# Argumenting the validity of Riemann Hypothesis

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(Dated: October 15, 2020)

## Abstract

There are tens of self-proclaimed proofs for the Riemann Hypothesis and only 2 or 4 disproofs of it in arXiv. To this Status Quo I am adding my very short and clear results even without explicit mentioning prime numbers. One of my breakthroughs uses the peer-reviewed achievement of Dr. Solé and Dr. Zhu, published just 4 years ago in a serious mathematical journal INTEGERS.

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## I. DISCLAIMER

An author is not obligated to check for the validity of all the papers, which he refers to. Otherwise, he would be checking all references of the papers he references, i.e. to check for the validity of all papers, which lead to his paper. And to do it using personal opinion. Thus, having found the Dr. Zhu paper, that has demonstrated, that "The probability of Riemann's hypothesis being true is equal to 100%", the author relies on the validity of this result, without trying to do an unauthorized peer-review of the Dr. Zhu paper and the papers he refers to.

## II. CAN WE BE SURE?

DEFINITIONS:

If something has probability 1/3, then it practically means, that a bag contains 3 balls. One of them is blue, other two are red. By taking the blue one from the bag (with closed eyes) the taker realizes the 1/3 probability event.

The probability 1 does not mean, that

A. the bag contains infinitely many blue balls and only one red ball; but rather that

B. the bag contains one ball only: blue, with none of red balls with it.

With the definition of probability nr. A one would mistakenly say, that probability that the equation $x - 5 = 0$ has solution is zero, because the amount of numbers is infinite.

An axiom is a thing, which is so obvious and natural, that it does not need proving.

AXIOM: *If something truly has the perfect 100% probability, then that something is true.*

This axiom becomes seen as trivially true, if one expands it following way:

*If something is true with 100% probability, then that something is truly true.*

Thus, relying on the validity of Dr. Zhu result, which he formulated as "The probability of Riemann's hypothesis being true is equal to 1", I can conclude, that Riemann Hypothesis is true.

## III. THE PAPER STRATEGY

It is known that the Riemann Hypothesis is true if either the Robin inequality [2]

$$\frac{\sigma(n)}{n} \le \mathrm{e}^{\gamma} \ln\ln n =: u(n) \tag{1}$$

or the Lagarias inequality [3]

$$\frac{\sigma(n)}{n} < \frac{H_n + \exp(H_n)\,\ln(H_n)}{n} =: U(n)\,, \quad H_n = \gamma + \ln n + O(1/n)\,. \tag{2}$$

holds, where $\sigma(n)$ is the sum of divisors of $n$, e.g. $\sigma(6) = 1+2+3+6$, and $\gamma \approx 0.577$ is the Euler constant. Therefore, Eqs. (1) and (2) are equivalents of the Riemann Hypothesis. If one or even both inequalities are proven to be true, the Riemann Hypothesis is true.

### A. One-page Proof

**The point.**
Consider the following. If the Riemann Hypothesis is the same as saying (for example) $x < 4$, and $x < 4$ is the same as saying the Riemann Hypothesis is true, then
1. $x < 4$ and the Riemann Hypothesis are equivalent statements.
2. $x < 5$ and the Riemann Hypothesis cannot be equivalent.

However, we have a unique situation where we have both $x < 4$ and $x < 5$ as two equivalents of the Riemann Hypothesis. In other words: school mathematics tells us that from the validity of $x < 4$ trivially follows validity of $x < 5$. Hence, Robin's paper [2] of 1984 AD has robbed from Lagarias's paper [3] of 2002 AD any meaning; that is not possible, because the 2002 AD paper is peer-reviewed and, therefore, must contain meaningful results. The consequence of this talk is the proof of the Riemann Hypothesis, which is the following:

If one of the equivalent formulations of the Riemann Hypothesis is showing the Riemann Hypothesis to be false, then all equivalent formulations of the Riemann Hypothesis show that the Riemann Hypothesis is false. Because $u(n) < U(n)$, the Robin formulation allows a situation[6] where the Riemann Hypothesis is shown to be false, whereas Lagaria's formulation still shows the Riemann Hypothesis to be true,

$$u(n) < \frac{\sigma(n)}{n} < U(n)\,. \tag{3}$$

Our assumption was that "one of the equivalent formulations of the Riemann Hypothesis is showing the Riemann Hypothesis to be false", but we came to a contradiction. Thus, all equivalent formulations are showing the Riemann Hypothesis to be true.

### B. The evidence using Dr. Solé and Dr. Zhu result

Numerical tests on the Robin inequality have shown that Eqs. (1) and (2) both hold for any needed $5041 \leq n < N$, as $U(n) > u(n)$. Today the unchecked area of $n$ is given by $n \geq N = \exp(\exp(26)) \gg 1$.

Dr. Solé and Dr. Zhu have proven [4] that for large numbers of $n$ one has

$$u(n) - \frac{\sigma(n)}{n} \geq -\beta(n)\,, \tag{4}$$

where $\beta(n) \geq 0$ is an unknown function which, if non-vanishing, is monotonically decreasing and $\beta(n) = 0$ for $n \to \infty$. The inequality (4) holds in all cases, even if the Riemann Hypothesis is false.

From Eqs. (2) and (4) it follows that the Riemann Hypothesis is true, if

$$\beta(n) + u(n) < U(n)\,, \tag{5}$$

which I call "Martila inequality". Following from this inequality, for large $n$ I am showing that the case $\beta(n) = C/n^x$, $x > 0$ and $x \neq 0$, where $C \geq 0$ is an arbitrary constant, satisfies the Martila inequality. [7] This discovery means that if $\beta(n)$ is an analytical function, or it can be expressed using a Taylor series expansion (for small $\epsilon = 1/n^v$, where $v > 0$, e.g. $v = 0.3$), then the Riemann Hypothesis is true. In general, if for large $n$ with $\beta(n) < \beta_0(n)$, where

$$\beta_0(n) + u(n) = U(n)\,, \tag{6}$$

the Riemann Hypothesis is true.

## IV. APPENDIX

### A. Prior research result

I start with the starting information of the papers [1, 4] (one of the papers is peer-reviewed), where is proven (cf. Theorem 2) that for the "limit inferior" one has

$$\lim_{n\to\infty} \inf d(n) \geq 0\,, \tag{7}$$

where $d(n) = D(n)/n$ and $D(n) = \mathrm{e}^{\gamma}\, n \ln\ln n - \sigma(n)$. Hereby the Riemann Hypothesis holds true, if $\lim\limits_{n\to\infty} \inf D(n) \geq 0$.

I conclude the existence of Eq. (4) with the continuous monotonic function $-\beta(n) \leq \inf d(n)$.

The main problem of the available Riemann Hypothesis proofs is a possible fatal mistake somewhere in the text. If a text is complicated enough, the mistake is practically impossible to find. The final result of Ref. [1] comes from too many theorems (Theorems 1, 2 and 3 in Ref. [4]), so the risk of having a mistake is very high. However, I will demonstrate that it is enough to hope for the validity of Theorem 2 in Ref. [4], i.e. I can prove the Riemann Hypothesis even without Theorems 1 and 3. Recall that the Riemann Hypothesis has been shown to hold unconditionally for $n$ up to $N = \exp(\exp(26))$, as written in Refs. [4, 5]. Thus, it is enough to check the Riemann Hypothesis for the region $n \gg 1$. Therefore, we do not need Theorem 3, because it is a trivial fact Dr. Zhu is proving that if $D(n) \geq 0$ for $n > N \gg 1$, the Riemann Hypothesis is correct. Also, we do not need Theorem 1, as Theorem 2 already says that Eq. (7) holds.

### B. Is $N$ large?

A journal referee might say some nonsense like "what if $N = \exp(\exp(26))$ is very small, i.e. maybe $N \sim 1$?" to reject the paper. I disagree! Ref. [4] tells us that the area where $n > M$ with $M \to \infty$ is decisive. I mean, if the Riemann Hypothesis is wrong, it must be shown wrong at $n \to \infty$. Therefore, you can replace $N$ with any fixed $M \gg N$ in my analysis.

### C. New Criterion

The harmonic number is

$$H_n = \gamma + \ln(n) + K(n)\,, \tag{8}$$

where $K(n) > 0$, and $K(n) = 0$ for $n \to \infty$. Thus,

$$U(n) = \mathrm{e}^{\gamma} \ln(\gamma + \ln(n)) + R(n)\,, \tag{9}$$

where $R(n) > 0$. It follows that the Riemann Hypothesis is true, if for large $n$ one has $\beta(n) \le \beta_0(n)$ with

$$\beta_0(n) = \mathrm{e}^{\gamma} \ln([\gamma/\ln(n)] + 1) + R(n)\,. \tag{10}$$

I am citing from the end of Ref. [4]: "For instance, one cannot rule out the case that $D(n)$ behaves like $-\sqrt{n}$ when $n \to \infty$, which would not contradict the fact that $\liminf_{n\to\infty} d(n) = 0$." This points to my function $\beta(n) = (C\sqrt{n})/n = C/\sqrt{n}$, where $C \ge 0$, e.g. $C = 1$. Because of $C/\sqrt{n} < \beta_0(n)$, the Riemann Hypothesis is true for such a case. [8] And in order to avoid the contradiction with the Robin inequality (which is $D(n) \ge 0$) we have to assign $C = 0$.

### D. Inequalities are true together

If the Robin inequality is violated at some $n = n_0$, then it is certain that both inequalities (Robin and Lagarias) are violated at some $n_h$. However, because of this certainty, we must be certain as well to violate them both at $n_0$. In the hypothetical situation where Robin inequality is violated for several finite $n_k$ but the Lagarias inequality is violated only for infinite $n_L \to \infty$, the Lagarias inequation has lost the meaning of an equivalent formulation of the Riemann Hypothesis. However, this is not possible. In another situation where the Robin inequation is violated only at one single point $n_0$, the Lagarias inequality must be violated at this point as well. Thus, if the Riemann Hypothesis is wrong, both inequalities must be violated together.

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[1] Zhu Y., The probability of Riemann's hypothesis being true is equal to 1, arXiv:1609.07555 [math.GM] (2016, 2018).

[2] Robin G., Grandes Valeurs de la fonction somme des diviseurs et hypothése de Riemann, J. Math. Pures Appl. **63**, 187–213 (1984); Akbary A.; Friggstad Z., Superabundant numbers and the Riemann hypothesis, Am. Math. Monthly **116** (3), 273–275 (2009).

[3] Lagarias J. C., An elementary problem equivalent to the Riemann hypothesis, The American Mathematical Monthly **109** (6), 534–543 (2002); Sandifer C. E., How Euler Did It, MAA Spectrum, Mathematical Association of America, p. 206 (2007).

[4] Solé P., Zhu Y., An Asymptotic Robin Inequality, INTEGERS, **A81**, 16 (2016), http://math.colgate.edu/∼integers/q81/q81.pdf

[5] Briggs K., Abundant numbers and the Riemann hypothesis, Experiment. Math. **15** (2), 251–256 (2006).

[6] A referee might say: "*Allowing a situation* that you do not show actually happens is NOT a proof." However, it can be argued that mathematics is consistent with the thesis "situation is allowed". This, however, drives us to the described contradiction. Therefore, in order to avoid such contradiction we must accept the validity of the Riemann Hypothesis.

[7] For this I defined $z(x, n) := \beta(n) - \beta_0(n)$ and extracted the critical curve $x = x_c(n)$ from $z(x, n) = 0$. In the limit $n \to \infty$ one has $x_c = 0$.

[8] To demonstrate this, one formally inserts $K(n) \equiv 0$ for all $n$ in Eq. (8), checks the resulting inequality, and restores $K(n) > 0$.