, but the situation gets much more complicated. One reason for this is the richer structure of semialgebraic sets compared to their algebraic counterparts over $\mathbb{C}$, and, in particular, the fact that $\chi(\mathcal{S})$ may not be dense even if it contains a generic polynomial. Also, the set of irreducible polynomials is larger over the reals, and proving the property

$$(t^2 + at + 1)^m \in \chi(\mathcal{S})$$

for every $a \in [-2, 2]$ may be a tedious task for a given sign pattern $\mathcal{S}$. We are going to avoid these obstructions with a specific construction of the pattern $\mathcal{S}$ which allows a better control over the space $\chi(\mathcal{S})$ as compared to the example in [55]. Similar constructions can be helpful to study other problems on the topic, which may include the restriction of Conjecture 1.1 to irreducible patterns [9, 11], the complex number analogue of this question in which the signs of the real and imaginary parts of the entries are considered separately [26, 43], and similar questions over different ground fields [4, 49, 54]. In order to reduce the complexity of our constructions and the amount of computation needed to check their validity, we decided to focus on Conjecture 1.1, which is arguably the most well studied problem on the topic. In fact, all calculations required to check our result can be made quite quickly even if we do not rely upon the work of any computer program.

The outline of our further considerations is as follows. Section 3 contains several basic definitions, standard techniques, and further remarks in algebraic geometry. In Section 4, we use the preparations of the previous section and give one condition that is sufficient to invalidate Conjecture 1.1. This condition depends on the existence of certain sign pattern $\mathcal{S}$ and algebraic set H that should satisfy several further assumptions. Section 5 is devoted to the construction of an appropriate algebraic set H, and, in Section 6, we describe the pattern $\mathcal{S}$ up to the existence of one additional construction that is called an *admissible* pattern. In Section 7, we provide a general framework of our construction of an admissible pattern, and, in Section 8, we confirm its validity and complete the proof of the main result.

3. Several basic results of algebraic geometry

To begin with, we specify several cases where we gain some notational simiplicity from the identification of a monic polynomial to the list of its coefficients.

Definition 3.1. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. The Euclidean topology on $\mathbb{F}_k$ is defined as the corresponding topology on $\mathbb{F}^k$ up to the identification of

$$t^k + c_{k-1}t^{k-1} + \ldots + c_1 t + c_0 \quad \text{and} \quad (c_0, c_1, \ldots, c_{k-1}).$$

Definition 3.2. A *polynomial mapping* $\Phi : \mathbb{C}_k \to \mathbb{C}$ is defined by the formula

$$\Phi(t^k + c_{k-1}t^{k-1} + \ldots + c_1 t + c_0) = \varphi(c_0, c_1, \ldots, c_{k-1})$$

whenever φ is a polynomial in $\mathbb{C}[x_1, \ldots, x_k]$.

The results of this section require several basic concepts and techniques from algebraic geometry. Although the main topic of the paper concerns the matrices over the reals, we need to consider algebraic sets and varieties over $\mathbb{C}$.

Definition 3.3. A subset $V \subset \mathbb{C}^k$ or $V \subset \mathbb{C}_k$ is called an *algebraic hypersurface* if V is the *zero locus* of some polynomial f of total degree at least one. In other words, V is the set of all points p such that $f(p) = 0$.

Definition 3.4. A set $V \subset \mathbb{C}^k$ or $V \subset \mathbb{C}_k$ is called *algebraic* if it can be written as the intersection of finitely many algebraic hypersurfaces.

Definition 3.5. If $k = 2$ in Definition 3.3, then V is called a *plane curve.* This curve is said to have *degree* d if the total degree of the corresponding polynomial f equals d. If f is irreducible, then such a curve is called *irreducible.*

Our construction requires a generalization of the following basic result, which is similar to the one used in the complex number counterexample [55].

Observation 3.6. *Let $p \in \mathbb{C}[x_1, \dots, x_v]$ be a nonzero polynomial with v variables and total degree d. If $F_1, \dots, F_v \subset \mathbb{C}$ are subsets of at least $d+1$ distinct numbers each, then there exist $\xi_1 \in F_1, \dots, \xi_v \in F_v$ such that $p(\xi_1, \dots, \xi_v) \neq 0$.*

Proof. This is immediate by the induction on v, since a nonzero polynomial of degree at most d cannot have more than d roots. □

We are ready to prove the desired generalization. As we explain after the proof, the assumption of the irreducibility of C in the following lemma is important.

Lemma 3.7. *There exists a function $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ with the following property. For all integers d and v, if*

(i) $C_1, \dots, C_v$ are irreducible plane curves of degree at most d each,
(ii) $f \in \mathbb{C}[x_1, y_1, \dots, x_v, y_v]$ is a polynomial of total degree at most d,
(iii) there exist $p_1 \in C_1, \dots, p_v \in C_v$ such that $f(p_1, \dots, p_v) \neq 0$,
(iv) $S_1 \subset C_1, \dots, S_v \subset C_v$ are subsets of cardinality at least $\varphi(d, v)$ each,

then there exist $\pi_1 \in S_1, \dots, \pi_v \in S_v$ such that $f(\pi_1, \dots, \pi_v) \neq 0$.

Proof. According to the condition (i), every such curve C_j is the zero locus of an irreducible polynomial $g_j \in \mathbb{C}[x, y]$. Up to taking a generic rotation of the coordinate system, we can assume that no curve C_j contains a straight line parallel to one of the coordinate axes $x = 0$, $y = 0$. In this case, for any j and

(3.1) for all $\xi \in \mathbb{C}$, there are at most d values of $\gamma \in \mathbb{C}$ such that $g_j(\xi, \gamma) = 0$.

Let $\mathcal{V}$ be the intersection of the zero loci of the v polynomials

$$g_1(x_1, y_1), \dots, g_v(x_v, y_v).$$

Then $\mathcal{V}$ is the product $C_1 \times \dots \times C_v$, and, since every polynomial g_j is irreducible, we conclude that $\mathcal{V}$ is an *irreducible variety* of dimension v, see the discussion of Lemma 1.54 in [33]. If H is the zero locus of f, then the condition (iii) implies $H \cap \mathcal{V} \neq \mathcal{V}$, which implies that the dimension of the algebraic set $H \cap \mathcal{V}$ is less than v by the irreducibility of $\mathcal{V}$, see [13, Chapter 9]. Therefore, the projection of a point in $H \cap \mathcal{V}$ onto the $(x_1, \dots, x_v)$ coordinates cannot be generic, and hence, by Theorem 3 in [13, Chapter 3, §2], the corresponding ideal I generated by

$$f(x_1, y_1, \dots, x_v, y_v),\ g_1(x_1, y_1),\ \dots,\ g_v(x_v, y_v)$$

contains a nonzero polynomial $\psi \in \mathbb{C}[x_1, \dots, x_v]$. If the total degree of ψ is δ, then, according to Observation 3.6, the conclusion of the lemma is true if every set S_j contains at least $\delta + 1$ points with different x-coordinates. In view of the condition (3.1), this means that the conclusion holds with $\varphi(v, d) \geqslant \delta d + 1$, so it remains to show that the optimal value of δ is bounded by a function of v and d. In fact, a Gröbner basis of the *elimination ideal* $I \cap \mathbb{C}[x_1, \dots, x_v]$ appears as a part of any Gröbner basis of I with the lexicographic ordering

$$y_1 > \dots > y_v > x_1 > \dots > x_v$$

as shown in standard texts on the topic [1, Theorem 2.3.4], and the total degree of any polynomial in any Gröbner basis of I cannot exceed

$$2\left(\frac{d^2}{2} + d\right)^{2^{2v-1}}$$

by the main result of the paper of Dubé [18]. □

Remark 3.8. As said above, the irreducibility assumption in the item (i) of Lemma 3.7 is important. In fact, the polynomial $f(x, y) = x$ is not an identical zero on a curve $xy = 0$ because of the point $(1, 0)$, but still f vanishes on the infinite family of the points of the form $(0, y)$ on this curve.

4. A characterization of almost sap patterns

Now we employ Lemma 3.7 to present a condition that guarantees that a given sign pattern is almost sap. According to Observation 2.6, this gives a sufficient condition for the invalidity of Conjecture 1.1. To begin with, we recall one additional definition that is commonly used in publications on the topic [11, 36].

Definition 4.1. The *inertia* of a nonzero polynomial p with real or complex coefficients is the triple (n_1, n_2, n_3), where n_1 is the number of roots of p with positive real part, n_2 is the number of roots of p with negative real part, and n_3 is the number roots of p with zero real part. The roots are counted with multiplicities.

Definition 4.2. We write $\mathcal{P}_k$, $\mathcal{N}_k$, $\mathcal{O}_k \subset \mathbb{R}_k$ to denote the sets of all polynomials in $\mathbb{R}_k$ which have the inertias $(n, 0, 0)$, $(0, n, 0)$, $(0, 0, n)$, respectively.

We proceed with the main result of the section.

Theorem 4.3. *A $2m \times 2m$ sign pattern $\mathcal{S}$ has to be almost sap if*

(o) $\mathcal{O}_{2m} \subset \chi(\mathcal{S})$

and there exists an algebraic set $H \subset \mathbb{C}_{2m}$ such that

(i) for all real numbers $p \neq 0$ and $q > 0$, we have $\left(t^2 + pt + q\right)^m \notin H$,

(ii) every polynomial in $\mathcal{P}_{2m} \cup \mathcal{N}_{2m}$ is contained in $\chi(\mathcal{S}) \cup H$.

Proof. We take a finite family F of polynomial mappings such that the intersection of the zero loci of all these mappings is H, and we write d to denote the largest total degree of the mappings in F. We need to check the conditions of Definition 2.5, and, to this end, we take some polynomial $\ell \in \mathbb{R}_q$ with arbitrarily large q, and we need to check that ℓ is divisible by some polynomial in $\chi(\mathcal{S})$.

Special case 1. If ℓ has $2m$ roots with zero real parts, counting roots with multiplicities, then the product of the corresponding irreducible factors of ℓ belongs to $\mathcal{O}_{2m}$ and hence to $\chi(\mathcal{S})$ by the assumption (o) in the lemma.

Using the result of the special case 1, we assume in the rest of this proof, without loss of generality, that every root of ℓ has real part nonzero. In other words, the inertia of ℓ is $(k_1, k_2, 0)$, and we can further drop the irreducible factors of ℓ corresponding to the smaller value between k_1 and k_2 to still have the product of the remaining factors of arbitrarily large degree, that is, of the degree that is not bounded by any function of $\mathcal{S}$ and d fixed in advance. The resulting product has inertia either $(k, 0, 0)$ or $(0, k, 0)$ for some k, and if k is odd, we get rid of one further real root to assume that k is even. Therefore, we can assume without loss of generality that the polynomial ℓ represents as the product of polynomials

$$\tau_j = t^2 + p_j t + q_j \tag{4.1}$$

with $q_j > 0$ and with p_j all of the same nonzero sign. Also, we denote by J the indexing set containing all possible values of j. It is clear that

$$\prod_{j \in J'} \tau_j \in \mathcal{P}_{2m} \cup \mathcal{N}_{2m}$$

for any subset $J' \subset J$ of cardinality m, and hence, in order to conclude the proof with the application of the assumption (ii), it remains to find such a set J' with

$$\prod_{j \in J'} \left(t^2 + p_j t + q_j\right) \notin H. \tag{4.2}$$

The possibility (4.2) is immediate from the assumption (i) if there are at least m different values of $j \in J$ for which the corresponding polynomials τ_j coincide. Otherwise, the ratio of at least $1/m$ of all the elements of J correspond to pairwise distinct polynomials of the form (4.1). Since m is a constant fixed in advance, we can assume without loss of generality that in fact $\tau_i \neq \tau_j$ whenever $i \neq j$.

We need one further technical notational convention. Namely, for any polynomial mapping $\pi : \mathbb{C}_{2m} \to \mathbb{C}$, we define $\overline{\pi} \in \mathbb{C}[x_1, y_1, \ldots, x_m, y_m]$ by the formula

$$\overline{\pi}(\xi_1, \gamma_1, \ldots, \xi_m, \gamma_m) = \pi\left(\prod_{i=1}^{m} \left(t^2 + \xi_i t + \gamma_i\right)\right).$$

Special case 2. Suppose that there exists a plane curve $C \subset \mathbb{C}^2$ of degree at most d such that the family C', which we define as the set of all points (p_j, q_j) that belong to C, has cardinality at least $|J| - m$. We can assume without loss of generality that C is irreducible because it has at most d irreducible components each of which has degree not exceeding the degree of C, and d is a constant fixed in advance. We also note that there exists a polynomial $\varphi \in F$ such that

$$\overline{\varphi}\left(\underbrace{C \times \ldots \times C}_{m \text{ times}}\right) \neq 0$$

because of the assumption (i), and hence we can apply Lemma 3.7 to find, since the set $C' \subset C$ is sufficiently large, a subfamily of m distinct points

$$(p_{j_1}, q_{j_1}), \ldots, (p_{j_m}, q_{j_m}) \in C'$$

that fulfill the condition

$$\overline{\varphi}\,(p_{j_1}, q_{j_1}, \ldots, p_{j_m}, q_{j_m}) \neq 0,$$

which means that the set $J' = \{j_1, \ldots, j_m\}$ satisfies the condition (4.2).

We proceed the argument. By the assumption (i), the family F should contain at least one nonzero polynomial mapping ψ. We can consider the correponding mapping $\overline{\psi}$ as a polynomial in the variables $x_1, y_1, \ldots, x_{m-1}, y_{m-1}$ with the coefficients in $\mathbb{C}[x_m, y_m]$, and we denote by $\sigma \in \mathbb{C}[x_m, y_m]$ one arbitrary nonzero coefficient arising in this way. Now we can assume that

$$\sigma(p_{j_m}, q_{j_m}) \neq 0, \quad \text{for some} \quad j_m \in J,$$

since otherwise all the points of the form (p_j, q_j) would lie on the curve $\sigma = 0$, and this situation is already considered in the special case 2. Therefore, the polynomial

$$\overline{\psi}_{m-1} := \overline{\psi}\,(x_1, y_1, \ldots, x_{m-1}, y_{m-1}, p_{j_m}, q_{j_m}) \in \mathbb{C}[x_1, y_1, \ldots, x_{m-1}, y_{m-1}]$$

is nonzero. We proceed with the application of the same argument to $\overline{\psi}_{m-1}$ which eventually leads us to indexes $j_1, \ldots, j_m$ in J such that

$$\overline{\psi}\,(p_{j_1}, q_{j_1}, \ldots, p_{j_m}, q_{j_m}) \neq 0,$$

and then the set $J' = \{j_1, \ldots, j_m\}$ satisfies the condition (4.2). □

5. A set H in Theorem 4.3

In the rest of our paper, we conclude the construction of our counterexamples, to which end we need to define the corresponding sign pattern $\mathcal{S}$ and algebraic set H as in Theorem 4.3. It turns out that the construction of H is somewhat more straightforward, and we proceed without the specification of the relevant value of m, which is assumed to be an arbitrary large integer in this section.

Definition 5.1. Let U be the set of all polynomials $f \in \mathbb{C}_{2m}$ for which there exist $\alpha, \beta \in \mathbb{C}$ such that the polynomial $f(t) - \alpha$ divides $t^2 - \beta$, and the quotient

$$\frac{f(t) - \alpha}{t^2 - \beta} \tag{5.1}$$

is either

(1) divisible by $(t-u)^2(t-v)^2$, for some $u, v \in \mathbb{C}$,
(2) divisible by $(t^2-u)(t^2-v)$, for some $u, v \in \mathbb{C}$.

Definition 5.2. We define H as the Euclidean closure of U.

Observation 5.3. *The set H is algebraic.*

Proof. Let U_1 be the image of the mapping $\mathbb{C}^{2m-2} \to \mathbb{C}_{2m}$ sending a vector

$$x = (a, b, u, v, w_0, \ldots, w_{2m-7})$$

to the polynomial

$$\left(t^{2m-6} + \sum_{j=0}^{2m-7} w_j t^j\right)(t^2 - b)(t-u)^2(t-v)^2 + a,$$

and let U_2 be the image of the mapping that sends x to

$$\left(t^{2m-6} + \sum_{j=0}^{2m-7} w_j t^j\right)(t^2 - b)(t^2-u)(t^2-v) + a.$$

We recall that the image of a polynomial mapping $\mathbb{C}^n \to \mathbb{C}^q$ is always *constructible* by a well known theorem of Chevalley, see [31] for a recent account, which means that the sets U_1 and U_2 are constructible. Furthermore, the *Zariski closure* coincides with the Euclidean closure for all constructible sets over the complex numbers, see Theorem 2.33 in [50], so the Euclidean closures of the sets U_1 and U_2 are algebraic. Since we have $U = U_1 \cup U_2$, it remains to apply a basic fact that the union of finitely many algebraic sets is algebraic, see Lemma 2 in [13, Chapter 1, §2]. □

The rest of this section is devoted to the condition (i) in Theorem 4.3.

Definition 5.4. For any polynomial $f \in \mathbb{C}[t]$, we define the *even part* f_{ev} and odd part f_{od} as the unique polynomials satisfying $f(t) = f_{\text{ev}}\left(t^2\right) + t\, f_{\text{od}}\left(t^2\right)$.

Example 5.5. If $f = t^{10} + t^3 + 2t^2 + 3t + 4$, then $f_{\text{ev}} = t^5 + 2t + 4$ and $f_{\text{od}} = t + 3$.

Observation 5.6. *The following conditions are equivalent for $f \in \mathbb{C}[t]$ and $b \in \mathbb{C}$:*

(1) the polynomial $(t^2 - b)$ divides f,
(2) $f_{ev}(b) = f_{od}(b) = 0$.

Proof. If $f_{\rm ev}(b) = p$, $f_{\rm od}(b) = q$, then there are $g, h \in \mathbb{C}[x]$ and $p, q \in \mathbb{C}$ such that

$$f_{\rm ev}(x) = (x - b)\, g(x) + p \text{ and } f_{\rm od}(x) = (x - b)\, h(x) + q,$$

and then

$$f(t) = (t^2 - b) \left(g\left(t^2\right) + t\, h\left(t^2\right)\right) + qt + p,$$

which means that the remainder of $f(t)$ modulo $t^2 - b$ is $qt + p$. □

We proceed with several lemmas on the sets U and H.

Lemma 5.7. *If $f \in U$, then $f_{od}(\beta) = 0$, $f_{ev}(\beta) = \alpha$ for the values in Definition 5.1.*

Proof. We use Observation 5.6. The first equality $f_{\rm od}(\beta) = 0$ follows because the odd parts of $f - \alpha$ and f coincide, and the second equality $f_{\rm ev}(\beta) = \alpha$ can be observed because the even part of $f - \alpha$ equals $f_{\rm ev} - \alpha$. □

Remark 5.8. In the following lemma, the condition $\deg f_{\rm od} = m - 1$ means that f has a nonzero number at the place of the $(2m - 1)$-th degree coefficient.

Lemma 5.9. *Let $f \in H$ be a polynomial with $\deg f_{od} = m - 1$. Then $f \in U$.*

Proof. The condition $f \in H$ means that there exists a sequence (f_n) of polynomials in U such that $f_n \to f$ as $n \to \infty$. We write (α_n) and (β_n) to denote the number sequences that correspond to (f_n) in terms of Definition 5.1. We get

$$(f_n)_{\rm od} (\beta_n) = 0 \quad \text{and} \quad (f_n)_{\rm ev} (\beta_n) = \alpha_n, \tag{5.2}$$

by Lemma 5.7, and we remark that, for all sufficiently large indexes n, we have

$$\deg f_{\rm od} = \deg (f_n)_{\rm od} = m - 1 \tag{5.3}$$

in view of the assumption in the formulation of the lemma. The equalities (5.3) can be used to show that the sequence (β_n) is bounded, which employs the fact that the roots of a monic polynomial depend continuously on its coefficients, see Theorem B in [32] for a precise statement. Therefore, the sequence (β_n) admits a converging subsequence, which allows us to assume without loss of generality that (β_n) has some limit β as $n \to \infty$. Passing to the limits in (5.2), we get

$$f_{\rm od}(\beta) = 0 \quad \text{and} \quad f_{\rm ev}(\beta) = \alpha, \tag{5.4}$$

where α is the limit of (α_n). In view of Observation 5.6, the equalities (5.4) imply that $f(t) - \alpha$ is divisible by $(t^2 - \beta)$, and, again by taking the limits, we get

$$\frac{f_n(t) - \alpha_n}{t^2 - \beta_n} \to \frac{f(t) - \alpha}{t^2 - \beta} \tag{5.5}$$

as $n \to \infty$. We recall that, directly from our application of Definition 5.1, we know that every left-hand side polynomial in (5.5) should be either

(1) divisible by $(t - u)^2(t - v)^2$, for some $u, v \in \mathbb{C}$, or
(2) divisible by $(t^2 - u)(t^2 - v)$, for some $u, v \in \mathbb{C}$.

One easily observes that the conditions (1) and (2) in Lemma 5.7 are preserved by taking the limit. Therefore, the polynomial on the right-hand side of (5.5) should satisfy at least one of the conditions (1) or (2). This means that the polynomial f satisfies the assumptions of Definition 5.1, and hence we have $f \in U$. □

We recall that the assumption (i) in Theorem 4.3 states that the polynomial

$$\pi(t) = \left(t^2 + pt + q\right)^m \quad \text{with real numbers} \quad p \neq 0, \quad q > 0 \tag{5.6}$$

does not belong to H. In view of Lemma 5.9, it suffices to prove $\pi \notin U$.

Lemma 5.10. *We have $\pi \notin U$ for the polynomial in* (5.6).

Proof. We argue by contradiction and assume the converse, which means that Definition 5.1 applies to π. Therefore, there exist $\alpha, \beta \in \mathbb{C}$ such that the polynomial $\pi - \alpha$ is divided by $t^2 - \beta$, and the quotient

$$\frac{\pi - \alpha}{t^2 - \beta} \tag{5.7}$$

is either

(1) divisible by $(t-u)^2(t-v)^2$, for some $u, v \in \mathbb{C}$,
(2) divisible by $(t^2-u)(t^2-v)$, for some $u, v \in \mathbb{C}$.

Since every pair of distinct roots of π sum to $-p \neq 0$, and since 0 is not a root of π by $q \neq 0$, the polynomial $(t^2 - \beta)$ cannot divide π with any $\beta \in \mathbb{C}$, and hence

$$\alpha \neq 0. \tag{5.8}$$

Now the expression

$$(\pi - \alpha)' = m(2t+p)\left(t^2 + pt + q\right)^{m-1} \tag{5.9}$$

for the derivative of $\pi - \alpha$ shows that every multiple root ρ of $\pi - \alpha$ should either be equal to $-p/2$ or be a root of $t^2 + pt + q = 0$. The latter would mean that ρ is a root of π, which is impossible by the condition (5.8). Therefore, such a ρ should be equal to $-p/2$ and be a simple root of the derivative (5.9), which means that $\pi - \alpha$ can have at most one multiple root, and this root cannot have multiplicity greater than two. Therefore, the condition (1) cannot be satisfied.

Now it remains to invalidate the condition (2) to reach the desired contradiction. It is clear from Definition 5.4 that

$$\pi(t) = \pi_{\text{ev}}\left(t^2\right) + t\,\pi_{\text{od}}\left(t^2\right) \quad \text{and} \quad \pi(-t) = \pi_{\text{ev}}\left(t^2\right) - t\,\pi_{\text{od}}\left(t^2\right),$$

which implies

$$2t\pi_{\text{od}}\left(t^2\right) = (t^2 + pt + q)^m - (t^2 - pt + q)^m$$

and if we take $t = x + iy$, with $x, y \in \mathbb{R}$, to be a root of $\pi_{\text{od}}\left(t^2\right) = 0$, we get

$$\left|(x+iy)^2 + p(x+iy) + q\right| = \left|(x+iy)^2 - p(x+iy) + q\right|,$$

where i is the imaginary unit. This simplifies to $px(q + x^2 + y^2) = 0$, and since we have $p \neq 0$, $q > 0$, we get $x = 0$, which means that the equation $\pi_{\text{od}}(\beta) = 0$ can only be satisfied if β is the square of a number in $i\mathbb{R}$. An application of Lemma 5.7 confirms that $\pi_{\text{od}}(\beta) = 0$, and hence we get

$$\beta \in \mathbb{R} \quad \text{and} \quad \beta \leqslant 0. \tag{5.10}$$

The consideration leading to (5.10) still applies if we replace β by any of the numbers u, v in the condition (2) above in this proof, so

(5.11) the polynomial $\pi - \alpha$ has at least six roots on $i\mathbb{R}$, counting multiplicities.

The set of all such roots, represented as $\xi + i\gamma$, with $\xi, \gamma \in \mathbb{R}$, lies on the curve

$$\left|q + p(\xi + i\gamma) + (\xi + i\gamma)^2\right| = r \tag{5.12}$$

with r being the m-th arithmetic root of $|\alpha|$. The formula (5.12) defines a bounded curve of degree four, so it cannot have more than four points on any straight line, and hence the polynomial $\pi - \alpha$ cannot have more than four roots on $i\mathbb{R}$ if we do not count multiplicities. However, during the consideration of the condition (1) we noticed that this polynomial can have at most one multiple root, and if this happens, the corresponding multiplicity equals two, and hence the polynomial $\pi - \alpha$ cannot have a total of more than five roots on $i\mathbb{R}$, counting multiplicities. Therefore, we reach a contradiction to the condition (5.11) and complete the proof. $\square$

We have just proved the validity of the condition (i) in Theorem 4.3.

Corollary 5.11. *We have $\pi \notin H$ for the polynomial in* (5.6).

Proof. Follows from Lemmas 5.9 and 5.10. $\square$

6. A sign pattern $\mathcal{S}$ in Theorem 4.3

This section specifies a sign pattern $\mathcal{S}$ relevant for Theorem 4.3 up to one auxiliary intermediate construction. We are still not ready to specify the corresponding value of m, but now we need to determine its parity.

Definition 6.1. We set $m = 2\mu + 1$ for some positive integer μ.

Definition 6.2. If (n_1, n_2, n_3) is the inertia of some polynomial as in Definition 4.1, then n_1 and n_2 are called its positive and negative *inertia indexes*, respectively.

Definition 6.3. A polynomial $\varphi \in \mathbb{R}_{4\mu}$ is called *balanced* if $\varphi(0) > 0$, and both the positive and negative inertia indexes of φ are between $2\mu - 4$ and $2\mu + 2$.

Definition 6.4. A polynomial $\varphi \in \mathbb{C}_{4\mu}$ is called *almost square-free* if φ has at most one multiple root which, if it exists, has multiplicity not greater than three.

Definition 6.5. A $4\mu \times 4\mu$ sign pattern $\mathcal{A}$ is called *admissible* if the following conditions are satisfied for all indexes i, j in $\{1, \ldots, 4\mu\}$:

(A1) the (i, j) entry of $\mathcal{A}$ is nonzero if $j = i + 1$,
(A2) the (i, j) entry of $\mathcal{A}$ is zero whenever $j \geqslant i + 2$,
(A3) any almost square-free balanced polynomial in $\mathbb{R}_{4\mu}$ belongs to $\chi(\mathcal{A})$.

We proceed with a construction of a pattern $\mathcal{S}$ relevant to Theorem 4.3. In what follows, the notation $M(i, j)$ stands for the (i, j) entry of a matrix M.

Definition 6.6. Let $\mathcal{A}$ be a $4\mu \times 4\mu$ admissible sign pattern. We define the sign pattern $\mathcal{S} = \mathcal{S}(A)$ of the size $(4\mu + 2) \times (4\mu + 2)$ as follows:

(S0) the upper left $4\mu \times 4\mu$ block of $\mathcal{S}(\mathcal{A})$ equals $\mathcal{A}$,
(S1) $\mathcal{S}$ has the plus sign at the positions $(4\mu, 4\mu + 1)$, $(4\mu + 1, 4\mu + 2)$,
(S2) $\mathcal{S}$ has the minus sign at the position $(4\mu + 2, 4\mu + 1)$,
(S3) $\mathcal{S}(4\mu + 2, 1) = \mathcal{A}(1, 2) \cdot \mathcal{A}(2, 3) \cdot \ldots \cdot \mathcal{A}(4\mu - 1, 4\mu)$,
(S4) all entries of $\mathcal{S}$ not specified in (S0)–(S3) are zero.

We can describe the set of polynomials realized by this new pattern.

Lemma 6.7. *Let $\mathcal{A}$ be an admissible $4\mu \times 4\mu$ sign pattern, and let $\mathcal{S}$ be the pattern as in Definition 6.6. A polynomial $h \in \mathbb{R}_{4\mu+2}$ belongs to $\chi(\mathcal{S})$ if and only if there exist positive real numbers a, b and a polynomial $g \in \chi(\mathcal{A})$ such that $h(t) = (t^2 + b)g(t) - a$.*

Proof. If we replace the $(4\mu + 2, 1)$ entry of $\mathcal{S}$ by a zero, then we get a block triangular sign pattern whose diagonal blocks

$$\mathcal{A} \quad \text{and} \quad \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}$$

realize the characteristic polynomials $g(t)$ and (t^2+b). The summands involving the $(4\mu+2,1)$ entry of $\mathcal{S}$ contribute an arbitrary negative number to the corresponding characteristic polynomial because the cofactor of this entry is the lower triangular matrix with the entries $(1,2),\ldots,(4\mu+1,4\mu+2)$ on the diagonal, and the sign of this contribution is determined by the condition (S3) in Definition 6.6. $\square$

Now we reduce Conjecture 1.1 to the existence of sparse admissible patterns.

Lemma 6.8. *If there exists a $4\mu\times4\mu$ admissible sign pattern $\mathcal{A}$ with less than 8μ nonzero entries, then Conjecture 1.1 is false.*

Proof. We apply Theorem 4.3 to the pattern $\mathcal{S}=\mathcal{S}(\mathcal{A})$ introduced in Definition 6.6 and algebraic set H constructed in Definition 5.2. The condition (i) in Theorem 4.3 is already confirmed in Corollary 5.11, so the remaining conditions are (o) and (ii). In the rest of the proof we check their validity, which would imply that $\mathcal{S}$ is almost sap by Theorem 4.3, and, since the number of nonzero entries of $\mathcal{S}$ equals four plus the number of nonzero entries of $\mathcal{A}$, that is, the latter number is less than $8\mu+4$, an application of Observation 2.6 would complete the argument.

In order to confirm the condition (o), we need to check that, if f is a monic polynomial of degree $4\mu+2$ with real coefficients and with all roots on the line $i\mathbb{R}$, then $f\in\chi(\mathcal{S})$. In fact, any such polynomial represents as

$$f(t)=\varphi(t^2) \tag{6.1}$$

for some polynomial φ of degree $2\mu+1$ with nonnegative real coefficients. For large positive α, the roots of the polynomial $\varphi+\alpha^{2\mu+1}$ have the form[1]

$$\alpha\,\varepsilon_j\,u_j(\alpha)\quad\text{with}\quad j\in\{1,\ldots,2\mu+1\}, \tag{6.2}$$

where $\{\varepsilon_1,\ldots,\varepsilon_{2\mu+1}\}$ is the family of all roots of -1 of the degree $2\mu+1$, and every $u_j(\alpha)$ is a function that approaches 1 as $\alpha\to\infty$. One of the roots in (6.2) should be real because the degree of φ is odd, and, in fact, this should be the root corresponding to an index k with $\varepsilon_k=-1$. Since the roots of the polynomial

$$g(t)=\frac{f(t)+\alpha^{2\mu+1}}{t^2+\alpha u_k(\alpha)} \tag{6.3}$$

come as the square roots of all but the k-th number in (6.2), the polynomial g has the positive and negative inertia indexes equal to each other. Similarly, since the numbers in (6.2) have nonzero imaginary parts unless $j=k$, the polynomial g has no roots on the line $i\mathbb{R}$, which implies, finally, that the inertia of g equals $(2\mu,2\mu,0)$, and hence g is balanced in terms of Definition 6.3. Similarly, the fact that the roots in (6.2) are simple implies that the polynomial g has no multiple roots, and hence g is almost square-free in the notation of Definition 6.4. Therefore, we get $g\in\chi(\mathcal{A})$ by the condition (A3) in Definition 6.5, and hence f belongs to $\chi(\mathcal{S})$ by Lemma 6.7.

We proceed with the condition (ii) in Theorem 4.3. We take

$$f\in\mathcal{P}_{4\mu+2}\cup\mathcal{N}_{4\mu+2} \tag{6.4}$$

and we note that, as long as a increases from 0 to $+\infty$, the inertia of the polynomial $f+a$ changes from either $(4\mu+2,0,0)$ or $(0,4\mu+2,0)$ at $a=0$ to one of

$$(2\mu+2,\,2\mu,\,0),\ \ (2\mu,\,2\mu+2,\,0)$$

[1]Again, we use the continuity of the roots of a monic polynomial with variable coefficients [32].

at every sufficiently large value of a. Also, the condition (6.4) implies

$$f(0) > 0, \tag{6.5}$$

and hence $f + a$ cannot have zero roots for nonnegative a. Therefore, the inertia of $f + a$ can change, as long as a moves from 0 to $+\infty$, only at those values of a which make $f + a$ have purely imaginary roots. Therefore, if we take D as the set of all positive a for which there exists a positive number b such that the polynomial $(t^2 + b)$ divides $f(t) + a$, we get that D is finite and non-empty, and the polynomial

$$g(t) = \frac{f(t) + \alpha}{t^2 + \beta}$$

corresponding to $\alpha = \max D$ satisfies the condition that

(6.6) both the positive and negative inertia indexes of g are at most $2\mu + 2$.

The situation splits into the two cases.

Case 1. If $f \in H$, then there is nothing to prove in the current condition (ii).

Case 2. If $f \notin H$, then $f \notin U$ by Definition 5.2, and hence the conditions (1) and (2) in Definition 5.1 are not satisfied by g. The failure of (1) shows that g is almost square-free in the notation of Definition 6.4. Also, the condition (6.6) together with the negation of the condition (2) in Definition 5.1 show that both the positive and negative inertia indexes of g are at least $2\mu - 4$, and, together with the inequality (6.5), which implies $g(0) > 0$, this shows that the polynomial g is balanced in terms of Definition 6.3. This implies $g \in \chi(\mathcal{A})$ from the item (A3) in Definition 6.5, and hence we get the desired condition $f \in \chi(\mathcal{S})$ from Lemma 6.7.

The cases 1 and 2 complete the proof of the condition (ii) in Theorem 4.3, which confirms the assertion of the current lemma. □

7. A sparse admissible pattern

Our next goal is to construct a pattern which has less than 8μ nonzero entries and satisfies the assumptions of Definition 6.5. We begin with two simple observations.

Observation 7.1. *Let S be an $n \times n$ sign pattern with even n. Then, for any polynomial f, we have $f(t) \in \chi(S)$ if and only if $f(-t) \in \chi(-S)$.*

Proof. Follows from the definitions of the characteristic polynomial and $\chi(S)$. □

Observation 7.2. *For any integer k, there exists a sign pattern $\mathcal{C}_k$ such that*

(C1) the (i, j) entry of $\mathcal{C}_k$ is nonzero if $j = i + 1$,
(C2) the (i, j) entry of $\mathcal{C}_k$ is zero if $j \geqslant i + 2$,
(C3) $\mathcal{C}_k$ has exactly $2k - 1$ nonzero entries,
(C4) $\chi(\mathcal{C}_k)$ contains all degree-k monic polynomials with all coefficients positive.

Proof. One can define $\mathcal{C}_k$ as a *signing* of the standard *companion matrix* pattern in a way similar to [19, 20]. Namely, one can take the plus signs at the positions

$$(1, 2),\ (2, 3),\ \ldots,\ (k - 1, k)$$

and the minus signs at the entries $(j, 1)$ with all $j \in \{1, \ldots, k\}$. This gives a total of $2k - 1$ nonzero entries, and all the other entries are set to be zero. □

We proceed with another important building block of our admissible pattern.

Assumption 7.3. The 6×6 sign pattern

$$\mathcal{W} = \begin{pmatrix} - & + & 0 & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 & 0 \\ 0 & 0 & + & - & 0 & 0 \\ 0 & 0 & + & 0 & + & 0 \\ + & 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & - & 0 \end{pmatrix}$$

possesses the following property. There exists a positive integer s such that, for any set U containing at least s distinct polynomials of the form

$$t^2 + pt + q \quad \text{with real numbers} \quad p > 0, \quad q > 0,$$

and for any set V containing at least s distinct polynomials of the form

$$t^2 + pt + q \quad \text{with real numbers} \quad p \neq 0, \quad q > 0,$$

there exist distinct $u_1, u_2 \in U$ and $v \in V$ such that $u_1 u_2 v \in \chi(\mathcal{W})$.

Our proof of Assumption 7.3 requires some further computational effort, so we decided to give it separately in Section 8. Now we are ready to confirm that this assumption is sufficient to disprove Conjecture 1.1.

Theorem 7.4. *Assumption 7.3 implies the invalidity of Conjecture 1.1.*

Proof. In view of Lemma 6.8, it suffices to construct a $4\mu \times 4\mu$ admissible sign pattern $\mathcal{A}$ with less than 8μ nonzero entries. To this end, we take $\mathcal{W}$ and s as in Assumption 7.3, we also take $\mathcal{C}_{4s}$ as in Observation 7.2, and we define

$$\mathcal{A} = \left(\begin{array}{c|c|c|c} \mathcal{W} & E_{6\times 6} & O_{6\times 4s} & O_{6\times 4s} \\ \hline O_{6\times 6} & -\mathcal{W} & E_{6\times 4s} & O_{6\times 4s} \\ \hline O_{4s\times 6} & O_{4s\times 6} & \mathcal{C}_{4s} & E_{4s\times 4s} \\ \hline O_{4s\times 6} & O_{4s\times 6} & O_{4s\times 4s} & -\mathcal{C}_{4s} \end{array}\right) \tag{7.1}$$

to be the sign pattern of the size $4\mu \times 4\mu$ with $4\mu = 4(2s+3)$. Here, the notation $E_{a\times b}$ stands for the $a \times b$ matrix which has the plus sign the $(a, 1)$ position and zeros everywhere else, and $O_{a\times b}$ is the $a \times b$ zero matrix. The conditions (A1) and (A2) in Definition 6.5 follow directly from our notation, and it is also clear that $\mathcal{A}$ has exactly $8\mu - 1$ nonzero entries.

Therefore, it remains to confirm the condition (A3) in Definition 6.5. To this end, we take a monic polynomial f of degree 4μ which is balanced in the notation of Definition 6.3 and almost square-free in terms of Definition 6.4. Since f is balanced, we can represent it as

$$f = \pi \cdot \nu \cdot \varphi_1 \cdot \varphi_2 \cdot \varphi_3 \cdot \varphi_4,$$

where π, ν are polynomials in $\mathbb{R}_{2\mu-4}$ with the inertias $(2\mu-4, 0, 0)$, $(0, 2\mu-4, 0)$, respectively, and

$$\varphi_j = t^2 + p_j t + q_j \quad \text{with} \quad p_j, q_j \in \mathbb{R} \text{ and } q_j > 0, \tag{7.2}$$

for $j \in \{1, 2, 3, 4\}$. Since, according to Definition 6.3, the polynomial f can have neither the positive inertia index nor the negative inertia index greater than $2\mu+2$, we get that

$$\text{there are } j_1, j_2 \in \{1, 2, 3, 4\} \text{ such that } p_{j_1} \geqslant 0 \text{ and } p_{j_2} \leqslant 0. \tag{7.3}$$

Furthermore, we use the representations of π and ν as the products of polynomials irreducible over $\mathbb{R}$, and we find sets Pos and Neg each consisting of $\mu - 2 = 2s+1$ polynomials of the form

$$t^2 + pt + q \quad \text{with} \quad q > 0$$

and $p > 0$ in the case of Pos, and with $p < 0$ in the case of Neg, such that ν is the product of all polynomials in Pos, and π is the product of all polynomials in Neg. Since the polynomial f was almost square-free, the elements of Pos and Neg cannot repeat, so they are sets indeed in the sense that all their elements are pairwise different. Also, we take an arbitrary splitting of these sets

$$\mathrm{Pos} = \mathrm{Pos}_1 \cup \mathrm{Pos}_2 \quad \text{and} \quad \mathrm{Neg} = \mathrm{Neg}_1 \cup \mathrm{Neg}_2$$

into the disjoint unions of the corresponding sets of cardinalities at least s each, which is possible because, as said above, we have $|\,\mathrm{Pos}\,| = |\,\mathrm{Neg}\,| = 2s+1$.

Now we are going to represent f as an element in $\chi(\mathcal{A})$. Since $\mathcal{A}$ has the corresponding block triangular form (7.1), we get

$$\chi(\mathcal{A}) = \chi(\mathcal{W}) \cdot \chi(-\mathcal{W}) \cdot \chi(\mathcal{C}_{4s}) \cdot \chi(-\mathcal{C}_{4s}), \tag{7.4}$$

and hence we need to represent f as the product of the four polynomials that belong to the four corresponding sets in (7.4). The situation splits into three cases, which depend on the signs of the numbers p_j in (7.2).

Case 1. If $p_1 \geqslant 0$, $p_2 \geqslant 0$, $p_3 \geqslant 0$, then, in view of the condition (7.3), we can assume without loss of generality that $p_4 \leqslant 0$. We use Assumption 7.3 and find distinct $\pi_1, \pi_2 \in \mathrm{Pos}_1$ and $\pi_3 \in \mathrm{Pos}_2$ such that $\pi_1\pi_2\pi_3 \in \chi(\mathcal{W})$. Using Assumption 7.3 again, but at this time taking into account Observation 7.1, we find distinct

$$\nu_1, \nu_2 \in \mathrm{Neg} \quad \text{and} \quad \pi_4 \in \mathrm{Pos} \setminus \{\pi_1, \pi_2, \pi_3\}$$

such that $\nu_1\nu_2\pi_4 \in \chi(-\mathcal{W})$. It remains to note that the products of all elements of

$$\{\varphi_1, \varphi_2, \varphi_3\} \cup \mathrm{Pos} \setminus \{\pi_1, \pi_2, \pi_3, \pi_4\} \quad \text{and} \quad \{\varphi_4\} \cup \mathrm{Neg} \setminus \{\nu_1, \nu_2\}$$

belong to $\chi(\mathcal{C}_{4s})$ and $\chi(-\mathcal{C}_{4s})$, respectively.

Case 2. If $p_1 \leqslant 0$, $p_2 \leqslant 0$, $p_3 \leqslant 0$, then the proof is similar to the previous case. In fact, we can assume without loss of generality that $p_4 \geqslant 0$, and we can find pairwise distinct polynomials $\pi_1, \pi_2 \in \mathrm{Pos}$ and $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathrm{Neg}$ such that $\pi_1\pi_2\nu_1 \in \chi(\mathcal{W})$ and $\nu_2\nu_3\nu_4 \in \chi(-\mathcal{W})$, and the products over the sets

$$\{\varphi_4\} \cup \mathrm{Pos} \setminus \{\pi_1, \pi_2\} \quad \text{and} \quad \{\varphi_1, \varphi_2, \varphi_3\} \cup \mathrm{Neg} \setminus \{\nu_1, \nu_2, \nu_3, \nu_4\}$$

belong to $\chi(\mathcal{C}_{4s})$ and $\chi(-\mathcal{C}_{4s})$, respectively.

Case 3. If $p_1 \geqslant 0$, $p_2 \geqslant 0$ and $p_3 \leqslant 0$, $p_4 \leqslant 0$, then, again, we complete the proof in a similar fashion. Using Assumption 7.3, we find distinct $\pi_1, \pi_2 \in \mathrm{Pos}_1$ and $\pi_3 \in \mathrm{Pos}_2$ such that $\pi_1\pi_2\pi_3 \in \chi(\mathcal{W})$, and also we find distinct $\nu_1, \nu_2 \in \mathrm{Neg}_1$ and $\nu_3 \in \mathrm{Neg}_2$ such that $\nu_1\nu_2\nu_3 \in \chi(-\mathcal{W})$, and then the products over the sets

$$\{\varphi_1, \varphi_2\} \cup \mathrm{Pos} \setminus \{\pi_1, \pi_2, \pi_3\} \quad \text{and} \quad \{\varphi_3, \varphi_4\} \cup \mathrm{Neg} \setminus \{\nu_1, \nu_2, \nu_3\}$$

belong to $\chi(\mathcal{C}_{4s})$ and $\chi(-\mathcal{C}_{4s})$, respectively.

Up to a possible relabeling of the indexes j in (7.2), the cases 1–3 cover all possibilities, and hence the proof is complete. $\square$

8. A proof of Assumption 7.3

Now we are going to prove Assumption 7.3 and finalize the paper. To this end, we need to study the set of polynomials realizable by the sign pattern $\mathcal{W}$ in Assumption 7.3. As we see later, the polynomials that are problematic from this point of view belong to the zero set of the following mapping.

Definition 8.1. We define the polynomial mapping Δ from $\mathbb{C}_6$ to $\mathbb{C}$ by the formula

$$\Delta(\psi) = c_1^3c_6 - c_1^3c_2c_4 + c_1^2c_2^2c_3 + c_1^2c_3c_4 - 2c_1c_2c_3^2 + c_3^3 \tag{8.1}$$

if $\psi = t^6 + c_1t^5 + c_2t^4 + c_3t^3 + c_4t^2 + c_5t + c_6$ is an arbitrary polynomial.

Lemma 8.2. *Let $\psi \in \mathbb{C}_6$ be a polynomial such that $\Delta(\psi) \neq 0$. Then*

$$\psi(t) + xt \text{ is not divisible by } t^4 + yt^2 + z$$

for any $x, y, z \in \mathbb{C}$.

Proof. If we write

$$\psi = t^6 + c_1t^5 + c_2t^4 + c_3t^3 + c_4t^2 + c_5t + c_6,$$

then the remainder of $\psi(t) + xt$ modulo $t^4 + yt^2 + z$ equals

$$(c_3 - c_1y)\,t^3 + (y^2 - z + c_4 - c_2y)\,t^2 + (c_5 + x - c_1z)\,t + (c_6 + yz - c_2z),$$

so this remainder is zero if and only if

$$\begin{cases} c_3 - c_1y = 0, \\ y^2 - z + c_4 - c_2y = 0, \\ c_5 + x - c_1z = 0, \\ c_6 + yz - c_2z = 0. \end{cases} \tag{8.2}$$

At this point, it is easy to eliminate x, y, z from the equations (8.2). In fact, the third of these equations allows one to express x as a function of the other variables, and x does not appear in any other equation. So we can remove this third equation, and we can further express y and z from the first and second equations in (8.2), respectively. Then we compare the expressions for y and z with the remaining fourth equation, and we get $\Delta(\psi) = 0$ as an outcome of the elimination. □

In the rest of this section, we use the notation Δ to denote the mapping in (8.1), and we write $\mathcal{W}$ to denote the sign pattern in Assumption 7.3.

Lemma 8.3. *We have $h \in \chi(\mathcal{W})$ if and only if*

$$h(t) - at = (t^2 + b)(t^2 + p_1t + q_1)(t^2 - p_2 + q_2)$$

for some positive a, b, p_1, p_2, q_1, q_2 in $\mathbb{R}$.

Proof. The proof is similar to Lemma 6.7. If we replace the $(5, 1)$ entry of $\mathcal{W}$ by a zero, then we get a block triangular sign pattern whose diagonal blocks are

$$\begin{pmatrix} - & + \\ - & 0 \end{pmatrix}, \begin{pmatrix} + & - \\ + & 0 \end{pmatrix}, \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix},$$

and these blocks realize the corresponding characteristic polynomials

$$(t^2 + p_1t + q_1),\ (t^2 - p_2 + q_2),\ (t^2 + b).$$

Finally, the summand involving the $(5, 1)$ entry of $\mathcal{W}$ contributes the at summand to the characteristic polynomial of a matrix with the pattern $\mathcal{W}$. □

Observation 8.4. *If $h \in \mathbb{R}_6$ is a polynomial with $h(0) > 0$, then, for any sufficiently large positive a, the polynomial $h(t) - at$ has inertia $(4, 2, 0)$.*

Proof. Due to the continuity of the roots of a monic polynomial as functions of its coefficients, it suffices to consider the polynomial $t^6 + h(0)$ instead of h. □

Lemma 8.5. *Let U, V be sets as in Assumption 7.3. The condition $u_1u_2v \in \chi(\mathcal{W})$ can be false for $u_1, u_2 \in U$ and $v \in V$ only if $\Delta(u_1u_2v) = 0$.*

Proof. We take $h = u_1u_2v$. Using the definitions of U, V in Assumption 7.3, we note that the inertia of h is either $(2, 4, 0)$ or $(0, 6, 0)$, and also we have $h(0) > 0$.

We proceed with the argument similar to a consideration in Lemma 6.8. Namely, as long as we increase the value of a from 0 to $+\infty$, the inertia of the polynomial $h(t) - at$ changes from either $(2, 4, 0)$ or $(0, 6, 0)$ at $a = 0$ to the inertia $(4, 2, 0)$ taken at every sufficiently large value of a by Observation 8.4. Therefore, we can take α as the supremum of the set of all positive a for which the positive inertia index of $h(t)-at$ is less than 4. Then, since the condition $h(0) > 0$ guarantees that the polynomial $h(t)-at$ has no zero roots for nonnegative a, the inertia of $h(t)-at$ can change, as long as a moves from 0 to $+\infty$, only at those values of a which make $h(t) - at$ have purely imaginary roots. Therefore, there exists some positive b such that the polynomial $(t^2 + b)$ divides $h(t) - \alpha t$, and their quotient

$$g(t) = \frac{h(t) - \alpha t}{t^2 + b}$$

has both the positive and negative inertia indexes not greater than 2. If g has roots on $i\mathbb{R}$, then we get $\Delta(h) = 0$ by Lemma 8.2, so it remains to consider the case that the inertia of g is $(2, 2, 0)$, which implies $h \in \chi(\mathcal{W})$ by Lemma 8.3. □

In order to complete the proof of the main result, we need several computational claims below. As we can see, Claims 8.6 and 8.7 are still straightforward and do not require any hard computation, but the proof of Claim 8.8 is somewhat more demanding. We hope that our comments on Claim 8.8 can allow one to check it easily on a computer and show the way how to perform this computation quite quickly even if we do not want to rely on the work of a computer.

Claim 8.6. *For all u, v in the set U as in Assumption 7.3, the polynomial mapping $\mathbb{C}_2 \to \mathbb{C}$ defined as $\pi \to \Delta(uv\pi)$ is not identically zero.*

Proof. According to Assumption 7.3, we can write

$$u = t^2 + pt + q,\ v = t^2 + p't + q',\ \pi = t^2 + xt + y \tag{8.3}$$

with positive real numbers p, p', q, q'. We note that $c_1^2c_2^2c_3$ is the only summand in the expression (8.1) with a nonzero contribution to the coefficient of x^2y^3 in $\pi \to \Delta(uv\pi)$. This coefficient equals $p + p' > 0$, which implies $\Delta(uv\pi) \neq 0$. □

Claim 8.7. *If u, v are distinct elements of the set U as in Assumption 7.3, then the polynomial mappings $\mathbb{C}_2 \to \mathbb{C}$ defined as*

$$\pi \to \Delta(u^2\pi) \quad \text{and} \quad \pi \to \Delta(v^2\pi) \tag{8.4}$$

are not linearly dependent.

Proof. We keep using the notation (8.3), and we conclude, similarly to Claim 8.6, that the coefficients of x^2y^3, x^3y^2, x^4y in $\Delta(u^2\pi)$ equal

$$2p, \quad 8p^2, \quad 10p^3 + 2pq,$$

respectively. Since $p \neq 0$ and $p' \neq 0$, the functions (8.4) can have linearly dependent pairs of the coefficients corresponding to (x^2y^3, x^3y^2) only if $p = p'$, but then the coefficient of x^4y invalidates the linear dependence because $u \neq v$. □

Claim 8.8. *Let $u \in U$, where U is the set in Assumption 7.3. Let $\pi = t^2 + xt + y$ with unknown x, y. Then $\Delta(u^2\pi)$ is irreducible as an element of $\mathbb{C}[x, y]$.*

Proof. Still using the notation (8.3), we note that the property $\Delta(\psi) = 0$ does not invalidate if we multiply every root of ψ by the same nonzero number. Therefore, we can assume without loss of generailty that $q = 1$. An inspection similar to Claims 8.6 and 8.7 shows that the total degree of $\Delta(u^2\pi)$ is five, and the degree with respect to y is three. Moreover, the monomial $2px^2y^3$ is the only one containing the third power of y with a nonzero coefficient. Therefore, if the polynomial $\Delta(u^2\pi)$ is reducible, one of the irreducible components should have the form either $x = s$ or $yx^k + \varphi_k = 0$, where $s \in \mathbb{C}$, $k \in \{0, 1, 2\}$, and $\varphi_k \in \mathbb{C}[x]$ is a polynomial of degree at most $k + 1$. This shows that one of the substitutions

$$x = s, \;\; y = s_1\, x + s_2 + s_3\, x^{-1} + s_4\, x^{-2} \quad \text{with} \quad s,\, s_1,\, s_2,\, s_3,\, s_4 \in \mathbb{C} \tag{8.5}$$

converts $\Delta(u^2\pi)$ into the identically zero rational function. But, in fact, we can reach a contradiction from the condition that this rational function is zero. This contradiction is immediate in the former possibility in (8.5), because the above mentioned coefficient of y^3 implies $s = 0$, but, nevertheless, the polynomial $\Delta(u^2\pi)$ is not divisible by x. The latter substitution (8.5) leads to a system with 5 variables s_1, s_2, s_3, s_4, p and 10 equations corresponding to the degree coefficients of x in the resulting rational function. From an inspection of the result of the computation, we immediately get $s_4 = 0$ from the (-4)-th degree coefficient. Also, the 0-th and 1-st degree coefficients allow one to express s_1 and s_2 as rational functions of s_3, p, and the situation reduces to a system of five non-trivial equations with variables s_3, p whose inconsistency can be checked with basic methods. □

We proceed with the main results.

Lemma 8.9. *Assumption 7.3 is true.*

Proof. Let U, V be sets as in Assumption 7.3. In view of Lemma 8.5, it is sufficient to find $u_1, u_2 \in U$ and $v \in V$ such that $\Delta(u_1u_2v) \neq 0$. We write $\mathcal{U}$ to denote the set of all pairs (p, q) such that $t^2 + pt + q \in U$, and also use the letter $\mathcal{V}$ to denote the corresponding set of pairs arising from V. We take

$$\delta(p_1, q_1, p_2, q_2, p_3, q_3) = \Delta\left((t^2 + p_1t + q_1)(t^2 + p_2t + q_2)(t^2 + p_3t + q_3)\right),$$

so now we need to find pairs $\mu_1, \mu_2 \in \mathcal{U}$ and $\nu \in \mathcal{V}$ such that

$$\delta(\mu_1, \mu_2, \nu) \neq 0. \tag{8.6}$$

Now we take two arbitrary distinct pairs $\mu' = (p', q')$ and $\mu'' = (p'', q'')$ in $\mathcal{U}$, and we consider the polynomial

$$\delta(\mu', \mu'', z), \tag{8.7}$$

which, in view of Claim 8.6, is not the zero function of a generic pair z. Also, we can assume without loss of generality that all the points in $\mathcal{V}$ belong to the zero locus of (8.7), because otherwise we can find a point $\nu \in \mathcal{V}$ such that $\delta(\mu', \mu'', \nu) \neq 0$, which corresponds to the formula (8.6) and completes the proof. Therefore, we have

$$\mathcal{V} \subset \Gamma_V \text{ with some plane curve } \Gamma_V \text{ of degree at most } d, \tag{8.8}$$

where d is the total degree of δ.

Now we proceed with a similar result on the set $\mathcal{V}$. We take an arbitrary pair $\nu_0 \in \mathcal{V}$, and we note that, in view of Claim 8.8, the condition

$$\delta(x, y, \nu_0) = 0$$

is not void as a function of generic pairs x, y. Therefore, if every $\mu_0 \in \mathcal{U}$ makes

$$\delta(x, \mu_0, \nu_0) = 0 \tag{8.9}$$

a void equation relative to x, then all the points in $\mathcal{U}$ should belong to some plane curve of degree at most d automatically. If this is not the case, which means that the equation (8.9) is not void, then, again assuming the invalidity of the desired result, we have this equation satisfied by every point in $\mathcal{U} \setminus \{\mu_0\}$. Therefore, we can assume without loss of generality that

$$\mathcal{U} \subset \Gamma_U \text{ with some plane curve } \Gamma_U \text{ of degree at most } d. \tag{8.10}$$

Also, we can assume without loss of generality that the curves Γ_U and Γ_V in (8.8) and (8.10) are irreducible because any of these curves has at most d irreducible components, and, since d is a fixed constant, one of the irreducible components of Γ_U should contain an arbitrarily large number of points of $\mathcal{U}$, and a similar conclusion holds for Γ_V and $\mathcal{V}$. Now we can assume without loss of generality that

$$\delta\left(\Gamma_U \times \Gamma_U \times \Gamma_V\right) = 0, \tag{8.11}$$

because, if this is not the case, then, in view Lemma 3.7, there exist $\nu \in \mathcal{V}$ and distinct $\mu_1, \mu_2 \in \mathcal{U}$ satisfying the condition (8.6), which is sufficient to complete the proof. The condition (8.11) implies that the curve Γ_V is a common irreducible component of the curves

$$\delta(\mu_1, \mu_1, z) = 0 \quad \text{and} \quad \delta(\mu_2, \mu_2, z) = 0, \tag{8.12}$$

for all distinct μ_1 and μ_2 in $\mathcal{U}$. Both curves (8.12) are irreducible by Claim 8.8, and the corresponding equations are not equivalent as shown in Claim 8.7. Therefore, the conditions (8.12) show that the set $\mathcal{V}$ lies in the intersection of two irreducible curves whose degrees are at most d, and hence the set $\mathcal{V}$ contains at most d^2 points. In particular, the set $\mathcal{V}$ cannot be arbitrarily large, which means that our previous considerations based on the invalidity of Assumption 7.3 lead to a contradiction. □

Therefore, we finalized the proof of Lemma 8.9 and confirmed Assumption 7.3. In view of Theorem 7.4, this completes our refutation of Conjecture 1.1.

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