Hopf fibrations in the language of 3D geometric algebra

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This short text is a continuation of a series on the applications of geometric algebra, related to the book [3]. As the book [3], this text is intended for young (or at least young at heart) and open-minded people.

Keywords:

Hopf fibration, *fiber*, *stereographic projection*, *rotor*, *spinor*, *unit sphere*, *geometric algebra*

An example Hopf fibration

The standard *unit* n - *sphere* S^n is the set of points $\{x_0,\ x_1,\cdots,\ x_n\}$ in $\mathbb{R}^{n+1}$ that satisfy the equation

$$x_0^2+x_1^2+\cdots+x_n^2=1.$$

The *dimension* of a unit sphere is the number of parameters we need to describe it. For example, for a unit circle S^1, we need just one parameter (an angle), which means that we can write

$$x_0=\cos\theta,\ x_1=\sin\theta\ .$$

The unit circle can also be described by unit complex numbers, $|z|=1$. Note that the product of two unit complex numbers is a complex number. In geometric algebra, we can represent a unit circle by a unit vector $e_1\cos\theta+e_2\sin\theta$ (e_1, e_2, and e_3 are orthonormal unit vectors); however, the product of unit vectors is not a vector. The solution is to use unit spinors, like $\cos\theta+e_1e_2\sin\theta$, where e_1e_2 is a new imaginary unit, due to $e_1e_2=-e_2e_1$. The product of spinors in exponential form gives an immediate meaning: angles just add up. It appears that we can represent S^3 in a similar way, using unit spinors (rotors) from *Cl*3 (instead of usually used *quaternions*, see in the text).

As an example of the *Hopf fibration*, consider the mapping $h:S^3\rightarrow S^2$, defined by

$$h(a,b,c,d)=\left(a^2+b^2-c^2-d^2,\ 2(ad+bc),\ 2(bd-ac)\right),\ a^2+b^2+c^2+d^2=1$$

(Hopf's original formula differs from that given here, see [2]; for a more general definition, see [7]). This means that a point from S^3 is mapped to a unit vector (check) that defines a point on S^2. Such a mapping can be connected to quaternions (see [4] and [7]), which means that we can formulate it in *Cl*3 (the even part).

Hopf fibrations in *Cl*3

Consider the rotor

$$R = a - bje_1 - cje_2 - dje_3$$

and let us find its effect on e_1

$$Re_1R^\dagger = \left(a^2 + b^2 - c^2 - d^2\right)e_1 + \left(2ad + 2bc\right)e_2 + \left(2bd - 2ac\right)e_3 .$$

It is clear that for $c = d = 0$ and $a^2 + b^2 = 1$ we have $Re_1R^\dagger = e_1$, which means that the set of points

$$C = \left\{\left(\cos\alpha,\ \sin\alpha,\ 0,\ 0\right) \middle| \alpha \in \mathbb{R}\right\}$$

in S^3 all map to e_1 via the Hopf map h.

Exercise 1: Show that the set C is the entire set of points that maps to e_1 via the Hopf map h.

The set C is a unit circle in a plane in $\mathbb{R}^4$. We can say that the set C is the *preimage set* $h^{-1}\left(e_1\right)$. We call the preimage set $h^{-1}\left(P\right)$ the *fiber* of the Hopf map over P.

Exercise 2: Show that for any point P in S^2, the preimage set $h^{-1}\left(P\right)$ is a circle in S^3 (see [4]).

Any point $p \in S^2$ can be represented by a unit vector, say $\hat{\mathbf{p}}$. Likewise, any point $P \in S^3$ can be represented by a *unit spinor* (*rotor*, see Sect. 1.9.13 in [3])), say R. Then the expression $R\hat{\mathbf{p}}R^\dagger$ can be seen as a rotation of the vector $\hat{\mathbf{p}}$, which means that we have a mapping $S^2 \to S^2$. However, we can also interpret it as a mapping $h_p : S^3 \to S^2$, defined by

$$h_{\hat{\mathbf{p}}}\left(R\right) = R\hat{\mathbf{p}}R^\dagger .$$

We call such a map the *Hopf fibration*. This means that there is not just one Hopf fibration, but there are infinitely many of them, one of which is $\hat{\mathbf{p}} = e_1$. In [5] and [6]), the reader can find an interesting definition with $\hat{\mathbf{p}} = e_3$ and application to *stereographic projection*.

Rotors in *Cl*3 are powerful and easy to imagine. As the Hopf fibration could play an important role in physics (for example, see [8]), a motivated reader could gain a great deal of pleasure translating the results of this topic to the language of geometric algebra.

References

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