From Complex Ginzburg-Landau Equation to Classical Field Theory

Ervin Goldfain

Abstract

Complex Ginzburg-Landau equation (CGLE) is a paradigm for the onset of chaotic patterns and turbulence in nonlinear dynamics of extended systems. Here we point out that the underlying connection between CGLE and the Navier-Stokes (NS) equation bridges the gap between fluid flows, on the one hand, and the mathematics of General Relativity (GR) and classical gauge theory, on the other. The analogy hints to a possible link between the transition from laminar to turbulent flows and the mass generation mechanism of quantum field theory (QFT).

Key words: chaos and complexity, complex Ginzburg-Landau equation, turbulence, Navier-Stokes equation, classical gauge theory, General Relativity.

1. Introduction

Building on the assertion that *complex dynamics* plays a critical role in foundational physics [1-3], the object of this brief report is to show that the roots of both classical gauge theory and GR may be traced back to the CGLE.

The report contains a couple of paragraphs and an Appendix section. First paragraph delves into the derivation of the NS equation from CGLE and the role of kinematic viscosity in the transition from laminar flows to turbulence. The second paragraph points out to research studies looking into the analogy between NS and classical field theory, namely GR and Maxwell's electrodynamics. The analogy suggests an unforeseen parallel between the onset of fluid turbulence via reduction of kinematic viscosity and the mass generation mechanism of QFT.

2. CGLE and the NS equations

We start by recalling that the CGLE encodes many key properties of out-of-equilibrium nonlinear dynamical systems with space-time dependence. As paradigm for the emergence of complex behavior, CGLE describes the generic onset of chaos, turbulence, and spatiotemporal patterns in extended systems [4-6]. It assumes the standard form

$$\partial_t z = \varepsilon z + (1 + i c_1)\Delta z - (1 - i c_3) z |z|^{2\sigma} \tag{1}$$

in which z is a complex-valued field, the parameters ε and σ are positive and the coefficients c_1 and c_3 are real. The *nonlinear Schrödinger equation* (NSE) is a particular embodiment of the CGLE in the limit $\varepsilon \to 0$, namely [11],

$$-i\partial_t z = c_1 \Delta z + c_3 |z|^{2\sigma} z \tag{2a}$$

To streamline the presentation, we work in 1+1 space-time and assume $\sigma = 1$. In its original formulation and natural units ($\hbar = 1$), the quantum-mechanical version of (2a) reads

$$i\frac{\partial}{\partial t} z(x,t) = [-\frac{1}{2m}\nabla^2 + V(x,t)]\, z(x,t) \tag{2b}$$

where $V(x,t)$ is the potential function. The so-called *Madelung transformation* enables one to turn (2b) into the quantum Euler equation for compressible potential flows [7]. In particular, taking the complex-valued field in the canonical form,

$$z(x,t)=\sqrt{\frac{\rho(x,t)}{m}}\exp[iS(x,t)] \tag{3}$$

and substituting it into (1)-(2) leads to

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{u})=0 \tag{4}$$

$$\frac{d\vec{u}}{dt}=\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}=-\frac{1}{m}\nabla(Q+V) \tag{5}$$

Here, $\vec{u}(x,t)$ denotes the flow velocity, $\rho=m|z|^2$ stands for the mass density and

$$Q=-\frac{1}{2m}\frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}} \tag{6}$$

is the so-called Bohm potential. The flow velocity and its associated probability current are given by

$$\vec{u}(x,t)=\frac{1}{m}\nabla S=-\frac{i}{m}\frac{\nabla z}{z} \tag{7}$$

$$\vec{j}=\rho\vec{u}=\frac{1}{2mi}[z^*(\nabla z)-z(\nabla z^*)] \tag{8}$$

Since the Schrödinger equation is conservative, the Madelung transformation naturally leads to the Euler equation, which is exclusively valid for *inviscid flows*. To account for fluid viscosity and arrive at the Navier-Stokes equation, one needs to either appeal to an extended version of the NSE containing non-conservative terms [7-8] or bring up the concept of *kinematic viscosity* – a concept linked to the mass of quantum particles [9] as in

$$\nu = \frac{1}{2m} \tag{9}$$

On account of (9) and for incompressible flows, the Navier-Stokes equation that mirrors (5) is given by

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} \tag{10}$$

where p is the pressure. An alternative expression for the Navier-Stokes equation (10) may be obtained using the identity

$$\vec{u} \cdot \nabla \vec{u} = \frac{1}{2} \nabla u^2 - \vec{u} \times \vec{\omega} \tag{11}$$

where $\vec{\omega} = \nabla \times \vec{u}$ represents the vorticity vector [10]. The corresponding Navier-Stokes equation reads

$$\frac{\partial \vec{u}}{\partial t} = -\vec{\omega} \times \vec{u} - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) + \nu \nabla^2 \vec{u} \tag{12}$$

Referring to (3), the phase of the field amounts to

$$S = m \int u \, dx \tag{13}$$

such that

$$z(x,t)=\sqrt{\frac{\rho(x,t)}{m}}\exp(-im\int u\,dx) \tag{14}$$

3. From CGLE to abelian gauge theory and GR

It is instructive to note that there is a wealth of literature dealing with the connection between the NS equation, on the one hand, and the formalism of classical electrodynamics and GR, on the other. The interested reader is invited to consult a cross-section of representative references listed under [12 - 19].

4. Concluding remarks

The brief analysis presented here falls in line with our previous findings where, under general boundary conditions, the long-run evolution of Renormalization Group flows is conjectured to converge on *strange attractors* [2, 21-22]. Supported by an underlying *multifractal structure*, these attractors provide realistic models for the onset of chaos in nonlinear dynamics, the transition to turbulence as well as for the phenomenology of the Standard Model near or above the electroweak scale [1, 3].

Relation (9) confirms that highly viscous fluids produce a hydrodynamic regime close to laminar flows. Interpreted in the context of the *minimal fractal manifold* [3], the kinematic viscosity assumes the role of a mass generation mechanism and corresponds to the inverse of the dimensional parameter

$$\varepsilon = 4 - D = O\left(m^2/\Lambda_{UV}^2\right) << 1$$

Stated differently,

$$\nu = \frac{1}{2m} = O(\varepsilon^{-1/2}) \tag{15}$$

APPENDIX

The aim of this Appendix section is to show that seemingly disparate concepts of quantum physics and classical field theory – namely, the Berry phase, gauge potentials and the connection coefficients of GR - share a common *geometric foundation*.

A) Berry phase in quantum physics

A quantum system adiabatically transported around a closed path C in the space of external parameters acquires a non-vanishing phase (*Berry phase*, BP in short). Since BP is exclusively path-dependent, it provides key insights into the geometric structure of quantum mechanics and quantum field theory (QFT). The BP concept is closely tied to *holonomy*, that is, the extent to which some of variables change as other variables or parameters defining a system return to their initial values.

Consider a quantum system described by the time-independent Hamiltonian $H(t)$, whose associated eigenstate is $|\psi(t)\rangle$ and which is embedded in a slowly changing environment. After a periodic evolution of the environmental parameters ($t \to t+T$), the eigenstate returns to itself, apart from a phase angle,

$$|\psi(t)\rangle = e^{i\alpha}|\psi(0)\rangle \tag{A1}$$

If ω denotes the eigenvalue of $|\psi(t)\rangle$, a generalization of the phase angle $\alpha = \omega T$ in units of $\hbar = 1$ is given by the "*dynamical phase*"

$$\gamma_d = \int_0^T \omega(t)\,dt = \int_0^T \langle\psi(t)|H(t)|\psi(t)\rangle\,dt \tag{A2}$$

Berry has shown that there is a time-independent (but contour dependent) supplemental "*geometric phase*" entering the phase angle, namely,

$$\alpha = \gamma_d + \gamma(C) \tag{A3}$$

where

$$\gamma(C) = \int_C \langle \psi \,|\, i\nabla\psi \rangle \, dr \tag{A4}$$

The dynamical phase γ_d encodes information about the duration associated with the cyclic evolution of the complex vector $|\psi(t)\rangle$. By contrast, (A4) encodes information about the geometry of the environment where the transport takes place.

B) The geometry of gauge and gravitational fields

The gauge field concept may be built from a straightforward geometric interpretation, according to [20]. Consider the parallel transport of a complex vector $|\psi\rangle$ round a closed rectangular loop. The difference between the value of $|\psi\rangle$ at the starting point ($|\psi\rangle_0$) and at the end point $|\psi\rangle_0 \rightarrow |\psi\rangle_f$ is given by

$$\Delta\psi = \psi_f - \psi_0 = -ig\,\Delta S^{\mu\nu} F_{\mu\nu}\psi \tag{B1}$$

in which $\Delta S^{\mu\nu}$ denotes the area enclosed by the rectangle and the strength of the gauge field is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \tag{B2}$$

Echoing the formation of the Berry phase, the effect of parallel transport is to induce a non-vanishing rotation of $|\psi\rangle$ in internal space proportional to the strength of the gauge field. Likewise, the curvature tensor of GR may be motivated through similar arguments. Taking a vector V^{κ} on a round trip by parallel transport, the difference between the initial and final components of the vector amounts to

$$\Delta V^{\kappa} = \frac{1}{2} R^{\kappa}_{\lambda\mu\nu} V^{\lambda} \Delta S^{\mu\nu} \tag{B3}$$

This equation faithfully replicates (B1) and signals the presence of a gravitational field, via the curvature tensor $R^{\kappa}_{\lambda\mu\nu}$. The geometric analogy between gauge theory and General Relativity is captured in the table below.

Gauge Theory	**General Relativity**
Gauge transformation	Coordinate transformation
Gauge group	Group of coordinate transformations
Gauge potential A_{μ}	Connection coefficient $\Gamma^{\kappa}_{\mu\nu}$
Field strength $F_{\mu\nu}$	Curvature tensor $R^{\kappa}_{\lambda\mu\nu}$

Comparison between the geometry of gauge and gravitational fields.

References

1. Available at the following site:

https://www.researchgate.net/publication/344227402_Complex_Dynamics_and_Foundational_Physics

2. Available at the following site:

https://www.researchgate.net/publication/339041409_From_Chaos_and_Strange_Attractors_to_Quantum_Theory

3. Available at the following sites:

http://www.aracneeditrice.it/aracneweb/index.php/pubblicazione.html?item=9788854889972

https://www.researchgate.net/publication/278849474_Introduction_to_Fractional_Field_Theory_consolidated_version

4. Available at the following site:

https://www.researchgate.net/publication/247813449_The_complex_Ginzburg-Landau_equation_as_a_model_problem#fullTextFileContent

5. https://arxiv.org/pdf/cond-mat/0106115.pdf

6. https://iopscience.iop.org/article/10.1209/0295-5075/82/11001/meta

7. https://www.mdpi.com/2311-5521/1/2/18

8. https://arxiv.org/pdf/1409.7036.pdf

9. https://arxiv.org/ftp/arxiv/papers/1301/1301.3586.pdf

10. https://my.mech.utah.edu/~pardyjak/me6700/vorticity.pdf

11. Available at the following site:

https://pdfs.semanticscholar.org/8867/0e1eb822da0246899c74ab71ebcbac3aed59.pdf

12. http://cds.cern.ch/record/435570/files/0004169.pdf

13. http://cds.cern.ch/record/517274/files/0109088.pdf

14. Available at the following site:

https://www.brown.edu/research/projects/crunch/sites/brown.edu.research.projects.crunch/files/uploads/Haralabos%20Marmanis%20thesis_0.pdf

15. https://arxiv.org/pdf/1003.0070.pdf

16. https://link.springer.com/article/10.1007%2Fs00006-005-0004-1

17. https://arxiv.org/abs/1104.5502

18. https://arxiv.org/pdf/1101.2451.pdf

19. https://arxiv.org/pdf/1004.2632.pdf

20. Ryder, L. H. "Quantum Field Theory", Cambridge Univ. Press, 1989.

21. Available at the following site:

https://www.researchgate.net/publication/331315283_Multifractal_Foundation_of_Effective_Field_Theory#fullTextFileContent

22. Available at the following site

https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.212.9798&rep=rep1&type=pdf