The non-periodic solution of a truly nonlinear oscillator with power nonlinearity

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Abstract

In this paper a well-known truly nonlinear oscillator with power nonlinearity mentioned to have only periodic solutions is investigated. It has been shown that such a proposition is not mathematically consistent as the equation may exhibit exact and general non-periodic solutions calculated for the first time using the generalized Sundman transformation.

Keywords: Truly nonlinear oscillator equation, periodic solution, power nonlinearity, generalized Sundman transformation.

Introduction

In the literature the differential equation

$$\ddot{x} + x + x^{1/3} = 0 \qquad (1)$$

has been studied for many years as a truly nonlinear conservative oscillator. The equation (1) is also mentioned to have only periodic solutions. However in a previous paper [1] we have shown that this equation admits exact non-periodic solutions so that the proposition following which its solutions are periodic is not consistent from mathematical point of view. For now, consider the differential equation with fractional nonlinearity

$$\ddot{x} + c\,x^{\ell} = 0 \qquad (2)$$

where $c = 1$, $\ell = 2n+1$, n being a positive integer, which has been for a long time studied in the literature as a truly nonlinear conservative oscillator [2-10] under the conditions that $x(0) = A$, and $\dot{x}(0) = 0$. As such the preceding authors and several others claimed that the equation (2) has only periodic solutions. In his study Gottlieb [6] generalizes the Mickens equation (2) by including the sign function as

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$$\ddot{x} + sign(x)|x|^{\alpha} = 0 \qquad (3)$$

where $0 < \alpha < 1$, and gives the so-called exact expression for various values of α of the angular frequency. Later the equation (3) has been investigated in the form [11-20]

$$\ddot{x} + c_{\alpha}^{2} x|x|^{\alpha-1} = 0 \qquad (4)$$

for $\alpha \geq 0$, and c_{α}^{2} is a real constant. These authors and many others claimed on the basis of phase plane analysis that all the solutions of the equation (4) are periodic. In this regard, Cveticanin [12] investigated the periodic properties of (4) and claimed to have computed its exact time period and approximate solution. In [14], Cveticanin and Poigany claimed to have solved the equation (4) by periodic Ateb function as discovered earlier by Rosenberg [21]. Such a solution as a periodic Ateb function is also computed by Cveticanin and Kovacic in [17]. In [15], Belendez and coworkers claimed to have determined the Fourier series expansion for the exact solution of (4). Using the so-called symmetrization method, Ghose Choudhury and his group claimed to have recovered the time period expression given by Cveticanin [12]. It is important to recall that the explicit solution obtained by Cveticanin and coworkers by means of the first integral of energy using initial conditions has been calculated under the hypothesis that all solutions of the equation (4) are periodic for $\alpha \geq 0$. The question is thus to ask whether there exists $\alpha \geq 0$ such that the solution of (4) is non-periodic. If such a value exists, then the calculation of the so-called exact periodic solutions performed previously in the literature for $\alpha \geq 0$, is not valid. In this way the proposition following which the equation (4) is a truly nonlinear conservative oscillator and has only periodic solutions, becomes non consistent from mathematical point of view. The present work predicts the existence of such a value of $\alpha \geq 0$, leading to non-periodic solutions of (4). To demonstrate this prediction we establish that the equation (2) where c and ℓ are now arbitrary parameters belongs to a general class of nonlinear differential equations previously formulated by Koudahoun et al. [22] (section 2) and calculate the exact and general solution of (2) (section 3) from which we show that for $\alpha = \frac{1}{3}, \frac{5}{3}, 3$, the equation (4) exhibits non-periodic solutions (section 4). Finally a conclusion of the work is given.

2- Statement of equation (2)

In their study Koudahoun et al. [22] applied the generalized Sundman transformation

$$y^{m}(\tau)=\int g^{\ell}(x)dx, \qquad d\tau=g^{\ell}(x)\left[\int g^{\ell}(x)dx\right]^{\frac{1}{m}-1}dt \tag{5}$$

where ℓ and m are arbitrary parameters, to the second-order linear differential equation

$$y''(\tau)+a\,y(\tau)=b \tag{6}$$

to obtain the general second-order nonlinear differential equations

$$\ddot{x}+ma\,g^{\ell}(x)\left(\int g^{\ell}(x)dx\right)^{\frac{2}{m}-1}-mb\,g^{\ell}(x)\left[\int g^{\ell}(x)dx\right]^{\frac{1}{m}-1}=0 \tag{7}$$

Setting $m=1$, leads to

$$\ddot{x}+a\,g^{\ell}(x)\left(\int g^{\ell}(x)dx\right)-b\,g^{\ell}(x)=0 \tag{8}$$

For $g(x)=x$, the equation (8) turns into

$$\ddot{x}+\frac{a}{\ell+1}x^{2\ell+1}-b\,x^{\ell}=0 \tag{9}$$

where $\ell\neq-1$.

Applying $a=0$, the equation (9) reduces to

$$\ddot{x}-b\,x^{\ell}=0 \tag{10}$$

which is identical to (2) for $c=-b$, and $\ell=2n+1$, with $b<0$. Now we may calculate the exact and general solution of (10) in the next section.

3- Exact and general solution of (10)

From (5) one may compute the solution $x(t)$ as follows

$$x(t)=(\ell+1)^{\frac{1}{\ell+1}}\left[y(\tau)\right]^{\frac{1}{\ell+1}} \tag{11}$$

For $a=0$, the solution of (6) may read

$$y(\tau)=\frac{1}{2}b\left(\tau^{2}+K_{1}\tau+K_{2}\right) \tag{12}$$

where K_1 and K_2 are arbitrary parameters so that the solution (11) becomes

$$x(t)=(\ell+1)^{\frac{1}{\ell+1}}\left(\frac{b}{2}\right)^{\frac{1}{\ell+1}}\left(\tau^2+K_1\tau+K_2\right)^{\frac{1}{\ell+1}} \tag{13}$$

where τ satisfies the quadrature

$$\int\frac{d\tau}{\left(\tau^2+K_1\tau+K_2\right)^{\frac{\ell}{\ell+1}}}=\left[\left(\frac{b}{2}(\ell+1)\right)^{\frac{\ell}{\ell+1}}\right]t+K_3 \tag{14}$$

where K_3 is an arbitrary parameter.

From the relation (14) we may show our prediction in the following section.

4- Exact and general solution for $\alpha=\frac{1}{3},\frac{5}{3},3$.

In this section we show the existence of non-periodic solutions of equation (4) for $\alpha=\frac{1}{3},\frac{5}{3},3$.

In the case $\alpha=\frac{1}{3}$, the equation (4) reduces to

$$\ddot{x}+c\,x^{1/3}=0 \tag{15}$$

The equation (15), as mentioned previously, is an interesting equation since it has been investigated for many years in the literature as a truly nonlinear oscillator [1-9] for $c=1$, which leads to

$$\ddot{x}+x^{1/3}=0 \tag{16}$$

The equation (16) has been also the subject of the thesis dissertation by Wilkerson [10]. The cases $\alpha=\frac{5}{3}$, and $\alpha=3$, have been solved in [12]. Let us now consider the equation (14). Given that K_1 and K_2 are arbitrary constants, it is always possible to write (14) as

$$\int\tau^{-\frac{2\ell}{\ell+1}}d\tau=\left[\left(\frac{b}{2}(\ell+1)\right)^{\frac{\ell}{\ell+1}}\right]t+K_3 \tag{17}$$

where $K_1=K_2=0$.

Integrating the left hand side of (17) leads to obtain

$$(\tau)^{\frac{1-\ell}{\ell+1}} = \left[\left(\frac{1-\ell}{\ell+1}\right)\left(\frac{b}{2}(\ell+1)\right)^{\frac{\ell}{\ell+1}}\right] t + K_4 \tag{18}$$

where $K_4 = \left(\frac{1-\ell}{1+\ell}\right) K_3$, and the exact and general solution (13) becomes

$$x(t) = \left(\frac{b(1-\ell)^2}{2(1+\ell)}\right)^{\frac{1}{1-\ell}} (t+K)^{\frac{2}{1-\ell}} \tag{19}$$

where K is a new arbitrary constant.

Substituting $\ell = \frac{1}{3}$, yields the solution (19) of (10) as

$$x(t) = \left(\frac{b}{6}\right)^{\frac{3}{2}} (t+K)^3 \tag{20}$$

Applying $b = -c$, the solution of (15) reads

$$x(t) = (-1)^{\frac{3}{2}} \left(\frac{c}{6}\right)^{\frac{3}{2}} (t+K)^3 \tag{21}$$

from which the desired solution of (16) takes the form

$$x(t) = (-1)^{\frac{3}{2}} \left(\frac{1}{6}\right)^{\frac{3}{2}} (t+K)^3 \tag{22}$$

that is

$$x(t) = -i \left(\frac{1}{6}\right)^{\frac{3}{2}} (t+K)^3 \tag{23}$$

where i is the pure imaginary number.

Case $\alpha = \frac{5}{3}$.

In this case the equation (4) gives

$$\ddot{x} + x^{5/3} = 0 \tag{24}$$

Using the exact and general solution (19) for $\ell = \frac{5}{3}$, one may obtain the solution of (24) in the form

$$x(t) = (-1)^{\frac{3}{2}}(12)^{\frac{3}{2}}(t+K)^{-3} \quad (25)$$

Case $\alpha = 3$.

This case yields as equation

$$\ddot{x} + x^3 = 0 \quad (26)$$

From (19), applying $\ell = 3$, one may obtain the solution of (26) as

$$x\,(t) = -\frac{i\sqrt{2}}{t+K} \quad (27)$$

It is worth to note that for $\alpha = 1$, the equation (4) gives the equation of the harmonic oscillator

$$\ddot{x} + c_1^2 x = 0 \quad (28)$$

for which the present theory gives as solution

$$x(t) = i\,c_1\, e^{i c_1 (t+K)} \quad (29)$$

In this situation, the real part of (29) that is

$$\mathrm{Re}[x\,(t)] = -c_1 \sin[c_1 t + K_0] \quad (30)$$

is also solution of (28), where $K_0 = c_1 K$. As can be seen, the amplitude of solution (30) depends on the frequency c_1 of the harmonic oscillator as found in a previous work [23]. It is easy to observe that the solutions (23), (25) and (27) are not periodic but consist of complex-valued functions. In this context we have shown the existence of $\alpha \geq 0$, such that the supposed and well-known truly nonlinear conservative oscillator (4) may exhibit non-periodic solutions. That being so a conclusion for the work may be given.

Conclusion

A well-known truly nonlinear oscillator mentioned in the literature to have only periodic solutions is investigated in this work. It has been shown that such a proposition is not consistent from mathematical point of view by exhibiting exact and general non-periodic solutions for this equation using the generalized Sundman transformation.

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