Elementary Proof that the Goldbach Conjecture is False

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Abstract

Christian Goldbach (March 18, 1690 – November 20, 1764) was a German mathematician. He is remembered today for Goldbach's conjecture.

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

On 7 June 1742, the German mathematician Christian Goldbach wrote a letter to Leonhard Euler (letter XLIII) in which he proposed the following conjecture:
Every even integer which is ≥ 4 can be written as the sum of two primes (the strong conjecture)
He then proposed a second conjecture in the margin of his letter:

Every odd integer greater than 5 can be written as the sum of three primes (the weak conjecture).

In number theory, Goldbach's weak conjecture, also known as the ternary Goldbach problem, states that every odd number greater than 5 can be expressed as the sum of three primes. (A prime may be used more than once in the same sum). In 2013, Harald Helfgott finally proved Goldbach's weak conjecture, a huge contribution to mathematics and number theory.

The “strong” conjecture has been shown to hold up through 4×10^{18}, but remains unproven for almost 300 years despite considerable effort by many mathematicians throughout history.

The author would like to give many thanks to Harald Helfgott for his proof of the weak conjecture, because this elementary proof of showing the strong conjecture is false, was completely dependent on Helfgott’s proof. Without Helfgott’s proof, this elementary proof would not be possible.

Proof

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes. In other words, $2k = p_1 + p_2$, where $k = 2, 3, 4, \ldots \infty$

The Goldbach Conjecture states that for every even integer N, and N > 2, then $N = P_1 + P_2$, where P_1, and P_2, are prime numbers.

For example, when N = 4, then 4 = 2 + 2, and since 2 is prime then the Goldbach Conjecture is satisfied. When N = 6, then 6 = 3 + 3, and since 3 is prime then the Goldbach Conjecture is satisfied again.

A proof of the strong Goldbach conjecture implies the ternary Goldbach conjecture, that is, all odd numbers greater than 5 are the sum of three primes. For example, in order to express an odd number $n > 5$ as the sum of three primes, subtract 3 and obtain an even number $n - 3 \geq 2$. If the strong conjecture is true, we can express $n - 3$ as a sum of two primes p_1, p_2; thus, since $n - 3$ is an even ≥ 2, then $n = (n - 3) + 3$ is the sum of the primes p_1, p_2 and 3, which is the sum of three prime numbers. Thus, proving the ternary Goldbach conjecture, if the strong conjecture is true. That is, for $n > 5$,

$$n = (n - 3) + 3$$

$$n = p_1 + p_2 + 3$$

While the weak Goldbach conjecture was finally proved, by Helfgott [1][2] in 2013, however the strong conjecture has remained unsolved. In this paper we shall use Helfgott's proof of the ternary Goldbach conjecture to prove the strong conjecture of even numbers is indeed false, opposite of what was expected.

Helfgott's proof of the ternary Goldbach conjecture does establish that every even number can be written as the sum of at most 4 primes. For example, subtract any odd prime number, p_4, from every even number, m, that is greater than the prime number being subtracted results with another odd number. That is, $m - p_4$ = an odd number. Now, to Helfgott's credit we can write the odd number, $m - p_4$, as the sum of three primes. This can be written as:

$$m - p_4 = p_1 + p_2 + p_3$$

$$m = p_1 + p_2 + p_3 + p_4$$

Thus, proving every even number can be written as the sum of at most 4 primes. However, to prove that the strong Goldbach conjecture we must reduce this improvement of sum of four primes down to the sum of two primes.

Let $n > 5$ be any odd number, then it is the sum of three primes p_1, p_2, p_3, then the ternary Goldbach conjecture can be written as follows:

$$n = p_1 + p_2 + p_3$$

Also, we assume p_1, p_2, and p_3, are all ≥ 2

Also, by definition every odd number can be written as follows:

$2k + 1$, where, k = 1, 2, 3, 4, … ∞, or k = every positive integer incremented by 1 at a time.

Therefore, the ternary Goldbach conjecture can be written as follows:

$$2k + 1 = p_1 + p_2 + p_3$$

Reducing, $2k = p_1 + p_2 + p_3 - 1$

Note that, by definition, every even number > 2 can be written as follows:

$2k$, where, $k = 2, 3, 4, \ldots \infty$

Therefore, every even number > 2 can be written as follows: $2k = p_1 + p_2 + p_3 - 1$

Since, $2k = p_1 + p_2 + p_3 - 1$

Then, $2k - (p_3 - 1) = p_1 + p_2$

Remember that $2k$ represents every even number > 2. This result is extremely important, as we have proven that $2k - (p_3 - 1) = p_1 + p_2$, which means that "every even number minus $(p_3 - 1)$ is equal to the sum of two prime numbers.

However, if the Goldbach conjecture is true, then every even integer greater than 2 can be expressed as the sum of two primes. Stated mathematically, $2k = p_1 + p_2$, where $k = 2, 3, 4, \ldots \infty$

However, we have just proven that $2k - (p_3 - 1) = p_1 + p_2$

Therefore, if the Goldbach conjecture is true, then:

$2k = p_1 + p_2 = 2k - (p_3 - 1)$

Therefore, $2k = 2k - (p_3 - 1)$

Then reducing, $p_3 - 1 = 0$

Therefore, $p_3 = 1$

However, p_3 is prime, however, $p_3 = 1$, and 1 is not a prime number, therefore we have a contradiction. Therefore, since p_3 must be any prime number and cannot be a constant number, we have proven by contradiction and counter example that indeed Goldbach conjecture is false.

Indeed, the author was surprised this simple contradiction of Goldbach conjecture would be possible. The author discovered this while trying to prove the Goldbach conjecture as true. However, it explains why Goldbach conjecture has not been able to be proven for over 258 years.

Again, the author expresses his eternal gratefulness to Harald Helfgott for his outstanding proof of the of the ternary Goldbach conjecture. Without Helfgott proof, the author's elementary proof that the strong Goldbach conjecture is false would not have been possible, it is totally dependent on Helfgott's proof. In other words, without Helfgott's proof of the ternary Goldbach conjecture, this simple proof would not be possible. Therefore, proof of the ternary Goldbach conjecture has led to the disproof of the strong Goldbach conjecture.

References:

1) Helfgott, H.A. (2013). "Major arcs for Goldbach'stheorem". arXiv:1305.2897 Freely accessible [math.NT].

2) Helfgott, H.A. (2012). "Minor arcs for Goldbach's problem". arXiv:1205.5252 Freely accessible [math.NT].

3) Guy, Richard K., Unsolved problems in number theory, (3rd edition). Springer-Verlag (2004).

4) From Euclid to Present: A Collection of Proofs regarding the Infinitude of Primes, Lindsey Harrison, December 14, 2013.

5) (Dun) W. Dunham,"Journey Through Genius: The Great Theorems of Mathematics," Penguin Books, John Wiley and Sons, 1990.