The Asymptotic Riemann Hypothesis

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Abstract

The Asymptotic Riemann Hypothesis is defined and it is showed that for a large class of functions, it is verified.

1 Introduction

It appears that the Riemann Hypothesis RH, the fact of the non trivial zeros of the Riemann zeta function are on the critical line, is a major challenge in mathematics due to its deep consequences in all domains of mathematics. Here, instead of trying to solve this difficult problem RH, we propose a nearby one, but simpler. Instead to show that all zeros are on the critical line, we propose that only asymptotically, when the imaginary part of the zeros are great, the zeros are one the critical line. It is the Asymptotic Riemann Hypothesis ARH. We don't demonstrate ARH but we show that for a deformation of the ksi function, namely the truncated integral, we have ARH.

2 The Asymptotic Riemann Hypothesis

The Riemann zeta function [R] is defined over the complex numbers by:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

It has a pole in $s = 1$.
The Asymptotic Riemann Hypothesis (ARH) is the following proposition:

$$\exists A \in \mathbf{R}, \forall s = x + iy \in \mathbf{C}, (x \in]0,1[) \wedge (y \geq A) \wedge (\zeta(s) = 0) \Rightarrow (x = \frac{1}{2})$$

If we replace the Riemann zeta function by a function F, we obtain ARH for F.

3 The Truncated Integral

The function ξ [E] [T] is defined with help of the Riemann zeta function and the Euler Gamma function:

$$\xi(s) = \frac{1}{\pi^{s/2}} \zeta(s) \Gamma(s/2)$$

We have for $Re(s) > 1$:

$$\xi(s) = \int_0^\infty t^{s/2-1}(\theta_3(t) - 1)dt$$

where θ_3 is the theta function:

$$\theta_3(t) = \sum_{n\in\mathbf{Z}} e^{-\pi n^2 t}$$

It verifies the functional equation:

$$\theta_3(t) = \frac{1}{\sqrt{t}}\theta_3(\frac{1}{t})$$

We have for $0 < Re(s) < 1$:

$$\xi(s) = 2\int_1^\infty [t^{s-1} + t^{-s}]f(t)dt$$

with:

$$f(t) = \theta_3(t^2) - 1 - \frac{1}{t}$$

So that we have also:

$$\xi(s) = \xi(1-s)$$

Now we define the truncated integral for a function f:

$$F(s,B) = \int_1^B [t^{s-1} + t^{-s}]f(t)dt$$

4 ARH for the Truncated Integrals

We have the following theorem:

Theorem: *For all function f, such that $f(t) \neq 0$ $\forall t$, and for all B, we have ARH for $F(s,B)$.*

Demonstration: We change the variables $t = e^u$:

$$F(s,B) = \int_0^{ln(B)} [e^{su} + e^{(1-s)u}]\tilde{f}(u)du = 0$$

with $\tilde{f}(u) = f(e^u)$. We separate the real and imaginary parts:

$$Re = \int_0^{ln(B)} cos(yu)[e^{xu} + e^{(1-x)u}]\tilde{f}(u)du = 0$$

$$Im = \int_0^{ln(B)} sin(yu)[e^{xu} - e^{(1-x)u}]\tilde{f}(u)du = 0$$

We apply the Taylor-Lagrange formula for the imaginary part, so that we factorize $(2x-1)$:

$$Im = (2x-1)\int_0^{ln(B)} sin(yu)ue^{cu}\tilde{f}(u)du = 0$$

with $c \in]x, 1-x[$. Now, we make integrations by parts for the real and imaginary parts:

$$Re = \frac{sin(yln(B))}{y}[B^x + B^{1-x}]f(B) -$$

$$-\int_0^{ln(B)} \frac{sin(yu)}{y}[(e^{xu} + e^{(1-x)u})\tilde{f}(u)]'du = 0$$

and

$$Im = (2x-1)[-\frac{cos(yln(B))}{y}ln(B)B^c f(B) +$$

$$+\int_0^{ln(B)} \frac{cos(yu)}{y}[ue^{cu}\tilde{f}(u)]'du = 0$$

A new integration by parts implies that we have uniformly in x:

$$Re = \frac{sin(yln(B))}{y}[B^x + B^{1-x}]f(B) + o(\frac{1}{y}) = 0$$

and

$$Im = (2x-1)[-\frac{cos(yln(B))}{y}ln(B)B^c f(B) + o(\frac{1}{y})] = 0$$

As the cos and sin cannot be small in the same time, it implies that:

$$\exists A, \forall y, (y \geq A) \Rightarrow (x = 1/2)$$

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5 Conclusion

We have showed ARH for the deformation of the ξ function $F(s, B)$. We can now define a function $s(B)$, by the implicit functions theorem, such that $F(s(B), B) = 0$ and $s(\infty)$ is a zero of the Riemann zeta function. The study of this function by analytic tool and holomorphic functions, may give informations if ARH can be solved. It seems that if the zero of zeta is off the critical line, $s(B)$ arrives on the critical line in a double point.

References

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[T] E.C.Titchmarsh, "The Zeta-Function of Riemann", Cambridge Univ.Press, London and New-York, 1930.