Too Many Trailing Zeros?

Ron Larham, Febuary 2020

Abstract

In this paper I discuss Question 8 from the Chalkdust 2019 Christmas card. In particular I investigate the ratio of the number of trailing zeros of a factorial to its number of digits.

The Puzzle

I set problems, that I find interesting, for colleagues at work. Recently I set a problem based on question 8 from the 2019 Chalkdust Christmas card [1], which basically asked for the number of trailing zeros of 243! (my version, as I wanted to use a larger factorial, asked for the trailing zeros of 1253!).

Such problems, if you know how, are reasonably simple to solve, though in this case I received no solutions. Each trailing zero arises from the pairing off of 2s and 5s in the prime decomposition of the factorial. There are always more 2s than 5s in this (which though obvious I prove as Lemma 2 below), so we are really just counting the 5s in the decomposition. This of course implicitly assumes the Fundamental Theorem of Arithmetic, that any positive integer may be written as a product of primes in essentially one way, see Hardy and Wright [3] (I am obliged to use this as a reference rather than any other of the innumerable possibilities as Hardy is my great grand supervisor at least according to the Maths Genealogy Project [2]).That is, any positive integer $n > 1$ may be written $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_N^{\alpha_N}$, p_i prime, $p_1 < p_2 < ... < p_N$, $\alpha_i > 0$ where the ps and αs are unique for n. I will refer to α_i as the number of times that p_i appears in the prime decomposition of n, and also as the exponent of p_i in the decomposition.

In order to provide more interesting information in the solution I wanted to include the number of digits in 1253! At which point I noticed that for this factorial the number of trailing zeros was about 10% of the number of digits. This lead to the question: what fraction of the digits of a factorial are trailing zeros. This fraction is plotted in figure 1, and the limit for large factorials is answered by Theorem 3, below.

In the following sections I provide proofs that there are more 2s than 5s in the prime decomposition of a factorial, and the general solution to such puzzles and develop appropriate bounds for the number of trailing zeros and digits of a factorial which allow me to determine the limit of the ratio of these.

Mathematical Analysis of the Puzzle

Lemma 1: In the prime decomposition of the factorial of a positive integer n the exponent, or the number of times it appears in the product, of a prime $p \le n$ is :

$$E_p(n) = \sum_{k=1}^{\lfloor \log_p(n) \rfloor} \lfloor n/p^k \rfloor$$

where $\lfloor x \rfloor$ denotes the floor function at x (the greatest integer less than or equal to x)

Proof: The first term in the sum counts the number of elements of $1, 2, ..n$ that are divisible by p, the second term the number of elements divisible by p^2, and so on up to the maximum power of p that divides n!. Summing these gives the exponent of p in the prime decomposition of $n!$. That is it counts each number in the factorial of the form ap^k where $p \nmid a$ exactly k times.$\Box$

Lemma 2: For any positive integer n The exponent of 2 in the prime decomposition of $n!$ is larger than the exponent of 5 in the decomposition.

Proof: For $n \geq 2$ truncating the series for $E_2(n)$ at the first term we see that the exponent of 2 in the prime decomposition of $n! : E_2(n) \geq \lfloor n/2 \rfloor$. Also:

$$E_5(n) = \sum_{k=1}^{\lfloor \log_5(n) \rfloor} \lfloor n/5^k \rfloor \leq \sum_{k=1}^{\lfloor \log_5(n) \rfloor} n/5^k < \sum_{k=1}^{\infty} n/5^k = n/4 \ldots (1)$$

But $n/4 < \lfloor n/2 \rfloor$ when $n \geq 2$. So for $n \geq 2$ the exponent of 2 in $n!$ is greater than that of 5. The case for $n = 1$ can be confirmed by hand.$\Box$

We may observe in numerical experiments that as $n \to \infty$ the ratio of 2s to 5s appears goes to 4.

Theorem 1: The number of trailing zeros of $n!$:

$$Z(n) = E_5(n) = \sum_{k=1}^{\lfloor \log_5(n) \rfloor} \lfloor n/5^k \rfloor \ldots (2)$$

Proof: This follows directly from Lemma 2. $\Box$

So for the problem as I set it, we have:

$$Z(1253) = \lfloor 1253/5 \rfloor + \lfloor 1253/5^2 \rfloor + \lfloor 1253/5^3 \rfloor = 250 + 50 + 10 + 2 = 312$$

which may be checked with Wolfram Alpha [4] using the query "number of trailing zeros of 1253!"

As an extension of Lemma 2, we may ask for any primes p, q what is limit of the ratio of $E_p(n)/E_q(n)$ as $n \to \infty$. This is addressed by Theorem 2.

Theorem 2: For any primes p and q, the limit of the ratio of the number times p appears in the prime decomposition of $n!$ to the number of times q appears in the decomposition:

$$\lim_{n \to \infty} \frac{E_p(n)}{E_q(n)} = \frac{q-1}{p-1}$$

Proof: As we may write $x = \lfloor x \rfloor + \theta_x$ where $\theta_x \in [0, 1)$ we have:

$$E_p(n) = \sum_{k=1}^{\lfloor \log_p(n) \rfloor} \lfloor n/p^k \rfloor = \left(\sum_{k=1}^{\lfloor \log_p(n) \rfloor} n/p^k \right) - \theta_{n,p} \times \lfloor \log_p(n) \rfloor$$

where $\theta_{n,p} \in [0,1)$. Then:

$$\frac{E_p(n)}{E_q(n)} = \frac{\left(\sum_{k=1}^{\lfloor \log_p(n) \rfloor} 1/p^k\right) - \theta_{n,p} \times \lfloor \log_p(n) \rfloor / n}{\left(\sum_{k=1}^{\lfloor \log_q(n) \rfloor} 1/q^k\right) - \theta_{n,q} \times \lfloor \log_q(n) \rfloor / n} \quad \ldots (3)$$

Now, as $\lim_{x\to\infty} \log_a(x)/x = 0$, for any a a positive integer, and $\sum_{k=1}^{\infty} 1/x^k = 1/(x-1)$,when $x > 1$, the limits as $n \to \infty$ of both the numerator and denominator on the right in (3) above, exist and are equal to $1/(p-1)$ and $1/(q-1)$ respectively. Therefore the limit of their ratio exists, and is equal to the ratio of the limits:

$$\lim_{n\to\infty} \frac{E_p(n)}{E_q(n)} = \frac{q-1}{p-1} \;\square$$

Hence, We may observe that as $n \to \infty$ the ratio of 2s to 5s in the prime decomposition of $n!$ goes to 4, which agrees with what numerical experiments lead me to expect.

Analysis of the Puzzle Extension
In order to provide more interesting information in the solution I wanted to include the number of digits in 1253!. As this was before I realised that Wolfram Alpha provided this information by default for factorials I decided to see what I could do with Stirling's formula (Abramowitz and Stegun 6.1.38 [5]):

$$n! = \sqrt{2\pi}\, n^{n+1/2} \exp(-n + \frac{\theta}{12n})$$

where $n > 0$ and $\theta \in (0,1)$.

*Lemma 3:*The number of digits d(n), of a positive integer n is:

$$d(n) = \lfloor \log_{10}(n) \rfloor + 1$$

Proof: Any k digit number n may be written:

$$n = 10^{k-1}x + y$$

where $x \in \{1, 2, ..., 9\}$ is the leading digit of n, and $y < 10^{k-1}$. So $10^{k-1} \leq n < 10^k$, and $k - 1 \leq \log_{10}(n) < k$, so:

$$\lfloor \log_{10}(n) \rfloor = k - 1.$$

Which gives $d(n) = \lfloor \log_{10}(n) \rfloor + 1 \;\square$

By Lemma 3, the number of digits of $n!$, $D(n)$ is:

$$D(n) = \lfloor \frac{1}{2} \log_{10}(2\pi) + (n + 1/2) \log_{10}(n) + \log_{10}(e)(-n + \frac{\theta}{12n}) \rfloor + 1$$

Since for large n, $0 < \log_{10}(e)\frac{\theta}{12n} \ll 1$, if we let:

$$D^*(n) = \lfloor \frac{1}{2} \log_{10}(2\pi) + (n + 1/2) \log_{10}(n)n - n \log_{10}(e) \rfloor + 1$$

we have

$$D^*(n) \leq D(n) \leq D^*(n) + 1 \ \ldots (4)$$

and usually $D^*(n) = D(n)$. We may also note that:

$$D^*(n) < (n + 1/2)\log_{10}(n)n - n\log_{10}(e) \quad \ldots (5)$$

Putting $n = 1253$ gives $D^*(n) = 3340$, also $D(1253)$ has the same value for all $\theta \in (0, 1)$ and so $D(n) = D^*(n) = 3340$. Which agrees with the value for the number of digits given by Wolfram Alpha.

At this point I noticed that $Z(1253) \approx D(1253)/10$ and started wondering about what happened to $Z(n)/D(n)$ as $n \to \infty$. Initially I used a numerical package to calculate and plot this ratio (see figure 1). Then proceeded with the following analysis.

Theorem 3. The limit of $R(n) = Z(n)/D(n)$ as $n \to \infty$ is zero.

Proof: Combining (1) and (2) we have for all $n \geq 2$

$$0 \leq Z(n) < \frac{n}{4}$$

And combining (4) and (5):

$$D(n) \geq D^*(n) > (n + 1/2)\log_{10}(n) - n\log_{10}(e)$$

So:

$$0 < R(n) < \frac{1}{4(\log_{10}(n) - \log_{10}(e))} = \frac{1}{4\log_{10}(n/e)}$$

And as $\lim_{n\to\infty}\log_{10}(n) = \infty$ this implies that:

$$\lim_{n\to\infty} R(n) = 0 \quad \square$$

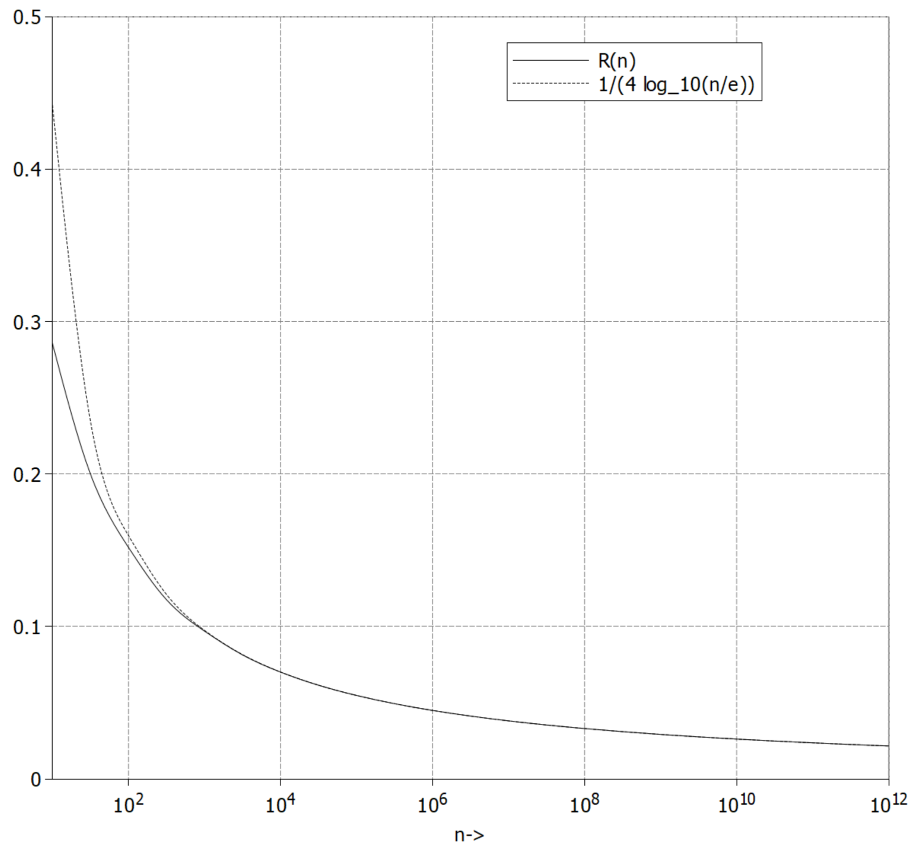

Figure 1: Plot of $R(n)$ the Ratio of Number of Trailing Zeros to Number of Digits of $n!$

References

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