An Untold Story of Brownian Motion

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Abstract

Although the concept of Brownian motion or Wiener process is quite popular, proving its existence via construction is a relatively deep work and would not be stressed outside mathematics. Taking the existence of Brownian motion in $C([0,1],\mathbb{R})$ "for granted" and following an existing implicit thread, we intend to present an explicit, simple treatment of the existence of Brownian motion in the space $C([0,+\infty[,\mathbb{R})$ of all continuous real-valued functions on the ray $[0,+\infty[$ with moderate technical intensity. In between the developments, some informative little results are proved.

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1 Introduction

Throughout, let $\mathbb{R}_+ \coloneqq [0,+\infty[$; let $C(\mathbb{R}_+)$ be the space of all continuous $\mathbb{R}$-valued functions on $\mathbb{R}_+$ equipped with the sigma-algebra $\mathscr{C}$ obtained by relativizing the product sigma-algebra of $\mathbb{R}^{\mathbb{R}_+}$ to $C(\mathbb{R}_+)$. Denote by $\pi_t : C(\mathbb{R}_+) \to \mathbb{R}, f \mapsto f(t)$ the natural projection on $C(\mathbb{R}_+)$ for all $t \in \mathbb{R}_+$. For every map f, the symbol $f^{-1)}$ denotes the preimage map induced by f.

A probability measure $\mathbb{W}$ defined on $\mathscr{C}$ is called *Wiener measure* (*over* $C(\mathbb{R}_+)$; necessarily unique) if and only if $\pi_0 = 0$ a.s.-$\mathbb{W}$ and $\mathbb{W}\circ\pi_t^{-1)}$ is the Gaussian distribution with mean 0 and variance t for all $t \in]0,+\infty[$ with the property that $0 \le t_1 < \cdots < t_k < +\infty$ implies

$$\mathbb{W}\circ(\pi_{t_k} - \pi_{t_{k-1}}, \dots, \pi_{t_2} - \pi_{t_1})^{-1)} = \otimes_{l=2}^{k} \mathbb{W}\circ(\pi_{t_l} - \pi_{t_{l-1}})^{-1)}$$

on the usual Borel sigma-algebra of $\mathbb{R}^{k-1}$. A random element of $C(\mathbb{R}_+)$ whose distribution is Wiener measure is called *Wiener process* or *Brownian motion* in $C(\mathbb{R}_+)$.

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In plain words, Wiener process in $C(\mathbb{R}_+)$ fixes the origin (almost surely), has Gaussian coordinates with mean zero and variance being the corresponding index, and has independent increments. We remark that the identity map of $C(\mathbb{R}_+)$, being evidently measurable-$\mathscr{C}$ and hence inducing $\mathbb{W}$ (having $\mathbb{W}$ as its distribution), is Brownian motion in $C(\mathbb{R}_+)$; thus the existence of Brownian motion in $C(\mathbb{R}_+)$ follows immediately from that of Wiener measure over $C(\mathbb{R}_+)$.

Proving the existence of Wiener measure over $C(\mathbb{R}_+)$ is far from elementary. The original work Wiener [5] utilizes Kolmogorov's extension theorem; one may also refer to Shiryaev [3] or Billingsley [1] for suitable preliminaries regarding the Wiener's approach. Tao [4] contains another approach relatively modern (self-similarity) that covers more general cases. Billingsley [2] proves the existence of Wiener measure over the metric space $C([0,1],\mathbb{R})$, equipped with the uniform metric, using Prokhorov's theorem, and implicitly leaves the existence of Wiener measure over $C(\mathbb{R}_+)$ as an exercise. At any rate, all the three representative approaches require a good deal of mathematical preliminaries.

However, we find that Billingsley's (implicit) approach would be more concrete and less technical for pedagogical purposes; moreover, under this approach the measurability of the space $C(\mathbb{R}_+)$ requires no extra care. The Billingsley's approach is thus technically more simple.

Taking the existence of Wiener measure over $C([0,1],\mathbb{R})$ "as given", we intend to prove the existence of Wiener measure over $C(\mathbb{R}_+)$ based on the stream of thought suggested in Billingsley [2].

2 Construction of Wiener Measure over $C(\mathbb{R}_+)$

To begin with, we roughly describe the main ideas of proving the existence of Wiener measure over $C([0,1],\mathbb{R})$ in Billingsley [2]. The existence of Wiener measure over $C([0,1],\mathbb{R})$ is thereafter taken as given.

2.1 Proof sketch for existence of Wiener measure over the metric space $C([0,1],\mathbb{R})$

Given a sequence of independent identically distributed random variables with zero mean and positive finite variance σ^2, construct for each $n \in \mathbb{N}$ a "Donsker process" being a random element of $C([0,1],\mathbb{R})$ obtained by linear interpolation, fixing the origin, between the $\frac{1}{\sigma\sqrt{n}}$-scaled cumulative sums of the first n components of the given sequence. It then can be shown that the finite-dimensional distributions of the Donsker processes converge weakly to those of the "now hypothetical" probability measure over $C([0,1],\mathbb{R})$ that we refer to as Wiener measure (considered here as a measure on the Borel sigma-algebra generated by the topology of $C([0,1],\mathbb{R})$ induced by the uniform metric).

It turns out that the sequence of Donsker processes is tight for some choice of the "ingredient" sequence of random variables from which we construct Donsker processes, and hence is precompact (relatively compact) by Prokhorov's theorem; there is then in particular some subsequence of the processes converging weakly to some probability measure over $C([0,1],\mathbb{R})$. But this limiting probability measure has the desired finite-dimensional distributions, and so it is Wiener measure.

2.2 From $C([0,1],\mathbb{R})$ to $C(\mathbb{R}_+)$

In checking the measurability of a map $f : \Omega \to C([0,1],\mathbb{R})$ on a given measurable space Ω, one may use for convenience the fact that the Borel sigma-algebra generated by the metric topology of $C([0,1],\mathbb{R})$ equals the cylinder sigma-algebra of $C([0,1],\mathbb{R})$; the measurability check then reduces to the easy check of the measurability of each function $f_t : \Omega \to \mathbb{R}$ whose value at every given ω is the value of the continuous function $f(\omega) : [0,1] \to \mathbb{R}$ at t. Thus far we have sketched what is explicitly said in Billingsley [2].

Indeed, the same philosophy is readily applicable to checking the measurability of a map with values in the measurable space $C(\mathbb{R}_+)$. To this end, we state a useful fact, which seems less well-documented:

Proposition 1. *If Θ is a nonempty set, and if S_θ is a separable metric space equipped with the Borel sigma-algebra $\mathscr{B}_{S_\theta}$ generated by the metric topology of S_θ for all $\theta \in \Theta$, then the cylinder sigma-algebra of the Cartesian product $\bigtimes_{\theta\in\Theta} S_\theta$ equals the product sigma-algebra $\bigotimes_{\theta\in\Theta} \mathscr{B}_{S_\theta}$ of $\bigtimes_{\theta\in\Theta} S_\theta$.*

Proof. Evidently from definition, the product sigma-algebra $\bigotimes_\theta \mathscr{B}_{S_\theta}$ is included in the cylinder sigma-algebra of $\bigtimes_\theta S_\theta$.

For the other inclusion relation, we recall that a countable product of second-countable spaces is still second-countable. Since, as can be shown, a separable metric space is precisely a second-countable metric space, every open set of a finite product $S_{\theta_1} \times \cdots \times S_{\theta_n}$ is a countable union of sets of the form $V_1 \times \cdots \times V_n$ with each V_i being a basic open set of S_{θ_i}. If $I \subset \Theta$ is finite with cardinality n, and if $\pi_I : (x_\theta)_{\theta\in\Theta} \mapsto (x_\theta)_{\theta\in I}$ on $\bigtimes_\theta S_\theta$, then

$$\begin{aligned}\bigcup_{(V_1,\dots,V_n)} \pi_I^{-1)}(V_1 \times \cdots \times V_n) &= \bigcup_{(V_1,\dots,V_n)} \bigcap_{i=1}^{n} \pi_i^{-1)}(V_i) \\ &\in \bigotimes_\theta \mathscr{B}_{S_\theta}\end{aligned}$$

for every countable collection of n-tuples $(V_1, \ldots, V_n)$ with each V_i being a basic open set of S_{θ_i}. Since the π_I-preimage of every given open set of $\bigtimes_{\theta \in I} S_\theta$ lies in $\bigotimes_\theta \mathscr{B}_{S_\theta}$, the sigma-algebra generated by π_I is included in $\bigotimes_\theta \mathscr{B}_{S_\theta}$; the desired inclusion then follows. □

We remark in passing that, since $\mathbb{R}$ is a separable metric space, the Borel sigma-algebra of $C([0,1], \mathbb{R})$ is by Proposition 1 also equal to the product sigma-algebra of $\mathbb{R}^{[0,1]}$ relativized to $C([0,1], \mathbb{R})$.

Since Wiener measure over $C([0,1], \mathbb{R})$ exists, let $\underline{W}$ be Wiener process in $C([0,1], \mathbb{R})$. If

$$W_t^\circ := \underline{W}_t - t\underline{W}_1$$

for all $t \in [0,1]$, then, since i) given any $\omega \in \Omega$, the function $t \mapsto \underline{W}_t(\omega) - t\underline{W}_1(\omega)$ is evidently continuous on $[0,1]$, and since ii) given any $t \in [0,1]$, the function $\omega \mapsto \underline{W}_t(\omega) - t\underline{W}_1(\omega)$ on Ω is evidently measurable (and hence a random variable), the map W° is a random element of $C([0,1], \mathbb{R})$. Indeed, the process W° is simply *Brownian bridge* in $C([0,1], \mathbb{R})$.

The interesting observation, as suggested in Problem 8.2 in Billingsley [2], is to consider the map $W : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ whose value is the function

$$t \mapsto (1+t)W_{t/(1+t)}^\circ$$

on $\mathbb{R}_+$. We will prove

Theorem 1 (Problem 8.2, Billingsley)**.** *If W° is Brownian bridge in $C([0,1], \mathbb{R})$, and if $W_t := (1+t)W_{t/(1+t)}^\circ$ for all $t \in \mathbb{R}_+$ on the same probability space, then $W \equiv (W_t)_{t \in \mathbb{R}_+}$ is Brownian motion in the measurable space $C(\mathbb{R}_+)$.* □

Since W by Theorem 1 induces exactly Wiener measure on $\mathscr{C}$, the desired existence matter is done.

In proving Theorem 1, it would be useful to apply a well-known fact, which may also serve as a nice exercise:

Proposition 2. *If $\mathbb{P}$ is a probability measure over $C(\mathbb{R}_+)$, and if $\mathbb{P}(\pi_0 = 0) = 1$, then $\mathbb{P}$ is Wiener measure if and only if $\mathbb{P} \circ (\pi_{t_1}, \ldots, \pi_{t_k})^{-1)}$ is the Gaussian distribution with mean zero and covariance matrix having each (i,j)-entry being $t_i \wedge t_j$ for all $0 \leq t_1 < \cdots < t_k < +\infty$.*

Proof. The "only if" part is standard.

The "if" part is technically simple, but seems relatively less articulated. That $\mathbb{P} \circ \pi_t^{-1)}$ is the Gaussian distribution with mean zero and variance t for all real $t > 0$ is evident. To see the independence of increments, recall that a Gaussian random vector has independent components if and only if the components are pairwise uncorrelated. If X is a random element of $C(\mathbb{R}_+)$ inducing $\mathbb{P}$, and if $0 \le t_1 < t_2 < t_3$ are real numbers, then the assumption implies that $\mathbb{E}(X_{t_3} - X_{t_2})(X_{t_2} - X_{t_1}) = t_2 - t_1 - t_2 + t_1 = 0$; the desired independence then follows. □

Thanks to Proposition 2, the existence of a random element of $C(\mathbb{R}_+)$ whose finite-dimensional distributions satisfy the characterization of Proposition 2 is equivalent to that of Wiener measure over $C(\mathbb{R}_+)$.

We now give

Proof of Theorem 1. It is a straightforward exercise to show that $\mathbb{E}W_t^\circ = 0$ for all $t \in [0,1]$ and that $\mathbb{E}W_s^\circ W_t^\circ = (s \wedge t)(1 - s \vee t)$ for all $s, t \in [0,1]$.

We remark that the function $t \mapsto (1+t)W^\circ_{t/(1+t)}(\omega)$ is continuous on $\mathbb{R}_+$ for all ω in the underlying probability space; moreover, it holds that $t \in \mathbb{R}_+$ implies that $(1+t)W^\circ_{t/(1+t)}$ is a random variable. Since W is then measurable-$\mathscr{C}$ by Proposition 1; the map W is indeed a random element of $C(\mathbb{R}_+)$.

Since a nonsingular linear transform of a Gaussian random vector is again Gaussian, Brownian bridge W° in $C([0,1], \mathbb{R})$ as previously defined is evidently a Gaussian process, i.e. has every finite-dimensional distribution being Gaussian. The function $t \mapsto t/(1+t)$ is a bijection; and so W is also a Gaussian process.

The proof is complete once we show that the finite-dimensional distributions of W satisfy the moment conditions required by the characterization given in Proposition 2. It is evident that $\mathbb{E}W_t = 0$ for all $t \in \mathbb{R}_+$. In addition, for all $s \le t$ we have

$$\begin{aligned}
\mathbb{E}W_s W_t &= (1+s)(1+t)\mathbb{E}W^\circ_{s/(1+s)} W^\circ_{t/(1+t)} \\
&= (1+s)(1+t)\left(\frac{s}{1+s} \wedge \frac{t}{1+t}\right)\left(1 - \frac{s}{1+s} \vee \frac{t}{1+t}\right) \\
&= (s+st) - \left(\frac{s^2(1+t)}{1+s} \vee st\right) \\
&= (s+st) - \left(\frac{s^2(1+t)}{1+s} \vee \frac{st+s^2t}{1+s}\right) \\
&= s + st - st \\
&= s.
\end{aligned}$$

But then the distribution of W is Wiener measure over $C(\mathbb{R}_+)$. □

References

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