The fine-structure constant from the golden ratio and pi
A puzzling formula

Bruno R Galeffi

Abstract: An approximation for α^{-1} to 99.99998% is determined from a straightforward formula using a combination of the golden ratio and π. An accuracy factor of 1.00000021354 is then formulated by means of the first two perfect numbers 6 and 28.

1. Introduction

Since it was introduced in the 1920s by Sommerfeld to account for the relativistic splitting of atomic spectral lines, the fine-structure constant (α) has likely become the most enigmatic (nearly mystical) fundamental constant. This dimensionless quantity characterizes the strength of the coupling of an elementary charged particle with the electromagnetic field through the formula $4\pi\varepsilon_0\hbar c\alpha = e^2$. In 1985, Feynman used these inspirational words to describe α in his book "QED: the strange theory of light and matter" [1]:

"…is it related to π or perhaps to the base of natural logarithms? Nobody knows. It's one of the greatest damn mysteries of physics: a magic number that comes to us with no understanding by man. You might say the 'hand of God' wrote that number, and 'we don't know how He pushed his pencil' …"

The fine-structure constant plays a central role in the Physics of the 21st century, allowing the testing of precise theories such as QED, or the possible drift of fundamental constants. The 2018 CODATA [2] recommended value for α^{-1} is:

$$\alpha^{-1} = 137.035\,999\,084\,(21) \tag{1}$$

In 2012, an experimental value for α was reported from rigorous measurement of g using a one-electron so-called "quantum cyclotron" apparatus, together with a calculation via the theory of QED that involved 12672 tenth-order Feynman diagrams [3-4]. The numerical value reported was:

$$\alpha^{-1} = 137.035\,999\,174\,(35) \tag{2}$$

2. Simple formulas approaching the value of α

The fine-structure constant is a dimensionless constant and as such, its numerical value is independent of the system of units used. However, determining the value of a constant from other constants such as e^-, c, h, and ε_0 having inherent incertitude can only provide incertitude on the result. Hence, the idea of discovering a simple expression establishing the fine-structure constant from other dimensionless constants such as π, e, or Φ (the golden ratio) is seductive. It would substantiate clear links between α and other existing constants, and possibly establish a definitive origin and accurate value for the fine-structure constant beyond experimental data. This idea led some to explore different formulas, mostly involving π [see 5-10 for recent papers].

A number of formulas have been developed with variable success, mostly intuitively, in order to reproduce the numerical value of α. A great deal of them try to exploit the integer part (137) in order to focus only on the fractional side. However, this approach does not provide any insight for the numerical value of α in its entirety and quintessence. Nonetheless, some of the known and noticeable formulas involving π or Φ are the following:

$$\alpha^{-1} = \sqrt{\pi^2 + 137^2} \simeq 137.036\ 016 \qquad [8]$$

$$\alpha^{-1} = \left(\frac{9}{16\pi^3}\sqrt[4]{\frac{\pi}{5!}}\right)^{-1} \simeq 137.036\ 082 \qquad [8]$$

$$\alpha^{-1} = 4\pi^3 + \pi^2 + \pi \simeq 137.036\ 304 \qquad [8]$$

$$\alpha^{-1} = \frac{360}{\phi^2} - \frac{2}{\phi^3} \simeq 137.035\ 628 \qquad [11]$$

3. A straightforward formula from φ and π

This article provides a new and straightforward combination of both the golden ratio and pi which closely approximates the fine-structure constant. It is believed that this combination of both φ and π in one unique formula to estimate α to such level of accuracy is reported for the first time. This value is ≈99.99998 % of the CODATA 2018 recommended value.

$$\alpha^{-1} = \left(\sqrt[3]{\phi}\left(\sqrt[3]{\phi}+2\right)\pi\right)^2 = 137.035\ 969\ 901 \qquad (3)$$

This formula is indeed quite mysterious in its simplicity, aesthetic and precision. But it reveals once again how, as already known, the ubiquitous "golden ratio" defined by Euclid circa 300BC [12] sneaks into most physical phenomena. Similarly, the omnipresent constant π, sometimes referred as Archimedes's constant, has always infiltrated mathematics, without anyone knowing why.

4. A bewildering accuracy factor of 1.000 000 213 54

It has been found that applying the following factor (4) to the expression in (3) uplifts the accuracy of the numerical value to a level similar to current estimates for α^{-1}. The factor found is expressed by:

$$\sqrt[136]{\sqrt[6]{6e^{-28}}+1} = 1.00000021354 \qquad (4)$$

Of great interest in the number 6 predominance in the expression of (4), as well as the exponent 28. In number theory, 6 is the first perfect number and 28 the second. On the other hand, 136 is equal to the sum of the first 16 integers.

Applying (4) to the expression (3) provides

$$\alpha^{-1} = 137.035\ 999\ 164 \qquad (5)$$

This numerical value found in (5) is well within incertitude of the experimental data provided by Aoyama et al. over the last 10 years [16-19], and also to the CODATA 2014 recommended value 137.035 999 139 (21). It is believed that the current true value of the so-called fine-structure constant is very close to that determined in (5).

Moreover, the numerical value found in (5) provides the following expression for the permeability of free space μ_0 :

$$\mu_0 = 4\pi \times 0.9999999999635 \times 10^{-7}\ \ \mathrm{H.m^{-1}} \qquad (6)$$

5. References

[1] RP Feynman, QED: The Strange Theory of Light and Matter, Princeton University Press, p. 129, (1985)

[2] PJ Mohr, BN Taylor, DB Newell (2019). "Fine structure constant" (https://physics.nist.gov/cgi-bin/cuu/Value? alph). CODATA Internationally recommended 2018 values of the fundamental physical constants. National Institute of Standards and Technology

[3] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio ; "Tenth-Order QED Contribution to the Electron g-2 and an Improved Value of the Fine Structure Constant"; https://arxiv.org/pdf/1205.5368.pdf

[4] Wikipedia online updated 28 Oct 2020; "Fine-structure constant"

[5] MA Sherbon "Golden Ratio Geometry and the Fine-Structure Constant"; Journal of Advances in Physics, 16(1), 362 -368 (2019)

[6] MA Sherbon; "Fine-Structure Constant from Golden Ratio Geometry" International Journal of Mathematics and Physical Sciences Research 5 (2):89-100 (2018)

[7] MA Sherbon; "Fine-structure constant from Sommerfeld to Feynman"; Journal of Advances in Physics 16 (1):335-343 (2019)

[8] G Dattoli; "The fine structure constant and numerical alchemy" ENEA -Tecnologie Fisiche e Nuovi Materiali, Centro Ricerche Frascati; RT/2009/FIM; and Ref therein

[9] Published Data of Robert Munafo online. https://mrob.com/pub/num/n-b137_035.html

[10] UD Jentschura, and I N´andori; "Attempts at a determination of the fine-structure constant from first principles: A brief historical overview"; https://arxiv.org/pdf/1411.4673.pdf

[11] R Heyrovska; "Golden Ratio Based Fine Structure Constant and Rydberg Constant for Hydrogen Spectra"; International Journal of Sciences; Volume 2, Issue May 2013

[12] Euclid. Elements, Book VI, Definition 3. c. 300BC