Composition of Relativistic Gravitational Potential Energy

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Abstract

A relativistic composition of gravitational redshift can be implemented using the Volterra product integral, agreeing with the redshift predicted by general relativity to first order. Using this composition as a model, new expressions are developed for gravitational potential energy, escape velocity, and a metric. Each of these expressions alleviates a perceived defect in its conventional counterpart. Unlike current theory, relativistic gravitational potential energy would be limited to rest energy and not limitless, escape velocity resulting from the composition would be limited to the speed of light, and the metric would be singularity-free. These ideal properties warrant investigation, at a foundational level, into relativistic compositions based on product integration.

Introduction

The motivation for reconsidering foundational assumptions about gravitation comes from recognizing that several known inconsistencies in general relativity seem to be linked to classical Newtonian gravitational potential energy.

First, there is the problem of infinity. This is exemplified by the theory of inflation which relies on general relativity to provide a limitless supply of energy, just like Newton's gravitation [1]. Gravitation is unique among physical theories in its acceptance of the infinite.

Second, escape velocity in general relativity cannot be considered a valid relativistic result, since it is predicted to exceed the speed of light. The fact that escape velocity is the same as Newtonian may indicate that classical principles have been mistakenly adopted without change at the foundation of the theory.

Third, given the previous two points, a problematic singular metric is more likely a symptom, rather than a cause, of inconsistency in the theory of general relativity.

Furthermore, gravitational redshift predicted by general relativity is different from gravitational redshift resulting from a relativistic composition, whereas one might expect them to be identical. This composition extends naturally from redshift to relativistic gravitational potential energy.

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A simple example of radial escape establishes algebraic dependence among the following:

$$\textit{Gravitational Force} \Leftrightarrow \textit{Potential Energy} \Leftrightarrow \textit{Escape Velocity} \Leftrightarrow \textit{Metric}$$

It will be shown that the relativistically composed version of gravitational potential energy would be capable of overcoming all the inconsistencies mentioned above, doing so in a manner that satisfies Mach's principle.

The required relativistic compositions are carried out using the Volterra product integral. Relativistically composed (or Machian) gravitational potential energy will be used to form a relativistic escape velocity. This will then be used to form Gullstrand-Painlevé coordinates which are completely determined by escape velocity. These metric coordinates can then be transformed into standard Schwarzschild coordinates.

Since the metric in general relativity can be determined by way of conventional relativistic escape velocity and Gullstrand-Painlevé coordinates, it is interesting to see what happens on substituting Machian escape velocity for Newtonian, the difference being a relativistic accounting of gravitational potential energy. The metric that emerges is the same as Brans-Dicke, which is free of singularities.

It is important to keep in mind that a second order test of gravitation has yet to be made.

Gravitational Redshift and the Volterra Product Integral

The total redshift, when it is influenced by more than one factor such as Doppler shift, gravitational shift and Hubble shift, can be found from the product,

$$1 + z_{\text{total}} \;=\; (1 + z_{\text{Doppler}})\,(1 + z_{\text{Gravity}})\,(1 + z_{\text{Hubble}}) \tag{1}$$

When the redshifts are small, the total redshift can be approximated by the sum of the constituent redshifts, but the multiplicative form must be maintained for full accuracy.

The Volterra product integral is a continuous version of the above product. It takes a function, chops it into infinitesimal elements, adds one to each infinitesimal element to form a factor, then multiplies these factors to form a product. For example, the continuous product of $1 + f(x)$ over the interval $[a, b]$, given the function $f(x) = x$, can be written as

$$\prod_a^b \Big(1 + f(x)\,\mathrm{d}x\Big) \;=\; \exp\Big(\int_a^b f(x)\,\mathrm{d}x\Big) \;=\; \exp\Big(\frac{b^2 - a^2}{2}\Big) \tag{2}$$

where $\prod$ stands for product integration. Functionally, the product integral is simply an ordinary integral which is then exponentiated. Conceptually, integration is like calculating simple interest, while product integration is like compound interest.

Consider the gravitational redshift at a radial distance, R, from an ideal solid sphere of mass, M. Given a test particle of mass m, the classical element of potential energy due to an infinitesimal spherical shell of matter is $\mathrm{d}u = -F(r)\,\mathrm{d}r$ where $F(r)$ is the force of gravity between a shell of radius r and the test particle at R. By the the Einstein equivalence principle, the redshift due to the shell is given by $\mathrm{d}z = -\mathrm{d}u/mc^2$, where the rest energy is taken to be constant. The total redshift, $\widetilde{z}$, can be composed relativistically as a product integral,

$$1 + \widetilde{z} \;=\; \prod_r \Big(1 + \mathrm{d}z\Big) \;=\; \exp\Big(\int_r \mathrm{d}z\Big) \;=\; \exp\Big(\frac{GM}{Rc^2}\Big) \qquad (3)$$

where the integral is taken over all the shells, following the classical derivation of potential energy. The composite redshift due to a mass M, at radial distance r, is then given by (4b), not the conventional relativistic (4a).

$$\text{(a) GR} : z \;=\; \Big(1 - \frac{2GM}{rc^2}\Big)^{-1/2} - 1 \qquad \text{(b) Mach} : \widetilde{z} \;=\; \exp\Big(\frac{GM}{rc^2}\Big) - 1 \quad (4)$$

Relativistic Gravitational Potential Energy

The corresponding relativistic gravitational potential energy can be obtained similarly, but must have opposite sign in the exponent to be consistent with the composition of relativistic gravitational redshift.

Following the gravitational redshift, relativistic gravitational potential energy is found to be

$$\widetilde{E}(r) \;=\; mc^2 \prod (1 + \mathrm{d}u/mc^2) \;=\; mc^2 \prod (1 - \mathrm{d}z) \;=\; mc^2 \exp\Big(\frac{-GM}{rc^2}\Big) \quad (5)$$

Simply attempting to form $\widetilde{E}(r)$ as $\prod(1 + \mathrm{d}u)$ would be improper. In (5), partitioning of $\mathrm{d}u = -mc^2\,\mathrm{d}z$ between mc^2 outside the product integral, and the dimensionless $-\mathrm{d}z$ inside, comes from a requirement that the product integral must be dimensionless in order to avoid exponentiated units.

Unlike Newtonian potential energy which is negative, relativistic gravitational potential energy is positive. Relativistic gravitational potential energy is an exponential map of the classical potential energy normalized by rest energy. In the absence of a gravitational field, relativistic gravitational potential energy is equal to rest energy. Gravitational potential energy is taken from that rest energy, and thus has a finite limit.

Classical potential energy is the first order term in a power series expansion:

$$\widetilde{E}(r) \;=\; mc^2 \left[1 - \frac{GM}{rc^2} + \frac{1}{2}\Big(\frac{GM}{rc^2}\Big)^2 - \ldots\right] \qquad (6)$$

Classical potential energy, U, is an approximation, $U \approx \widetilde{U}$.

$$\text{(a) GR} :\; U \;=\; \frac{-GMm}{r} \qquad \text{(b) Mach} :\; \widetilde{U} \;=\; mc^2 \left[\exp\Big(\frac{-GM}{rc^2}\Big) - 1\right] \quad (7)$$

A discussion of normalization by rest energy, and some very elementary examples to compare with classical gravitation can be found in [2].

Mach's principle [3] posits that rest energy of an object can be viewed as gravitational potential energy due to the elevation of that object from distant matter. Hence, the term Machian will be used to describe any quantities relating to gravity resulting from these compositions, but it is particularly appropriate in reference to relativistic gravitational potential energy.

Scale Factor

It is convenient to adopt a more compact notation. Based on gauge theoretic dimensional variability as outlined in Appendix A, the dimensionless scale factor for energy in general relativity, and in the Machian composition, can be given respectively by

$$\text{(a) GR :}\;\; \sigma(r) = \left(1 - \frac{2GM}{rc^2}\right)^{1/2} \qquad \text{(b) Mach :}\;\; \widetilde{\sigma}(r) = \exp\left(\frac{-GM}{rc^2}\right) \tag{8}$$

Second order terms in their power series have opposite sign, opening the possibility for decisive testing. Machian energy, $\widetilde{E} = mc^2\,\widetilde{\sigma}$, has a parallel in general relativity, $E = mc^2\,\sigma$.

Machian Escape Velocity from Potential Energy

Machian escape velocity can be derived from the condition that kinetic energy balances potential energy. Relativistic gravitational potential energy gives a radial escape velocity limited to the speed of light, as might be expected from a relativistic theory, whereas this condition is violated in both classical theory and general relativity. This can be demonstrated using a similar technique for both general relativity and the Machian composition.

In a gravitational field, the rest energy, $\widetilde{E}(\infty) = mc^2$, falls to $mc^2\,\widetilde{\sigma}$. The diminished rest energy is increased kinetically by the Lorentz factor according to the relativistic expression $mc^2\,\widetilde{\sigma}\,\gamma$ with $\gamma = (1 - v^2/c^2)^{-1/2}$. The condition for escape is $mc^2\,\widetilde{\sigma}\,\gamma = \widetilde{E}(\infty)$, or

$$\text{(a) GR :}\;\; \sigma\,\gamma \;=\; 1 \qquad \text{(b) Mach :}\;\; \widetilde{\sigma}\,\gamma \;=\; 1 \tag{9}$$

Solving (9b) for velocity, the Machian scale factor gives the escape velocity (10b) which is limited to c. Solving equation (9a) with the general relativity scale factor, assuming $E = mc^2\,\sigma$ similarly, would give the escape velocity as (10a) which is the same as Newtonian escape speed and can exceed the speed of light.

$$\text{(a) GR :}\; v_{\text{esc}}(r) = (2GM/r)^{1/2} \qquad \text{(b) Mach :}\; \widetilde{v}_{\text{esc}}(r) = c(1 - \widetilde{\sigma}^2)^{1/2} \tag{10}$$

See Appendix B for a composition of Machian escape velocity from gravitational acceleration.

Machian Metric

For Machian gravitational potential and its associated escape velocity, the coordinate system of Gullstrand and Painlevé [4] gives the metric in geometrized units ($c = G = 1$) as

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + (\mathrm{d}r + \widetilde{\beta}\mathrm{d}\tau)^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{11}$$

where τ is proper time in the frame of an object in free fall, initially at rest at infinity. These coordinates are determined by escape velocity, $\widetilde{\beta} = (1 - \widetilde{\sigma}^2)^{1/2}$.

Since $\widetilde{\beta}$ is less than the speed of light for any non-zero radius, there is no absolute event horizon. In static Schwarzschild coordinates, instead of $g_{tt} = -\sigma^2$ that term becomes $g_{tt} = -\widetilde{\sigma}^2$, and the resulting metric equation is then given by

$$\mathrm{d}s^2 = -\widetilde{\sigma}^2\mathrm{d}t^2 + \widetilde{\sigma}^{-2}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{12}$$

where t is proper time in the frame of a motionless object. The only difference from the metric in general relativity is the scale factor. The Machian metric is the same as the Brans-Dicke metric.

Discussion

It is interesting that Brans-Dicke theory [5] was formulated with the Machian idea that G should not be treated as a constant, but rather as a field that varies with the density of surrounding matter. However, inertial and gravitational mass in this theory would differ slightly, by a presently undetectable amount.

Brans-Dicke includes an explicit term to enforce Machian relations, but the relativistic composition of gravitational potential energy is implicitly Machian. Therefore, instead of pursuing Brans-Dicke, it seems reasonable to attempt to change general relativity so that there is a correspondence to Machian potential energy instead of classical gravitational potential energy. This change alone might resolve the inconsistencies listed in the Introduction.

As mentioned above, the product integral functions as an exponential map for gravitational energy. This appears to be an essential relativistic composition which is missing from general relativity.

Appendix A

Dimensional Variability in General Relativity

Bowler [6] has shown that general relativity is a gauge theory in which the fundamental dimensions of length, time and mass vary radially with the dimensionless scale factor respectively as $L = L_o\sigma$, $T = T_o\sigma^{-1}$ and $M = M_o\sigma^{-3}$. As shown in Table 1, which demonstrates the variability of some physical quantities, energy becomes

$$E(r) \;=\; E_o\,\sigma(r) \tag{13}$$

in the presence of a gravitational field at radius, r, compared to the original energy, E_o, sufficiently far from the field where $\sigma(\infty) = 1$. The radial variability corresponds to the escape example under consideration.

Table 1: Dimensional Variability in General Relativity

	Radial	Transverse
Length: L	$L_o\,\sigma$	L_o
Time: T	$T_o\,\sigma^{-1}$	$T_o\,\sigma^{-1}$
Energy: E	$E_o\,\sigma$	$E_o\,\sigma$
Mass: M	$M_o\,\sigma^{-3}$	$M_o\,\sigma^{-1}$
Velocity: v	$v_o\,\sigma^2$	$v_o\,\sigma$
Acceleration: a	$a_o\,\sigma^3$	$a_o\,\sigma^2$
Momentum: p	$p_o\,\sigma^{-1}$	p_o
Force: f	f_o	$f_o\,\sigma$
Newton: G	$G_o\,\sigma^8$	$G_o\,\sigma^3$
Planck: h	h_o	h_o
$-GM/rc^2$: Φ	Φ_o	Φ_o

The table is a straightforward dimensional analysis having no context. For example, energy could refer to potential energy, or the energy of a photon in a gravitational field.

Energy contraction or time dilation could explain Pound-Rebka results. The Shapiro time delay experiment is consistent with the table's radial variability of velocity (of light) in a gravitational field.

Planck's constant and products of complementary pairs like Energy×Time or Momentum×Length have no scale factor, nor does the measure of gravitational field strength, Φ.

The difference between radial and transverse scale factors for mass is a bit puzzling, but not relevant to energy balance in the radial escape example.

Perhaps the most curious thing in the table is the radial scale factor to the *eighth* power for Newton's G.

Appendix B

Machian Escape Velocity as Relativistic Composition

In the above presentation, Machian escape velocity was determined by a balance between potential and kinetic energies. Alternatively, Machian escape velocity can be composed relativistically from the point of view of an observer in a stationary frame viewing events in a gravitationally accelerated frame. There are related assumptions.

Gravitational force will be applied in the rest frame where measurements are made. Also, time dilation and length contraction due to free fall velocity of an object in a gravitational field are assumed not to be involved, in keeping with the river model of gravity and Gullstrand-Painlevé coordinates,

A relativistic velocity can be composed by considering time as intervals of Δt. Let $g\Delta t$ be the change in velocity that would be brought about in one time interval by an acceleration, $g = -GM/r^2$, expected from the force of gravity under classical Galilean assumptions. Taking g as constant over the interval, a relativistic velocity for interval n can be composed approximately as

$$v_n \;=\; \frac{v_{n-1} + g\Delta t}{1 + v_{n-1}\, g\Delta t/c^2} \tag{14}$$

and the distance traveled can be approximated similarly as

$$r_n \;=\; r_{n-1} + v_n \Delta t \tag{15}$$

Given appropriate initial values, the resulting velocity corresponds to Machian escape velocity associated with relativistic potential energy calculated using the product integral. To verify that this composition of velocity produces Machian escape velocity, the recursion (14) can be rearranged to give an acceleration

$$a \;=\; \lim_{\Delta t \to 0} \frac{v_n - v_{n-1}}{\Delta t} \;=\; g\,\gamma^{-2} \tag{16}$$

which is the same as the time derivative of Machian escape velocity (10b).

I.e., given $\widetilde{v}_{\text{esc}} \;=\; c\,(1 - \widetilde{\sigma}^2)^{1/2}$, and recalling that $\gamma^{-1} \;=\; \widetilde{\sigma}$ for escape, the acceleration (16) can be found from

$$a \;=\; \frac{\mathrm{d}\widetilde{v}_{\text{esc}}}{\mathrm{d}t} \;=\; \frac{\mathrm{d}\widetilde{v}_{\text{esc}}}{\mathrm{d}\widetilde{\sigma}}\,\frac{\mathrm{d}\widetilde{\sigma}}{\mathrm{d}r}\,\frac{\mathrm{d}r}{\mathrm{d}t} \tag{17}$$

where these derivatives are given by

$$\frac{\mathrm{d}\widetilde{v}_{\text{esc}}}{\mathrm{d}\widetilde{\sigma}} \;=\; \frac{-c\,\widetilde{\sigma}}{(1 - \widetilde{\sigma}^2)^{1/2}} \qquad \frac{\mathrm{d}\widetilde{\sigma}}{\mathrm{d}r} \;=\; \frac{GM\,\widetilde{\sigma}}{r^2 c^2} \qquad \frac{\mathrm{d}r}{\mathrm{d}t} \;=\; \widetilde{v}_{\text{esc}} \tag{18}$$

References

[1] Steinhardt P., Inflationary cosmology on trial. (2011) http://www.youtube.com/watch?v=IcxptIJS7kQ at 33-35 minutes, in response to a question from the audience about infinite energy for inflation.

[2] Walker C., Calculation of gravitational potential by the method of multiple redshifts. Unpublished (2010) http://sites.google.com/site/revisingnewton (shells2010dec29.pdf)

[3] Bondi H., Cosmology. Cambridge University Press, Cambridge (1952)

[4] Hamilton A. and Lisle J., The river model of black holes, Am. J. Phys. 76, 519 (2008) arXiv:gr-qc/0411060

[5] Brans Carl H., The roots of scalar-tensor theory: an approximate history. (2005) arXiv:gr-qc/0505063

[6] Bowler M.G., Gravitation and Relativity. Pergamon Press, Oxford (1976)

Related Material

1) One consequence of Machian gravitation is that cosmological inflation would be untenable. As an alternative to inflationary cosmology, the redshift of light can be viewed as evidence of a quantum mechanical harmonic oscillator by Planck's hypothesis, in which light energy decays exponentially by losing a quantum of energy, hH, every cycle. It is proposed that the primary obstacle to tired light posed by supernova data can be overcome by complementarity between distant time dilation and received light energy.

Uncertainty and complementarity in the cosmological redshift (2015)
https://fqxi.org/community/forum/topic/2292

2) Machian gravitational relativity, discussed above in Appendix B, implies a rest frame compatible with quantum mechanics in which to compose Machian escape velocity. This universal rest frame could correspond to a plenum of energy populated by fundamental quanta at the inferred zero point of electromagnetic radiation, $hH/2$. The properties of space, if not space itself, could be due to these quanta.

A Tale of Two Relativities (2017)
https://fqxi.org/community/forum/topic/3071