Frame of Reference Moving Along a Line in Quantum Mechanics

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Abstract

We consider a frame of reference moving along a line with respect to an inertial frame of reference. We expect the rate of total energy expectation value not to depend on the order of transformations of a transformation that is a composition of a translation and a transformation that has a frame of reference accelerating with a time dependent acceleration with respect to an inertial frame of reference.

1 Introduction

Consider a frame of reference $\mathcal{F}'$ with coordinates x', y', z', t' and an inertial frame of reference $\mathcal{F}$ with coordinates x, y, z, t. The coordinates of the frames being related by

$$x' = x - f(t) \qquad y' = y \qquad z' = z \qquad t' = t \tag{1}$$

The origin of $\mathcal{F}'$ will then be moving with acceleration $\ddot{f}(t)$ with respect to $\mathcal{F}$. With respect to $\mathcal{F}$ consider a quantum system of a particle with mass m in a potential $V(x, y, z, t)$. For the wave function $\psi(x, y, z, t)$ with respect to $\mathcal{F}$ and corresponding wave function $\psi'(x', y', z', t')$ with respect to $\mathcal{F}'$ we have

$$|\psi'(x', y', z', t')|^2 = |\psi(x, y, z, t)|^2 \tag{2}$$

Consequently there is a real valued function $\beta(x, y, z, t)$ such that

$$\psi'(x', y', z', t') = e^{-\frac{i}{\hbar}\beta(x,y,z,t)}\psi(x, y, z, t) \tag{3}$$

Different ψ may have different β.

2 Schrödinger equations

With respect to $\mathcal{F}$ the wave function ψ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x, y, z, t) + V(x, y, z, t)\psi(x, y, z, t) = i\hbar\frac{\partial\psi}{\partial t}(x, y, z, t) \tag{4}$$

With repect to $\mathcal{F}'$ we have an additional force $m\ddot{f}(t')$ and hence additional potential $m\ddot{f}(t')x' + V_0(t')$. Consequently the wave function ψ' satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla'^2\psi'(x', y', z', t') + \left[V'\Big(x' + f(t'), y', z', t'\Big) + m\ddot{f}(t')x' + V_0(t')\right]\psi'(x', y', z', t') = i\hbar\frac{\partial\psi'}{\partial t'}(x', y', z', t') \tag{5}$$

Now

$$V' = V \qquad \nabla' = \nabla \qquad \frac{\partial}{\partial t'} = \dot{f}\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \tag{6}$$

and on substituting (3) in (5) and using (1), (4), and (6) gives

$$\left[\frac{i\hbar}{2m}\nabla^2\beta + \frac{1}{2m}(\nabla\beta)^2 + m\ddot{f}(x-f) + V_0 - \dot{f}\frac{\partial\beta}{\partial x} - \frac{\partial\beta}{\partial t}\right]\psi + \frac{i\hbar}{m}\left[\nabla\beta\cdot\nabla\psi - m\dot{f}\frac{\partial\psi}{\partial x}\right] = 0 \tag{7}$$

Adding and subtracting (7) and its complex conjugate gives the two equations

$$\begin{aligned} 2\left[\frac{1}{2m}(\nabla\beta)^2 + m\ddot{f}(x-f) + V_0 - \dot{f}\frac{\partial\beta}{\partial x} - \frac{\partial\beta}{\partial t}\right]A^2 \quad &+ \quad \frac{i\hbar}{m}\nabla\beta\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) \\ &- \quad i\hbar\dot{f}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) = 0 \qquad (8) \\ A\nabla^2\beta + 2\nabla A\cdot\nabla\beta \quad &= \quad 2m\dot{f}\frac{\partial A}{\partial x} \qquad (9) \end{aligned}$$

where $A^2 = \psi^*\psi$.

3 Solution to equations

Let $\gamma(x, y, z, t)$ be any real valued function satisfying the homogeneous equation of (9)

$$A\nabla^2\gamma + 2\nabla A\cdot\nabla\gamma = 0 \tag{10}$$

hence

$$\nabla\cdot(A^2\nabla\gamma) = A(A\nabla^2\gamma + 2\nabla A\cdot\nabla\gamma) = 0 \tag{11}$$

With respect to $\mathcal{F}$ choose the potential V and a wave function ψ for this potential such that for any t the set of points where ψ is zero is a sphere S. Let the ball B be S and the set of points interior to S. Assume there is $p_1 \in B$ and $p_1 \notin S$ such that $\nabla\gamma(p_1) \neq 0$. We then have $A^2(p_1)\nabla\gamma(p_1) \neq 0$. There is a curve in B with tangent vector $\nabla\gamma$ and containing p_1. Following this curve from $p_1 \in B$ we will reach a $p_2 \in B$ where either γ is a maximum on B or p_2 is a point of S. In either case $A^2(p_2)\nabla\gamma(p_2) = 0$. By $A^2(p_2)\nabla\gamma(p_2) = 0$ and (11) we have $A^2(p_1)\nabla\gamma(p_1) = 0$. This is a contradiction hence $\nabla\gamma = 0$ for all points of B. Let U_0 be the set of points where $\nabla\gamma = 0$. We have $B \subset U_0$. Assume $U_0 \neq \mathbb{R}^3$. There is then a curve with tangent vector $\nabla\gamma$ that contains a point p_3 on the boundary of U_0 and a point $p_4 \notin U_0 \supset S$ hence $A^2(p_4)\nabla\gamma(p_4) \neq 0$. Now $A^2(p_3)\nabla\gamma(p_3) = 0$ and by (11) we would have $A^2(p_4)\nabla\gamma(p_4) = 0$. This is a contradiction so we must have $U_0 = \mathbb{R}^3$. Consequently $\nabla\gamma = 0$ for all points. We can conclude, for a ψ that is zero on S for all t, that γ depends only on t.

Since γ depends only on t there is then a function $F(t)$ such that the solution to (9) has form

$$\beta(x, y, z, t) = m\dot{f}(t)x + F(t) \tag{12}$$

Substituting this in (8) gives

$$\dot{F} = V_0 - mf\ddot{f} - \frac{1}{2}m\dot{f}^2 \tag{13}$$

4 Rates of total energy expectation values

Total energy expectation values are

$$\bar{E}' = i\hbar \int \psi'^* \frac{\partial \psi'}{\partial t'} d\tau' \qquad \bar{E} = i\hbar \int \psi^* \frac{\partial \psi}{\partial t} d\tau \tag{14}$$

We have from (3), (12), (13), and 14)

$$\frac{d\bar{E}'}{dt'} = \dot{V}_0 - m\dot{f}\dddot{f} + m\dddot{f} \int \psi^* x \psi d\tau + \frac{d}{dt} \int \psi^* x \psi d\tau + i\hbar \ddot{f} \int \psi^* \frac{\partial \psi}{\partial x} d\tau + i\hbar \dot{f} \frac{d}{dt} \int \psi^* \frac{\partial \psi}{\partial x} d\tau + \frac{d\bar{E}}{dt} \tag{15}$$

Now results of measurements do not depend on V_0 hence for $d\bar{E}'/dt'$ we must have $\dot{V}_0 = 0$. Consequently we have for the rate of total energy expectation value

$$\frac{d\bar{E}'}{dt'} = -m\dot{f}\dddot{f} + m\dddot{f} \int \psi^* x \psi d\tau + \frac{d}{dt} \int \psi^* x \psi d\tau + i\hbar \ddot{f} \int \psi^* \frac{\partial \psi}{\partial x} d\tau + i\hbar \dot{f} \frac{d}{dt} \int \psi^* \frac{\partial \psi}{\partial x} d\tau + \frac{d\bar{E}}{dt} \tag{16}$$

Perform the transformation

$$\hat{x} = x - b \qquad \hat{y} = y \qquad \hat{z} = z \qquad \hat{t} = t \tag{17}$$

and then

$$x' = \hat{x} - f(\hat{t}) \qquad y' = \hat{y} \qquad z' = \hat{z} \qquad t' = \hat{t} \tag{18}$$

Also perform the transformation

$$\tilde{x} = x - f(t) \qquad \tilde{y} = y \qquad \tilde{z} = z \qquad \tilde{t} = t \tag{19}$$

and then

$$x' = \tilde{x} - b \qquad y' = \tilde{y} \qquad z' = \tilde{z} \qquad t' = \tilde{t} \tag{20}$$

We expect (16) to be the same for these two composition of transformations but it is not if $\dddot{f} \neq 0$. The values of $d\bar{E}'/dt'$ for the two different orders of composition of transformations differ by $m\dddot{f}b$.

5 Conclusion

The rate of total energy expectation value depends on the order transformations are performed for a transformation that is a composition of a translation and a transformation that has a frame of reference moving along a line with a time dependent acceleration with respect to an inertial frame of reference.

References

[1] Physics Essays, September 2008, June 2013

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