Some relations among Pythagorean triples

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Abstract

Some relations among Pythagorean triples are established. The main tool is a fundamental characterization of the Pythagorean triples through a cathetus that allows to determine the relationships between two Pythagorean triples with an assigned cathetus a and b and the Pythagorean triple with cathetus $a \cdot b$.

1 Introduction

Let x, y and z be positive integers satisfying

$$x^2 + y^2 = z^2.$$

Such a triple (x, y, z) is called a Pythagorean triple and if, in addition, x, y and z are co-prime, it is called a primitive Pythagorean triple. First, let us recall a recent novel formula that allows to obtain all Pythagorean triples as follows.

Theorem 1.1. *([1])* (x, y, z) *is a Pythagorean triple if and only if there exists* $d \in C(x)$ *such that*

$$x = x, \quad y = \frac{x^2}{2d} - \frac{d}{2}, \quad z = \frac{x^2}{2d} + \frac{d}{2}, \tag{1.1}$$

Key words and phrases: Pythagorean triples, Diophantine equations.
AMS (MOS) Subject Classifications: 11D61.

with x positive integer, $x \geq 1$, and where

$$C(x) = \begin{cases} D(x), & \text{if } x \text{ is odd,} \\ \\ D(x) \cap P(x), & \text{if } x \text{ is even,} \end{cases}$$

with

$$D(x) = \left\{ d \in \mathbb{N} \quad \text{such that } d \leq x \text{ and } d \text{ divisor of } x^2 \right\} ,$$

and if x is even with $x = 2^n k$, $n \in \mathbb{N}$ and $k \geq 1$ is a fixed odd number, with

$$P(x) = \left\{ d \in \mathbb{N} \quad \text{such that } d = 2^s l, \text{ with } l \text{ divisor of } x^2 \text{ and } s \in \{1, 2, \ldots, n-1\} \right\} .$$

In [2] we found relations between the primitive Pythagorean triple (x, y, z) generated by any predeterminated positive odd integer x using (1.1) and the primitive Pythagorean triple generated by x^m with $m \in \mathbb{N}$ and $m \geq 2$. In [2] we took care of relations only for the case in which the primitive triple (x, y, z) is generated whith $d \in C(x)$ only with $d = 1$ and the primitive triple (x^m, y', z') is generated with $d_m \in C(x^m)$ only with $d_m = 1$ obtaining formulas that give us y' and z' directly from x, y, z.

Theorem 1.2. *([2]) Let (x, y, z) be the primitive Pythagorean triple generated by any predeterminated positive odd integer $x \geq 1$ using* (1.1) *with $z - y = d = 1$ and let (x^m, y', z') be the primitive Pythagorean triple generated by x^m, $m \in \mathbb{N}$, $m \geq 2$, using* (1.1) *with $z' - y' = d_m = 1$, we have the following formulas*

$$\begin{aligned} y' &= y \left[1 + \sum_{p=1}^{m-1} x^{2p} \right] , \\ z' &= y \left[1 + \sum_{p=1}^{m-1} x^{2p} \right] + 1 , \end{aligned} \tag{1.2}$$

for every $m \in \mathbb{N}$ and $m \geq 2$.
Moreover, we have

$$z \left[(-1)^{m-1} + \sum_{p=1}^{m-1} (-1)^{m-1-p} x^{2p} \right] = \begin{cases} y' & \text{if } m \text{ is even,} \\ z' & \text{if } m \text{ is odd,} \end{cases} \tag{1.3}$$

and

$$z \left[(-1)^{m-1} + \sum_{p=1}^{m-1} (-1)^{m-1-p} x^{2p} \right] + (-1)^{m-2} = \begin{cases} z' & \text{if } m \text{ is even,} \\ y' & \text{if } m \text{ is odd.} \end{cases} \tag{1.4}$$

This was the first step to investigate on other relations between Pythagorean triples.

We want to find relations between the Pythagorean triple (a, a_1, a_2), (b, b_1, b_2), $(a \cdot b, y, z)$ generated by a, b, $a \cdot b$ respectively using (1.1) with $a_2 - a_1 = d_1 \in C(a)$, $b_2 - b_1 = d_2 \in C(b)$, $z - y = d_3 \in C(a \cdot b)$ to obtain formulas that give us y, z and d_3 directly from a_1, a_2, b_1, b_2, d_1, d_2.

2 Results

The following theorem holds.

Theorem 2.1. *Let (a, a_1, a_2), (b, b_1, b_2), $(a \cdot b, y, z)$ be the Pythagorean triples generated by a, b, $a \cdot b$ respectively using* (1.1) *with $a_2 - a_1 = d_1 \in C(a)$, $b_2 - b_1 = d_2 \in C(b)$, $z - y = d_3 \in C(a \cdot b)$. Then*

$$y = a_1 b_2 + a_2 b_1, \qquad z = a_1 b_2 + a_2 b_1 + d_1 d_2, \tag{2.1}$$

and moreover,

$$y = a_1 b_1 + a_2 b_2 - d_1 d_2, \qquad z = a_1 b_1 + a_2 b_2, \tag{2.2}$$

with $d_3 = d_1 \cdot d_2 \in C(a \cdot b)$.

Proof. To prove (2.1) we verify that

$$z^2 - y^2 = (a \cdot b)^2, \tag{2.3}$$

with y and z given in (2.1).
To do this, consider the Pythagorean triples generated by a and b respectively using (1.1)

$$\begin{aligned} &a, \qquad a_1 = \frac{a^2 - d_1^2}{2d_1}, \qquad a_2 = \frac{a^2 + d_1^2}{2d_1}, \qquad d_1 \in C(a), \\ &b, \qquad b_1 = \frac{b^2 - d_2^2}{2d_2}, \qquad b_2 = \frac{b^2 + d_2^2}{2d_2}, \qquad d_2 \in C(b). \end{aligned} \tag{2.4}$$

Writing (2.3) with y and z given from (2.1), we have

$$(a_1 b_2 + a_2 b_1 + d_1 d_2)^2 - (a_1 b_2 + a_2 b_1)^2 = (a \cdot b)^2;$$

that is,

$$d_1^2 d_2^2 + 2a_1 b_2 d_1 d_2 + 2a_2 b_1 d_1 d_2 = a^2 b^2,$$

and using (2.4) we obtain

$$d_1^2 d_2^2 + 2\frac{a^2 - d_1^2}{2d_1}\frac{b^2 + d_2^2}{2d_2} d_1 d_2 + 2\frac{a^2 + d_1^2}{2d_1}\frac{b^2 - d_2^2}{2d_2} d_1 d_2 = a^2 b^2,$$

$$2d_1^2 d_2^2 + (a^2 - d_1^2)(b^2 + d_2^2) + (a^2 + d_1^2)(b^2 - d_2^2) = 2a^2 b^2,$$

from which it is easy to see that

$$2a^2 b^2 = 2a^2 b^2.$$

As a result, (2.3) is an identity with y and z given from (2.1) and that the triple

$$a \cdot b \qquad a_1 b_2 + a_2 b_1 \qquad a_1 b_2 + a_2 b_1 + d_1 d_2$$

is a Pythagorean triple with $d_3 = (d_1 \cdot d_2) \in C(a \cdot b)$.
Therefore, (2.1) holds.
To prove (2.2), using (2.1) and (1.1) we consider

$$\begin{aligned} y &= a_1 b_2 + a_2 b_1 = \frac{(a \cdot b)^2 - (d_1 \cdot d_2)^2}{2d_1 d_2} = \frac{a^2 b^2 - d_1^2 d_2^2}{2d_1 d_2} = \frac{(a_2^2 - a_1^2)(b_2^2 - b_1^2) - d_1^2 d_2^2}{2d_1 d_2} \\ &= \frac{d_1(a_2 + a_1) d_2 (b_2 + b_1) - d_1^2 d_2^2}{2d_1 d_2} = \frac{(a_2 + a_1)(b_2 + b_1) - d_1 d_2}{2}; \end{aligned}$$

that is,

$$2a_1 b_2 + 2a_2 b_1 = a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1 - d_1 d_2.$$

Then

$$y = a_1 b_2 + a_2 b_1 = a_1 b_1 + a_2 b_2 - d_1 d_2.$$

Since $z - y = d_1 d_2$,

$$z = a_1 b_1 + a_2 b_2.$$

So the triple

$$a \cdot b \qquad y = a_1 b_1 + a_2 b_2 - d_1 d_2 \qquad z = a_1 b_1 + a_2 b_2$$

is a Pythagorean triple with $d_3 = (d_1 \cdot d_2) \in C(a \cdot b)$.

Therefore, (2.2) holds as well.
Consequently, formulas (2.1) and (2.2) have thus been proved with $d_3 = (d_1 \cdot d_2) \in C(a \cdot b)$. □

We note that from (2.1) it is possible to find easily the formulas (2.1) in which $d = d_m = 1$ almost in the case $m = 2$ and $m = 3$ while for $m > 3$ it is more difficult for calculus. In fact, if we consider the Pythagorean triple $(a, a_1, a_1 + 1)$ and having by (1.1) $a_1 = \frac{a^2-1}{2}$ from which $2a_1 + 1 = a^2$, then using (2.1) we obtain for $m = 2$

$$y' = a_1b_2 + a_2b_1 = 2a_1(a_1 + 1) = a_1(1 + 1 + 2a_1) = a_1(1 + a^2),$$

$$z' = a_1(1 + a^2) + 1,$$

for $m = 3$

$$\begin{aligned} y' &= a_1b_2 + a_2b_1 = 2a_1^2(1 + a^2) + a_1(1 + a^2) + a_1 = a_1[2a_1(1 + a^2) + (1 + a^2) + 1] \\ &= a_1[1 + (1 + a^2)(1 + 2a_1)] = a_1[1 + (1 + a^2)a^2] = a_1[1 + a^2 + a^4] \end{aligned}$$

$$z' = a_1[1 + a^2 + a^4] + 1,$$

which is formulas (1.2) in the cases $m = 2$ and $m = 3$.

For future work, it may be interesting to study the relationships between Eisenstein Triples after the result found in [3] which gives a characterization of Eisenstein Triples through a side of the triangle.

References

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