How Likely is it for Countably Many Almost Sure Events to Occur Simultaneously?

Yu-Lin Chou*

Abstract

Given a countable collection of almost sure events, the event that at least one of the events occurs is "evidently" almost sure. It is, however, not so trivial to assert that the event for every event of the collection to occur is almost sure. Measure theory helps to furnish a simple, definite, and affirmative answer to the question stated in the title. This useful proposition seems to rarely, if not never, occur in a teaching material regarding measure-theoretic probability; our proof in particular would help the beginning students in probability theory to get a feeling of almost sure events.

Keywords: almost sure event; probability measure; set of full measure
MSC 2020: 97-01; 60A10

1 Probability and Measure

Thanks to Kolmogorov's deep, fruit-bearing insight, the long-time unsettled object — the concept of probability — is "embedded" in mathematics via measure theory, we may then enjoy the wonderful ramifications. For a non-technical (i.e. not intended only for experts) treatment of measure-theoretic probability without loss of rigor, we refer the reader to Billingsley [1], which may be comparable, in terms of style, to An Introduction to the Theory of Numbers by Hardy and Wright [3].

In view of the astonishing results following from the "paradigm shift" of probability theory — defining probability as a measure, the measure-theoretic approach to probability is then not an indoctrination nor a move only for the sake of using mathematics. Until an essentially superior treatment of probability is discovered, the measure-theoretic approach is a best option available. We thus recommend the measure-theoretic approach

*Yu-Lin Chou, Institute of Statistics, National Tsing Hua University, Hsinchu 30013, Taiwan, R.O.C.; Email: y.l.chou@gapp.nthu.edu.tw.

to the reader's mind, not because of some mathematical magnate saying so, nor because of schools teaching so, nor because of many mathematicians doing so, but because of the sheer elegance, pleasure, and power of the measure-theoretic approach. A common component that captures the elegance, pleasure, and power of the measure-theoretic approach is the fact that the defining properties of a measure, either taken in the original sense of Carathéodory (outer-measure-first; e.g. Federer [2]) or in the sense of Radon (countably-additive-measure-first; e.g. Billingsley [1]), are so few while the implications are so far-reaching.

We will approach the concerned question — how likely it is for countably many almost sure events to occur simultaneously — in a measure-theoretic way. The yet convinced reader will perhaps change their mind after perusing, for instance, the cited Billingsley's work.

2 Agreements and Heuristics

For the efficiency and clarity of our incoming argument, let us agree in advance on some terminologies.

If $(\Omega, \mathscr{F}, P)$ is a probability space, by an (P-)*almost sure event* we mean precisely a set $A \in \mathscr{F}$ such that $P(A) = 1$. Although some authors (e.g. Billingsley [1]) refer to an almost sure event as support of the underlying probability measure, we find the terminology *ad hoc* and not that informative for our purposes. To connect the relation "$P(A) = 1$" with the phrase "the event A occurring with probability 1 (almost surely)", the reader may write $A = \{\omega \in \Omega \mid \omega \in A\}$ or $A = \{\omega \in \Omega \mid \mathbb{1}_A(\omega) = 1\}$. Here $\mathbb{1}_A$ denotes the indicator (characteristic) function $\Omega \to \{0, 1\}$ of A. We employ instead the term "indicator function" as the term "characteristic function", although usual in analysis, refers to another major object in probability and statistics, i.e. the Fourier transform (that version with "e^{itx}") of a probability density function.

Some events that may look like not so "almost sure" are indeed almost sure. For instance, consider the standard normal (standard Gaussian) distribution

$$B \mapsto \int_B \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, \mathrm{d}x$$

defined on the Borel sigma-algebra $\mathscr{B}_{\mathbb{R}}$ generated by the standard topology of $\mathbb{R}$. If P denotes the standard Gaussian distribution, then, as the (Lebesgue) integral of the standard normal density $x \mapsto \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is $= 1$ and hence finite, the measure P is absolutely continuous with respect to Lebesgue measure. Since the set $\mathbb{Q}$ of rationals is countable, we have $P(\mathbb{R} \setminus \mathbb{Q}) = 1$. So the set of irrationals is an almost sure event with respect to the standard Gaussian distribution, although $\mathbb{Q}$ is dense in $\mathbb{R}$.

If $(\Omega, \mathscr{F}, \mathbb{M})$ is a measure space, we say that $\mathbb{M}$ is a *finite measure* if and only if $\mathbb{M}(\Omega) < +\infty$. If $A \in \mathscr{F}$, the set A is said to be *of full* ($\mathbb{M}$-)*measure* if and only if $\mathbb{M}(A) = \mathbb{M}(\Omega)$. Thus, if $\mathbb{M}$ is a probability measure, then a set of full $\mathbb{M}$-measure is precisely an $\mathbb{M}$-almost sure event. For our purposes, the case where $\mathbb{M}$ is trivial — i.e. where $\mathbb{M}$ is the zero of the (Abelian) monoid of finite measures on $\mathscr{F}$, equipped with the addition $(\mathbb{M}, \underline{\mathbb{M}}) \mapsto (\mathbb{M}(A) + \underline{\mathbb{M}}(A))_{A \in \mathscr{F}}$, whose stability and associativity are ensured jointly by the monotone convergence theorem and the fact that limit operation preserves the usual addition — is not excluded.

By a *countable* set we mean precisely a set whose cardinality is no greater than that of $\mathbb{N}$. Thus a finite set and the set of even integers ≥ 1 are countable.

3 Answer

Our line of reasoning will be made in a slightly more general framework — a finite measure space. This is always (mathematically) proper as for every measurable space $(\Omega, \mathscr{F})$ there is some probability measure on $\mathscr{F}$; any Dirac measure $\mathbb{D}^\omega : A \mapsto \mathbb{1}_A(\omega), \mathscr{F} \to \{0, 1\}$ with $\omega \in \Omega$ serves the purpose.

Let $(\Omega, \mathscr{F}, \mathbb{M})$ be a finite measure space; let $\mathscr{A} \subset \mathscr{F}$ be a countable collection of sets of full measure. Since $\mathbb{M}$ is a finite measure, since $A \subset \cup\mathscr{A} \subset \Omega$, and since $\cup\mathscr{A} \in \mathscr{F}$, we have

$$\mathbb{M}(A) \leq \mathbb{M}(\cup\mathscr{A}) \leq \mathbb{M}(\Omega)$$

for all $A \in \mathscr{A}$. As every $A \in \mathscr{A}$ is by assumption of full measure, we have $\mathbb{M}(\cup\mathscr{A}) = \mathbb{M}(\Omega)$. In particular, if $\mathbb{M}$ is a probability measure, it follows that the event for some element of $\mathscr{A}$ to occur is almost sure. This backs up the first statement made in the abstract.

What about $\cap\mathscr{A}$? First of all, the set $\cap\mathscr{A}$ lies by the countability assumption in $\mathscr{F}$; so it makes sense to investigate $\mathbb{M}(\cap\mathscr{A})$. To being with, we remark that $\{\cap\mathscr{A}, \cup_{A\in\mathscr{A}} A^c\}$ is a partition of Ω; it follows that

$$\mathbb{M}(\cap\mathscr{A}) + \mathbb{M}(\cup_{A\in\mathscr{A}} A^c) = \mathbb{M}(\Omega).$$

But $\mathbb{M}(\cup_{A\in\mathscr{A}} A^c) \leq \sum_{A\in\mathscr{A}} \mathbb{M}(A^c) = 0$, we have

$$\mathbb{M}(\cap\mathscr{A}) = \mathbb{M}(\Omega).$$

In particular, if $\mathbb{M}$ is a probability measure, then the event for the elements of $\mathscr{A}$ to occur simultaneously is an almost sure event!

References

[1] Billingsley, P. (1995). *Probability and Measure*, third edition. Wiley.

[2] Federer, H. (1996). *Geometric Measure Theory*, reprint of the first edition. Springer.

[3] Hardy, G.H. and Wright, E.M. (2008). *An Introduction to the Theory of Numbers*, sixth edition. Oxford University Press.