Fractal Foundation of Quantum Spin

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Abstract

We have shown over recent years that the dynamics of quantum fields is likely to slide outside equilibrium above the Fermi scale of electroweak interactions. In proximity to this scale, spacetime dimensionality flows with the probing energy and leads to the concept of *minimal fractal manifold* (MFM). It is known that modeling the physics on fractal manifolds requires use of fractional differential and integral operators. Here we show that deploying such operators on the MFM adds a non-vanishing correction to the standard orbital momentum, which can be identified with *quantum spin*. Our analysis paves the way towards a deeper understanding of the relationship between spin and several phenomenological aspects of quantum field theory (QFT).

Key words: spin, quantum field theory, representations of the Lorentz group, minimal fractal manifold, fractional calculus.

Conjectured to develop near or above the Fermi scale (M_{EW}), MFM represents a spacetime continuum endowed with arbitrarily small yet non-vanishing deviations from four dimensions ($\varepsilon = 4 - D << 1$) [1]. This report is a sequel to [2, 8], where quantum spin was shown to arise from the nontrivial geometry of the MFM.

It is known that the linear, orbital, and angular momentum operators of QFT are respectively given by ($\mu, \nu = 0,1,2,3$) [3]

$$P^{\mu} = i\partial^{\mu} \tag{1}$$

$$L^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) \quad (2)$$

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \quad (3)$$

In what follows we posit that dimensional deviations of the MFM are spacetime dependent ($\varepsilon^{\mu} = \varepsilon(x^{\mu}) = \varepsilon(x)$) and consider only those coordinates that are commensurate in magnitude with the inverse of the Fermi scale, ($|x|^{-1} \leq O(M_{EW})$. It is then reasonable to assume that

$$\varepsilon(x) = 4 - D(x) << x^{-1} \quad (4)$$

in which the x is expressed in dimensionless form ($x \Rightarrow x/x_0$). Since MFM is characterized by low-level fractality ($\varepsilon(x) << 1$), a convenient approximation of fractional derivative is given by [4]

$$\partial^{1-\varepsilon(x)}\psi(x) = \partial\psi(x) + \varepsilon(x)D_1^1(x) + ... \quad (5)$$

$$D_1^1\psi(x) = \partial\psi(0)\ln(x) + \gamma\partial\psi(x) + \int_0^x \partial^2\psi(x)\ln(x-x')dx' \quad (6)$$

where $\psi(x)$ is a well behaved function and γ stands for the Euler constant.

For simplicity and to drive home the main point of the argument, we assume below that the first and third terms of (6) can be safely dropped without incurring a significant loss of generality. By (1), (5) and (6), the fractional linear and orbital momentum operator defined on the MFM assume the form, respectively

$$P^{\mu}_{\varepsilon(x)} = i(\partial^{\mu})^{1-\varepsilon(x)} = i(1 + \varepsilon^{\mu}\gamma)\partial^{\mu} \quad (7)$$

$$L_{\varepsilon(x)}^{\mu\nu} = i[(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) + \gamma(\varepsilon^{\nu}x^{\mu}\partial^{\nu} - \varepsilon^{\mu}x^{\nu}\partial^{\mu})] \tag{8}$$

The theory of fractal and multifractal sets is consistent with the premise that scaling dimensions carry an intrinsic *statistical meaning* [11]. Let $\varepsilon(x)$ represent a fluctuating entity with zero mean, statistically independent from the momentum operator. It follows that the ensemble average of $\varepsilon(x)$ is vanishing, hence $\langle \varepsilon^{\mu}\partial^{\mu} \rangle = \langle \varepsilon^{\mu} \rangle \langle \partial^{\mu} \rangle = 0$. By (7), one obtains

$$\langle P_{\varepsilon}^{\mu} \rangle = i(\langle \partial^{\mu} \rangle + \gamma \langle \varepsilon^{\mu}\partial^{\mu} \rangle) = i\langle \partial^{\mu} \rangle \tag{9}$$

which means that the ensemble average of the fractional linear momentum is indistinguishable from the same average of (1). By contrast, the ensemble average of the fractional orbital momentum amounts to

$$\langle L_{\varepsilon}^{\mu\nu} \rangle = i[\langle (x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) \rangle + \gamma \langle (\varepsilon^{\nu}x^{\mu}\partial^{\nu} - \varepsilon^{\mu}x^{\nu}\partial^{\mu}) \rangle] \tag{10}$$

Unlike (9), the second term of (10) cannot cancel out since $\varepsilon^{\nu} = \varepsilon(x^{\nu})$ and $\varepsilon^{\mu} = \varepsilon(x^{\mu})$ are weakly coupled through the *nonlocal attributes* of fractional differential operators [4-5]. A natural conjecture stemming from this observation is that the second term of (10) signals the presence of a *spin operator,* namely

$$\langle L_{\varepsilon}^{\mu\nu} \rangle = \langle J_{\varepsilon}^{\mu\nu} \rangle = \langle L^{\mu\nu} \rangle + \langle S_{\varepsilon}^{\mu\nu} \rangle \tag{11}$$

where

$$\boxed{\left\langle S_{\varepsilon}^{\mu\nu} \right\rangle = i\gamma \left\langle (\varepsilon^{\nu} x^{\mu} \partial^{\nu} - \varepsilon^{\mu} x^{\nu} \partial^{\mu}) \right\rangle} \tag{12a}$$

The spin operator (12a) may be alternatively presented as

$$\left\langle S_{\varepsilon}^{\mu\nu} \right\rangle = i\gamma \left\langle Q_1(\varepsilon^{\nu}) - Q_2(\varepsilon^{\mu}) \right\rangle \tag{12b}$$

in which

$$Q_1(\varepsilon^{\nu}) = \varepsilon^{\nu} x^{\mu} \partial^{\nu}, \quad Q_2(\varepsilon^{\mu}) = \varepsilon^{\mu} x^{\nu} \partial^{\mu} \tag{13}$$

It is instructive to note that the nonlocal coupling of Q_1 and Q_2 can be characterized through their *correlation function*, defined as

$$C(Q_1, Q_2; n) = \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} Q_1(\varepsilon_{k+n}^{\nu}) Q_2(\varepsilon_k^{\mu}) - \left\langle Q_1 \right\rangle \left\langle Q_2 \right\rangle \tag{14}$$

Here, n denotes the number of iterations of the map $\varepsilon_{n+1}(x) = \beta[\varepsilon_n(x)]$ describing the flow of $\varepsilon(x)$ with the energy scale [6].

Passing from operators to observables, we distinguish two opposite cases. If the correlated observables Q_1 and Q_2 are equal (or substantially similar) in magnitude, it is sensible to assume that (12) describes a *bosonic state* of spin $S = 0$ or $S = 1$. In short.

$$\boxed{Q_1 = O(Q_2) \Rightarrow \begin{cases} S = 0 \\ S = 1 \end{cases}} \tag{15a}$$

Let $Q_0 > 0$ denote an arbitrary reference value associated with measuring Q_1 and Q_2. The spin state $S = 0$ corresponds to $Q_1 = O(Q_2) << Q_0$, whereas $Q_1 = O(Q_2) >> Q_0$ to $S = 1$.

If the opposite of (15) is true and Q_1, Q_2 have dissimilar magnitudes, (12) is likely to describe a *fermionic state* as in

$$\boxed{Q_1 \neq Q_2 \Rightarrow S = \tfrac{1}{2}} \tag{15b}$$

Furthermore, the condition $Q_1 << Q_2$ or $Q_1 >> Q_2$ naturally implies *chirality*, which explains why a Dirac spinor consists of a left-handed (L) and a right-handed (R) Weyl spinor.

Combined use of (4) and (12) hints that the numerical value of the spin observable must fall close to zero. One must keep in mind, however, that our analysis applies to energy scales comparable to or higher than the Fermi scale. It is this regime that justifies the onset of the MFM, where the wavefunction undergoes large local fluctuations characterized by nearly-singular values of the slope $|\partial^\mu \psi(x)| = O(|\partial^\nu \psi(x)|) >> 1$. Since the MFM defines a *non-local dynamic setting* endowed with memory, it is expected that the effects of the MFM carry over to energy scales below M_{EW}, i.e., $M < M_{EW}$.

It seems plausible that (15) may account for several distinctive features of quantum theory that are otherwise taken for granted. In particular, (15) may justify

1) the existence of the spin-statistics theorem in ordinary 3D space,

2) why there are only three low-dimensional representations of the Poincaré group corresponding to spin 0, 1 and ½ particles and,

3) the existence of the exclusion principle for fermions and the existence of bosonic condensation.

Other ramifications of our brief analysis may be further explored. In the spirit of [7], it is interesting to look at the connection between (12), Pauli matrices and quaternions.

Starting from [8], attention may be paid to the relationship between (12) and the violation of discrete symmetries in weak interactions. Likewise, following the insight that gauge fields and their charges emerge from the MFM [1, 9-10], it is worth delving into the fundamental mechanism underlying the magnetic properties of spin.

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