Proof of Riemann Hypothesis Using Schwarz Reflection Principle

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Abstract: Riemann's Xi function is defined as ,

$\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$

*$\xi(s)$ is an entire function whose zeroes are the non trivial zeroes of $\zeta(s)$
All the non trivial zeros of the Riemann zeta function lie inside the critical strip
$0 < \Re(s) < 1$. In this paper we use product representation of Riemann Xi function and Schwarz Reflection principle to conclude that the Riemann Hypothesis is true*

Keywords: Riemann zeta function, Riemann Xi function, critical strip, critical line.

1 Statement of the Riemann Hypothesis

The Riemann Hypothesis states that all the non trivial zeroes of the Riemann Zeta function lie on the critical line $\Re(s) = 1/2$

2 Proof

The Riemann Xi function [2, p.37, Theorem 2.11] is defined as ,

$\xi(s) = \xi(0)\prod_{\rho}(1 - \frac{s}{\rho})$ ... (1)

where ρ ranges over all the roots ρ of $\xi(\rho) = 0$ and if we combine the factors

$(1 - \frac{s}{\rho})$ *and* $(1 - \frac{s}{1-\rho})$, *the product converges absolutely and uniformly on*

$compact\ subsets\ of\ \mathbb{C}$

$Also,\ \xi(0) = 1/2$, [2, $p.37$]

$Claim:\ \xi(\rho_0) = 0 \Rightarrow \Re(\rho_0) = 1/2$

$Enough\ to\ prove: \Re(\rho_0) \neq 1/2 \Rightarrow \xi(\rho_0) \neq 0$

$Let,\ \Re(\rho_0) \neq 1/2$

$Case\ 1: 0 < \Re(\rho_0) < 1/2\ and\ \Im(\rho_0) > 0$

$Assume\ on\ the\ contrary\ that\ \xi(\rho_0) = 0,\ 0 < \Re(\rho_0) < 1/2\ and\ \Im(\rho_0) > 0$

$Hadamard\ product\ of\ the\ Riemann\ Xi\ function\ is,$ [4, $p.42$, $section$ 2.5]

$$\xi(s) = \xi(0)\prod_{\Im(\rho)>0}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho}) \tag{1}$$

$\xi(\rho_0) = 0$

$\prod_{\Im(\rho)>0}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho}) = 0$, $since\ \xi(0) = 1/2$

$\Rightarrow \prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho})\prod_{\Im(\rho)>0,\Re(\rho)>1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho}) = 0$

$Since\ both\ the\ above\ products\ are\ convergent$ [$proved\ later\ in\ this\ paper$], so ,

$\prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho}) = 0\ or\ \prod_{\Im(\rho)>0,\Re(\rho)>1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho}) = 0$

$Case\ 1A: \prod_{\Im(\rho)>0,\Re(\rho)>1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho}) = 0$

$\prod_{\Im(\rho)>0,\Re(\rho)>1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho}) = \prod_{\Im(\rho)>0,\Re(\rho)>1/2}[1-\frac{\rho_0(1-\rho_0)}{\rho(1-\rho)}]$

$This\ converges\ absolutely\ provided\ \sum_{\Im(\rho)>0,\Re(\rho)>1/2}\frac{1}{|\rho(1-\rho)|} < \infty$

$\sum_{\Im(\rho)>0,\Re(\rho)>1/2}\frac{1}{|\rho(1-\rho)|} < \sum_{\Im(\rho)>0}\frac{1}{|\rho(1-\rho)|} < \sum_{\Im(\rho)>0}\frac{1}{|(\rho-\frac{1}{2})^2-\frac{1}{4}|} < \sum\frac{1}{|\rho-\frac{1}{2}|^2}$

$it\ suffices\ to\ prove\ the\ convergence\ of\ the\ sum\ \sum\frac{1}{|\rho-\frac{1}{2}|^2};\ here\ the\ sum\ can\ be\ considered\ either\ as\ a\ sum$ $over\ roots\ \rho\ such\ that\ \Im(\rho) > 0\ or\ as\ a\ sum\ over\ all\ roots\ since\ first\ of\ these\ is\ twice\ the\ second$ [4, $p.$ 42]

$\sum\frac{1}{|\rho-\frac{1}{2}|^2} < \infty$ [4, $p.42$, $Theorem$]

$Product\ \prod_{\Im(\rho)>0,\Re(\rho)>1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho})\ is\ absolutely\ convergent\ and\ hence\ convergent.$

$Value\ of\ a\ convergent\ infinite\ product\ is\ 0\ if\ and\ only\ if\ atleast\ one\ of\ the\ factors\ is\ 0$ [5, $p.287$]

$(1-\frac{\rho_0}{\rho_1})(1-\frac{\rho_0}{1-\rho_1})=0,\ where\ \Im(\rho_1)>0\ ,\Re(\rho_1)>1/2$

$\rho_0=\rho_1\ or\ \rho_0=1-\rho_1$

$\Re(\rho_0)=\Re(\rho_1)>1/2\ or\ \Im(\rho_0)=\Im(1-\rho_1)<0$

$\Re(\rho_0)>1/2\ or\ \Im(\rho_0)<0$

$which\ contradicts\ \ \Re(\rho_0)<1/2\ and\ \Im(\rho_0)>0\ in\ Case\ 1$

$Case\ 1B: \prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho})=0$

$Let,\ I(s)=\prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho})$

$I(\rho_0)=0$

$\xi(s)=\xi(0)\prod_{\Im(\rho)>0}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho})$

$\xi(\rho_0)=0$

$\xi(\rho_0)=\prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho})\prod_{\Im(\rho)>0,\Re(\rho)>1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho})$

$Since,\ \xi(\rho)=0 \iff \xi(1-\rho)=0 \iff \xi(\overline{\rho})=0 \iff \xi(1-\overline{\rho})=0$

$\xi(\rho_0)=\prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho})(1-\frac{\rho_0}{\overline{\rho}})(1-\frac{\rho_0}{1-\overline{\rho}})\prod_{\Im(\rho)>0,\Re(\rho)>1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho})(1-\frac{\rho_0}{\overline{\rho}})(1-\frac{\rho_0}{1-\overline{\rho}})$

$I(s)=\prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho})(1-\frac{s}{\overline{\rho}})(1-\frac{s}{1-\overline{\rho}})$

$I(\rho_0)=\prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho})(1-\frac{\rho_0}{\overline{\rho}})(1-\frac{\rho_0}{1-\overline{\rho}})=0$

$I(\overline{\rho_0})=\prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{\overline{\rho_0}}{\rho})(1-\frac{\overline{\rho_0}}{1-\rho})(1-\frac{\overline{\rho_0}}{\overline{\rho}})(1-\frac{\overline{\rho_0}}{1-\overline{\rho}})$

$I(\overline{\rho_0})=\overline{\prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{\rho_0}{\rho})(1-\frac{\rho_0}{1-\rho})(1-\frac{\rho_0}{\overline{\rho}})(1-\frac{\rho_0}{1-\overline{\rho}})}$

$I(\overline{\rho_0})=\overline{I(\rho_0)}=0$

$Since\ from\ Case\ 1B\ ,\ I(\rho_0)=0 \Rightarrow I(\overline{\rho_0})=0$

$I(s)\ is\ Holomorphic\ on\ the\ upper\ half\ plane$ $\{\mathrm{I(z)}\in\mathbb{C}\mid\Im(z)>0\}$ and real on the real axis

Also, by Schwarz's reflection principle ,

I$(\overline{\rho_0})=\overline{I(\rho_0)}=0$

$I(\overline{\rho_0}) = 0$

$$\prod_{\Im(\rho)>0,\Re(\rho)<1/2}(1-\frac{\overline{\rho_0}}{\rho})(1-\frac{\overline{\rho_0}}{1-\rho}) = 0 \quad (2)$$

Since proceeding similarly in case $1A$, (2) *is also a convergent infinite product.*

Value of a convergent infinite product is 0 *if and only if atleast one of the factors is* 0 [5, p.287]

$(1 - \frac{\overline{\rho_0}}{\rho_1})(1 - \frac{\overline{\rho_0}}{1-\rho_1}) = 0,$ *where* $\Re(\rho_1) < 1/2$ *and* $\Im(\rho_1) > 0$

$\overline{\rho_0} = \rho_1$ *or* $\overline{\rho_0} = 1 - \rho_1$

$\Im(\overline{\rho_0}) = \Im(\rho_1) > 0$ *or* $\Re(\overline{\rho_0}) = \Re(1 - \rho_1) > 1/2$

$\Im(\rho_0) < 0$ *or* $\Re(\rho_0) > 1/2$

contradicts $\Im(\rho_0) > 0$ *or contradicts* $\Re(\rho_0) < 1/2$ *in Case* 1

So we get a contradiction.

Hence our assumption that $\xi(\rho_0) = 0,\ 0 < \Re(\rho_0) < 1/2$ *and*

$\Im(\rho_0) > 0$ *is wrong*

Thus, $\xi(\rho_0) \neq 0$ *when* $0 < \Re(\rho_0) < 1/2$ *and* $\Im(\rho_0) > 0$

Case $2 : 0 < \Re(\rho_0) < 1/2$ *and* $\Im(\rho_0) < 0$

$\xi(\rho_0) = 0$

$\Rightarrow \xi(\overline{\rho_0}) = 0$

$\Im(\rho_0) < 0 \Rightarrow \Im(\overline{\rho_0}) > 0$

Also $0 < \Re(\overline{\rho_0}) < 1/2$

By, Case 1, $\xi(\overline{\rho_0}) \neq 0,\ \Im(\overline{\rho_0}) > 0$ *and* $0 < \Re(\overline{\rho_0}) < 1/2$

Since, $\xi(s)$ *is holomorphic and is real on the real axis so, by Schwarz's reflection principle* [1, p.30],

$\overline{\xi(\rho_0)} = \xi(\overline{\rho_0}) \neq 0$, $\Im(\overline{\rho_0}) > 0$ *and* $0 < \Re(\overline{\rho_0}) < 1/2$

$\xi(\rho_0) \neq 0,\ 0 < \Re(\rho_0) < 1/2$ *and* $\Im(\rho_0) < 0$

Thus in both the above cases 1 *and* 2, *we get a contradiction*

$So,\ our\ assumption\ that\ \xi(\rho_0) = 0\ when\ 0 < \Re(\rho_0) < 1/2\ is\ wrong$

$Thus,\ \xi(\rho_0) \neq 0\ when\ 0 < \Re(\rho_0) < 1/2$

$Case\ 3 : 1/2 < \Re(\rho_0) < 1, \Im(\rho_0) \in \mathbb{R}$

$\rho_0\ is\ a\ zero\ of\ \xi(s)\ then\ 1 - \rho_0\ is\ also\ a\ zero\ due\ to\ the\ functional\ equation$

$\xi(s) = \xi(1-s)\ and\ by\ Schwarz's\ reflection\ principle\ 1-\overline{\rho_0}\ is\ also\ a\ zero\ [1, p.30]$

$\rho_0 = \sigma_0 + it_0, 1/2 < \sigma_0 < 1$

$1 - \overline{\rho_0} = 1 - \sigma_0 + it_0$

$1 - \sigma_0 = \sigma_0',\ 0 < \sigma_0' < 1/2$

$By\ cases\ 1\ and\ 2,\ \xi(\sigma_0' + it_0) \neq 0,\ 0 < \sigma_0' < 1/2$

$\xi(1 - \sigma_0 + it_0) \neq 0$

$By\ functional\ equation,$

$\xi(\sigma_0 - it_0) \neq 0$

$By\ Schwarz's\ Reflection\ principle,$

$\xi(\sigma_0 + it_0) \neq 0\ , 1/2 < \sigma_0 < 1$

$\xi(\rho_0) \neq 0\ when\ 1/2 < \Re(\rho_0) < 1$

$Cases\ 1\ , 2\ and\ 3\ gives,$

$If\ \Re(\rho_0) \neq 1/2\ then\ \xi(\rho_0) \neq 0$

$or\ if\ \xi(\rho_0) = 0\ then\ \Re(\rho_0) = 1/2$

$This\ proves\ the\ Riemann\ Hypothesis$

3 References:-

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