Thermal ionization in stellar cores based on 3-dimensional Fermi gas model.

Shiladittya Debnath.

Dept. of Electronics and Communication Engineering, West Bengal University of Technology; Kolkata, India. Email – sd.debnath157@gmail.com

Abstract: This paper examines the thermal ionization in the core of a main-sequence stable star, with an assumption that the main-sequence stellar core behaves like an ideal 3-Dimensional (D) Fermi gas. This assumption has been based on the fact that the stellar core persists as a region of very high temperature, typically in a range between $15 \times 10^6\mathrm{K}$ and density near 150 g cm^{-3},like that of our Sun where the classical gas description fails and Fermi-Dirac (F-D) distribution becomes important. Finally we compare our ionization equation with the Saha's thermal ionization equation based on classical Maxwell-Boltzmann (M-B) distribution. The final non-relativistic mathematical calculation provides us with the result that the ionization fraction is exponentially proportional to the Fermi energy and has volume dependency, under the 3-D ideal Fermi gas consideration.

Key Words: Thermal Ionization, 3-D Fermi gas, stellar core.

1. Introduction :

At the super high temperature inside a star's core, the electrons are ionized from their parent atoms and an energetic state of charged particle is formed, known as plasma. The constituents of this plasma state are identical particles with half integral spins. These particles are fermions which obey the Pauli's exclusion principle, according to which no two fermions can have all identical quantum energy states. The statistical consideration of Fermi-Dirac (F-D) distribution in stellar cores was first realized by Prof. R.H Fowler (1926) and letter developed under relativistic transformation by Prof. S. Chandrasekhar (1931; 1932) to apply it in the structure and stability of white-dwarfs. Because of very high density at the core, fermions (in this case electrons) are packed very closely to each other creating a degenerate electron gas which in addition to thermo-nuclear pressure, generates an electron degeneracy pressure (although it is negligible in the core of a main-sequence stable star) which counteracts the inward gravitational collapse to maintain the so called hydrostatic equilibrium of the stellar core. The thermal ionization formulation in turn provides us with an initial understanding of the mechanism of energy transfer (in addition to another important factor of conduction method which will be discussed in future research) from the star's core to the inter-atmospheric envelopes. Here, in the following discussions we have considered the outer envelope of the stellar core as 3-D Fermi gas in a weakly degenerate state where the dominant particle distribution is governed by the F-D distribution and formulated a non-relativistic thermal ionization equation for a main-sequence stellar core.

2. Thermal ionization in main-sequence stellar cores:

The main-sequence stars are characterized by their energy generating mechanism deep inside their cores where primarily four hydrogen nuclei combine to form a helium nucleus through thermo-nuclear fusion. Once a star of mass comparable to that of the Sun forms, after a period of 10 million years, the star's core reaches a state of thermal equilibrium and becomes radiative. This results in the energy transportation by radiation (also through conduction) rather than convection .Thus, based on the above argument, we provide a thermal ionization equation to understand the radiative energy properties in a main-sequence stellar core.

3, Methodology

In this section we are going to discuss some important quantum-statistical nature of isotropic 3-D

Fermi gas and will proceed to derive a thermal ionization equation for a main-sequence stellar core non-relativistically.

3.1 Fundamental ideas regarding a 3-D Fermi gas:

A 3-D isotropic homogenous and non-relativistic Fermi gas is called a Fermi sphere. This can be considered as a 3-D infinite square potential (i.e. a cubical box) well of infinite length L, where the potential can be given as $\widetilde{V}(x,y,z) = \begin{cases} 0, \forall (x,y,z) \in (-\frac{L}{2},\frac{L}{2}) \\ \infty, \text{ for any other value of } x \end{cases}$. (i)

For this model, applying the standard formulation of quantum mechanics, it can be shown that the energy for energy levels n_x, n_y *and* n_z is

$E_n = E_0 + \left(\frac{\hbar^2\pi^2}{2mL^2}\right)\left(n_x^2 + n_y^2 + n_z^2\right)$ (ii), where$\forall\left(n_x, n_y, n_z\right) \in \mathbb{Z}^+$, for the n^{th} energy level.E_0 is the ground-state potential energy, m is the mass of the single fermion and $\hbar$ is the reduced Plank's constant. Because fermions obey Pauli's exclusion principle, for N Fermions with ½ integral spin in the square well potential, two fermi particles cannot have exactly all similar quantum numbers. The 3-D Fermi energy for a Fermi sphere is given by $\mathrm{E_F^{3-D}} \equiv \mathrm{E_F} = \frac{\hbar^2}{2\mathrm{m}}\left(\frac{3\pi^2 N}{\mathrm{V}}\right)^2$, where $V =$ volume when replaced by $L^2 \rightarrow V^{\frac{2}{3}}$. Hence the total energy of a Fermi sphere for N fermions is given by (as given in *En. wikipedia.org.2020. Fermi Gas*)

$E_F^T \equiv \frac{3}{5}E_F N + E_0 N = \left(\frac{3}{5}E_F + E_0\right)N$ (iii).

Within a thermodynamic limit (i.e. the total number of particles *N* are so large that the quantum energy number *n* may be treated as a continuous variable. In this case, the overall number density profile in the box is indeed uniform), the degenerate degree can be calculated as

$g(E) = \frac{1}{V}\frac{\partial N(E)}{\partial E} = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}(E - E_0)^{\frac{1}{2}}$ (iii-a)

where the number of particles as a function of energy N(E) is obtained by substituting equation (iii) by a varying energy $(E - E_0)$ as $N(E) = \frac{V}{3\pi^2}\left[\frac{2m}{\hbar^2}(E - E_0)\right]^{\frac{3}{2}}$. (iii-b)

Thus, we begin our derivation based upon the above stated assumption regarding the 3-D Fermi gas model of the stellar core.

3.2 Derivation of thermal ionization equation for 3-D Fermi gas model :

The Fermi energy for the thermodynamic limit in 3-D Fermi gas is given according to equation (iii) with an assumption $E_F^T \equiv E_F^{(0)}$, where $E_F^{(0)}$ is the ground-state Fermi energy. Let the probability (entropy) function be defined as $S(N_e, N)$for a Fermi gas that has N_e electrons out of *N* Fermi particles in a given ensemble. The grand canonical ensemble as calculated by *Kelly (2002)* for such a case can be written as

$Z_q = \sum_{\{q\}} \exp[-\beta(E_q - \mu_F N_q)]$ (iv),

where $\beta = \frac{1}{k_b T}$ and k_b is the Boltzmann's constant and T is the absolute temperature of the system. Where {*q*} indexes the ensemble of all possible microstate for electrons (*e*), protons (*p*) and hydrogen atoms (*H*).

Thus for e, $Z_e = \sum_{\{e\}} \exp[-\beta(E_e - \mu_F N_e)]$ (v).

Similar expressions could be found out for H and p. The chemical potential μ_F (Fermi level) of the three-dimensional ideal Fermi gas as given by *Kelly (2002)* is related to the zero temperature Fermi energy E_F by a Sommerfeld Expansion

$\mu_F = E_0 + E_F[1 - \frac{\pi^{12}}{12}\left(\frac{k_B T}{E_F}\right)^2 - \frac{\pi^4}{80}\left(\frac{k_B T}{E_F}\right)^4 + \cdots]$ (vi)

Considering a semi-classical limit, the entropy (probability) function for indistinguishable fermions as calculated by *Ghosh et al. (2019)* is

$$S(N_e, N) = \left(\frac{Z_e}{N_e!}\right)\left(\frac{Z_p}{N_p!}\right)\left(\frac{Z_H}{N_H!}\right) \quad \text{(vii).}$$

For the exact momentum distribution calculation of a Fermi gas, we assume that the fermions have an anti-symmetric wave function $\psi_{ij(i\neq j)}$ which are the functions in Fermi space C_k and $C_k^{\beth}$(for Bose field). Thus the resulting wave functions for fermions are as $\psi_{ij}(C_k, C_k^{\beth}) = \psi_i(C_k)\,\psi_j(C_k^{\beth}) - \psi_i(C_k^{\beth})\psi_j(C_k)$.

The exact momentum distribution calculated by *Setlur (2020)* is

$$< C_k\, C_k^{\beth} > = n_F(k) + (2\pi k_F)\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}[\Lambda_{k-\frac{q_1}{2}}(-q_1)/\{(2\omega_R(q_1)\left(\omega_R(q_1) + \omega_{k-\frac{q_1}{2}}(-q_1)\right)^2\left(\frac{m^3}{q_1^4}\right)(\cosh(\lambda(q_1)) - 1)\}] - (2\pi k_F)\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}[\Lambda_{k+\frac{q_1}{2}}(-q_1)/\{(2\omega_R(q_1)\left(\omega_R(q_1) + \omega_{k+\frac{q_1}{2}}(-q_1)\right)^2\left(\frac{m^3}{q_1^4}\right)(\cosh(\lambda(q_1)) - 1)\}] \quad \text{(viii)}$$

where $\lambda(q) = (2\pi q)(\frac{1}{v_q})$, where v_q is the fermi velocity of *q* particle. And $\omega_R = \left(\frac{|q|}{m}\right)\sqrt{\frac{\left(k_F+\frac{q}{2}\right)^2 - \left(k_F-\frac{q}{2}\right)^2\exp\left(-\frac{\lambda}{2}\right)}{1-\exp(-\lambda(q))}}$, $\Lambda_k \equiv \frac{V}{(2\pi\hbar)^d}$ where *V* is the volume of *d* dimensional phase (ϕ)-space and k_F is the radius of the Fermi sphere. Thus from equation (viii) (which can be found in a detailed form in *Setlur's 2019* paper given in the reference), we can say that the Fermi momentum ($p_F = E_F/k_B$) is continuous and the sum of the grand canonical distribution function is the integral

$$Z_i = \int g_F^d(\varepsilon_i)e^{\left[-\frac{\frac{p^2}{2m_e}}{k_BT}\right]}(d^nxd^np)/h^3 \quad \text{(ix),}$$

$$p_F = \sqrt{2mE_F} => \frac{p_F^2}{2m} = E_F$$

where g_F^d is the (statistical weight) degenerate fermi level , ε_i are the individual fermions' energy states. We now proceed to calculate Z_i within a thermodynamical limit in a 6-dimensional (*d*) ϕ-space, so we can assume *n* = 3. Thus d^3x is the volume *V* and d^3p is the momentum, $(dp_xdp_ydp_z) \equiv d^3p = 4\pi p^2dp$. Now, taking $p_F \to p$, equation (ix) can be re-written as

$$Z_i = \frac{4\pi g_F^d}{h^3}\iint p^2\exp[-\frac{\frac{p^2}{2m}}{k_BT}]\,d^3xd^3p\ ,$$ which implies that

$$=> Z_i = \frac{4\pi g_F^d}{h^3}\int d^3x\int_0^{\infty}p^2\exp\left[-\frac{\frac{p^2}{2m}}{k_BT}\right]dp \quad \text{(x).}$$

Let's take $t^2 = \frac{p^2}{2mk_BT}$ $=> 2tdt = \frac{p}{mk_BT}dp.$ $=> 2tdt\,mk_BT = pdp.$ (xi)

and making equation (xi) as substitution, equation (x) can be re-written as

$$Z_i = \frac{4\pi g_F^dV}{h^3}(2mk_BT)^{\frac{1}{2}}\int_0^{\infty}(2mk_BT)t^2e^{-t^2}dt^2\ .$$

$$=> Z_i = \frac{4\pi g_F^dV}{h^3}(2mk_BT)^{\frac{3}{2}}\sqrt{\left(\frac{\pi}{4}\right)} = \frac{g_F^dV}{h^3}(2mk_BT)^{\frac{3}{2}} \quad \text{(xii).}$$

Because electrons and protons are both fermions the degeneracy for electrons and protons is $g_{F(e)}^d \equiv g_{F(p)}^d = 2$. Thus, based on equation (xii) the grand canonical partition function for *e* and *p* can be written as $Z_e = \frac{2V}{h^3}(2m_ek_BT)^{\frac{3}{2}}$ and $Z_p = \frac{2V}{h^3}\left(2m_pk_BT\right)^{\frac{3}{2}}$ (xiii).

Now, the derivation for Z_H is identical as of previous one for *e* or *p* except for fact that hydrogen atoms are not fermions so the inclusion of Fermi ($E_F = E_F^{(0)}$) (binding) energy is necessary in equation (xii) as the derivation is carried out in a Fermi sphere as a 3-D Fermi gas along with the zero-level Bohr energy and taking $-I \equiv \frac{z}{n^2}(-13.6\ ev)$. The degenerate state for *H* which if calculated as $g^d_{F(H)} = \sum_{l=0}^{n-1}(2l+1)$ where l is the angular momentum quantum number. This degeneracy equation results in series expansion in n^2 which for excited hydrogen atom is 2 i.e. *n=2* thus $n^2 = 4$. Hence the degeneracy for excited hydrogen atom is given by $g^d_{F(H)} = 4$, so, the grand canonical partition function for hydrogen atom is

$$Z_H = \frac{4V}{h^3}(2m_H k_B T)^{\frac{3}{2}} e^{\left[-\frac{E_F+(-I)}{k_B T}\right]} \qquad \text{(xiv).}$$

Going back to equation (vii) and taking logarithms of both the sides and using the Stirling's approximation we have

$$\ln S = N_e \ln Z_e + N_p \ln Z_p + N_H \ln Z_H - N_p \ln N_p - N_p \ln N_p - N_p \ln N_p + N_e + N_p + N_H \quad \text{(xv).}$$

The system is electronically neutral under the influence of electrons and ions, so we can write $N_e \equiv N_p\ and\ N_H = N - N_e$. Then differentiating equation (xv), we have

$$\frac{d(lnS)}{dN_e} = 0, \text{which implies } \ln[Z_e Z_p (N - N_e)] - \ln[Z_H N_e^2] = 0 \qquad \text{(xvi).}$$

Which implies $\frac{Z_P Z_e}{Z_H} = \frac{N_e^2}{N-N_e}$ (xvii).

Now, taking Z_e, Z_p and Z_H from equations (xiii) and (xiv) and substituting in equation (xvi), we find

$$\frac{[\frac{2V}{h^3}(2m_e k_B T)^{\frac{3}{2}}]\ [\frac{2V}{h^3}(2m_e k_B T)^{\frac{3}{2}}]}{\frac{4V}{h^3}(2m_e k_B T)^{\frac{3}{2}} e^{\left[-\frac{E_F+(-I)}{k_B T}\right]}} = \frac{N_e^2}{N-N_e} \qquad \text{(xviii).}$$

Now, simplifying the equation (xviii) and cancelling the similar terms and taking $m_p \approx m_H$, we find

$$\frac{V}{h^3}(2m_e k_B T)^{\frac{3}{2}} e^{\left[\frac{E_F+(-I)}{k_B T}\right]} = \frac{N_e^2}{N-N_e},$$

which implies $\frac{N_e^2}{N-N_e} = \frac{V}{h^3}(2m_e k_B T)^{\frac{3}{2}} e^{\left[\frac{E_F+(-I)}{k_B T}\right]}$, (xix)

$$\Rightarrow \frac{N_e^2}{N-N_e} = V\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}} \exp\left(\frac{E_F+(-I)}{k_B T}\right). \qquad \text{(xx).}$$

Where, N is a function of T as $N(T) = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{V,T}$ and $\Omega(V,T,\mu)$ is the grand canonical potential and defined as $\Omega = -g^d_F \Lambda_k k_B T \int \ln\left[1 + e^{\beta(\mu-\varepsilon(p))}\right] d^d p$, where $\beta = \left(\frac{1}{k_B T}\right)$ is Boltzmann's coefficient. As moving towards the upper atmospheric zones of high temperature and relatively low density range, and taking the classical limit of M-B distribution in equation (xv) and ; putting $E_F \rightarrow 0$ and $E \equiv -I$ in H partition function and taking $\chi = \frac{n_e}{n}$, we obtain the stellar atmospheric thermal ionization version of Saha's equation(which in details is given in *Saha's 1920* paper)

$$\frac{\chi^2}{1-2\chi} = \frac{1}{n}\left(\frac{2\pi m_e}{h^2}\right)^{\frac{3}{2}} \exp\left(-\frac{I}{k_B T}\right). \qquad \text{(xxi)}$$

Result and Discussion:

From equation (xx), we have derived the expression for thermal ionization for a main-sequence stellar core considering it as an ideal and weakly degenerate 3-D Fermi gas which is mathematically similar to that of the Saha's equation (xxi). Comparing equations (xx) and (xxi), we find a difference due the fact that in the Fermi version the ionization factor is exponentially proportional to the zero (ground)-level Fermi energy which in contrast for Saha's equation is proportional to the zero-level Bohr energy for the classical Maxwellian gas. We also find that ionization energy for the F-D distribution is more energetic (and vice-versa) as compared to the semi-classical ionization formulation. However, limitations arise when we compare a complex stellar core with a 3-D ideal Fermi gas, neglecting all other phenomenon like such as plasma collisions, magnetohydrodynamic

(MHD) turbulence and relativistic motion of ionized particles. As argued by *Adams et al.(2004)*, pressure ionization due to relativistic electron degeneracy can safely be ignored in the above presented discussion because in the core of a main-sequence star, the hydrogen and helium is fully ionized (thermally) and is weakly degenerate, however they are more prominent in giant-stars of 0.4-1.5 times the Solar mass, where the helium core becomes degenerate before it is hot enough for helium to start fusion and also in collapsed core like that of a white dwarf. This is a theoretical model which proceeded by considering simple assumptions regarding the quantum-statistical behaviour of a stellar core ionization but it provides a first-step towards the modification of actual stellar-core thermal ionization behaviour for a main-sequence star

Conclusion:

In conclusion, we summarize the fact that considering a stellar core as weakly degenerate 3-D fermi (ideal and homogenous) gas, we have devised a theoretical model where we find that the degree of ionization at the star's core depends on the net volume associated with the fermi gas and energy associated with such an ionization process is exponentially proportional to the Fermi energy for the constituent particles. In future, we consider to conduct a research paper considering the relativistic ionization and thermal conduction properties along with MHD in interior of main-sequence stars on the basis of 3-dimensional Fermi gas model.

Acknowledgement:

In acknowledgement, the author is highly grateful to Dr.Brajagopal Majumder, Ph.D. (Retd.)Reader. Professor of Physics, Tripura Gotv.College; for crosschecking all the mathematical equations and for many other helpful discussions.

Data availability statement:

No new data were generated or analysed in support of this research.

As this is completely a recent theoretical initialization therefore, there are no new data associated with this article.

References :

Fowler, R.H., 1926. On Dense Matter. MNRAS, 87(2), pp.114–122.

Chandrasekhar, S.,1932.Some Remarks on the State of Matter in the Interior of Stars *Zeitschrift fuer Astrophysik* **5**, no. 5 ,pp. 321-27

Chandrasekhar, S.,1931. The Maximum Mass of Ideal White Dwarfs ,*The Astrophysical Journal* **74**, no. 1,pp: 81-82.

Adams, C., Laughlin, G. and Graves, G. J. M.,2004.*Red Dwarfs and the End of the Main Sequence*, Revista Mexicana de Astronomía y Astrofísica,[online]. Available at: https://ui.adsabs.harvard. edu/abs/2004RMxAC..22...46A.

Garg, S., Bansal, R. and Ghosh, C., 2019. *Thermal Physics: With Kinetic Theory, Thermodynamics And Statistical Mechanics*. 2nd ed. McGraw Hill, pp.13.53-13.55.

Setlur, G., 2020. *Exact Momentum Distribution Of A Fermi Gas In One Dimension*. [online] arXiv.org. Available at: http://arxiv.org/abs/cond-mat/9705219.

Arulsamy, A., 2020. *Ionization Energy Based Fermi-Dirac Statistics*. [online] arXiv.org. Available at: http://arxiv.org/abs/cond-mat/0410443.

Baierlein, R., 1999. *Thermal Physics*. Cambridge, U.K.: Cambridge University Press.

En. wikipedia.org.2020. *Fermi Gas*.[online] Available at: https://en.wikipedia.org/wiki/Fermi_gas.

Saha, M., 1920. *Ionisation in the Solar Chromosphere*. *Nature*, 105(2634), pp.232-233.

Kelly,James.,1996-2002. *Statistical Mechanics of Ideal Fermi System.*[online] Available at : https://web.archive.org/web/20180412225816/http://www.uam.es:80/personal_pdi/ciencias/jgr/pdfs/fermi.pdf

Eisberg, R. and Resnick, R., 1974. Quantum Physics Of Atoms, Molecules, Solids, Nuclei And Particles. New York: Wiley.