EXTENSIONS OF SOME TRIGONOMETRIC DOUBLE ANGLE AND PRODUCT FORMULAE

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ABSTRACT: In this paper, proofs of extensions of some Trigonometric double angle and Product formulae involving sine and cosine functions are presented.

Keywords: *Trigonometric double angle formulae, Trigonometric Product formulae, binomial expansion.*

1. INTRODUCTION

The main objective of this paper is to extend the following Trigonometric double angle and Trigonometric Product formulae:

(1.1) $$2\text{Sin}x\text{Cos}x = \text{Sin}2x$$

(1.2) $$2\text{Cos}^2x = 1 + \text{Cos}2x$$

(1.3) $$\text{Sin}P - \text{Sin}Q = 2\text{Cos}\left(\frac{P+Q}{2}\right)\text{Sin}\left(\frac{P-Q}{2}\right)$$

(1.4) $$\text{Cos}P + \text{Cos}Q = 2\text{Cos}\left(\frac{P+Q}{2}\right)\text{Cos}\left(\frac{P-Q}{2}\right)$$

2. EXTENSIONS

(1.1) can be extended as follow:

(2.1) $$2^n Cos^n axSin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

(1.2) can be extended as follow:

(2.2) $$2^n Cos^n axCos(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x$$

(1.3) can be extended as follow:

(2.3) $$2^n Cos^n\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}Sin((P+Q)k-nQ)$$

(1.4) can be extended as follow:

(2.4) $$2^n Cos^n\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) = \sum_{k=0}^{n}\binom{n}{k}Cos((P+Q)k-nQ)$$

3. PROOFS

To proof (2.1) and (2.2), note that,

(3.1) $$(r+t)^n = \sum_{k=0}^{n}\binom{n}{k}r^{n-k}t^k$$

If we let $r = e^{\left(\frac{m}{n}\right)ix}$, $t = e^{\left(2a+\frac{m}{n}\right)ix}$, we can see from (3.1) that,

$$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \sum_{k=0}^{n}\binom{n}{k} e^{\left(\frac{m}{n}\right)(n-k)ix}.e^{\left(2a+\frac{m}{n}\right)ikx}$$

$$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \sum_{k=0}^{n}\binom{n}{k} e^{\left(m-\left(\frac{m}{n}\right)k+2ak+\left(\frac{m}{n}\right)k\right)ix}$$

(3.2) $$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \sum_{k=0}^{n}\binom{n}{k} e^{(2ak+m)ix}$$

We can see from (3.2) that,

$$\left(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix}\right)^n = \left(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax}+e^{iax})\right)^n$$

Also. we can see from (3.2) that,

$$\sum_{k=0}^{n}\binom{n}{k} e^{(2ak+m)ix} = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x + iSin(2ak+m)x)$$

So, from (3.2), we see that,

$$\left(e^{\left(a+\frac{m}{n}\right)ix}(e^{-iax}+e^{iax})\right)^n = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x + iSin(2ak+m)x)$$

$$\left(2e^{\left(a+\frac{m}{n}\right)ix}\left(\frac{e^{iax}+e^{-iax}}{2}\right)\right)^{n} = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x \; + \; iSin(2ak+m)x)$$

(3.3) $$2^{n}e^{(an+m)ix}\left(\frac{e^{iax}+e^{-iax}}{2}\right)^{n} = \sum_{k=0}^{n}\binom{n}{k}(Cos(2ak+m)x \; + \; iSin(2ak+m)x)$$

Note that,

$$\left(\frac{e^{iax}+e^{-iax}}{2}\right) = \text{Cos(a)x}$$

Also note that,

$$e^{(an+m)ix} = Cos(an+m)x + iSin(an+m)x$$

So, from (3.3), we can see that,

$$2^{n}(Cos(an+m)x + iSin(an+m)x)\text{Cos}^{n}ax = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x + i\sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

(3.4) $$2^{n}\text{Cos}^{n}axCos(an+m)x + i(2^{n}\text{Cos}^{n}axSin(an+m)x) = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x + i\sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

Equating the real and imaginary parts of (3.4), we see that,

(3.5) $$2^{n}Cos^{n}axSin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x$$

This completes the proof of (2.1).

(3.6) $$2^{n}Cos^{n}axCos(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x$$

This completes the proof of (2.2).

If we set $m \; = \; ny - an \; and \; x \; = 1$ in (3.5)and (3.6), *we see that*,

(3.7) $$2^{n}Cos^{n}axSin(ny) = \sum_{k=0}^{n}\binom{n}{k}Sin((2k-n)a+ny)$$

(3.8) $$2^{n}Cos^{n}axCos(ny) = \sum_{k=0}^{n}\binom{n}{k}Cos((2k-n)a+ny)$$

If we set $a \; = \; \left(\frac{P+Q}{2}\right), \; y \; = \; \left(\frac{P-Q}{2}\right) \; in$ (3.7), *we see that*,

$$\begin{aligned} 2^{n}Cos^{n}\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) &= \sum_{k=0}^{n}\binom{n}{k}Sin((2k-n)\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right)) \\ &= \sum_{k=0}^{n}\binom{n}{k}Sin((2k)\left(\frac{P+Q}{2}\right)-n\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right)) \\ &= \sum_{k=0}^{n}\binom{n}{k}Sin((2k)\left(\frac{P+Q}{2}\right)+n\left(\frac{-p-Q+P-Q}{2}\right)) \\ 2^{n}Cos^{n}\left(\frac{P+Q}{2}\right)Sin\left(\frac{n(P-Q)}{2}\right) &= \sum_{k=0}^{n}\binom{n}{k}Sin((P+Q)k-nQ) \end{aligned}$$

This completes the proof of (2.3).

Also, if we set $a \; = \; \left(\frac{P+Q}{2}\right), \; y \; = \; \left(\frac{P-Q}{2}\right) \; in$ (3.8), *we see that*,

$$\begin{aligned} 2^{n}Cos^{n}\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) &= \sum_{k=0}^{n}\binom{n}{k}Cos((2k-n)\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right)) \\ &= \sum_{k=0}^{n}\binom{n}{k}Cos((2k)\left(\frac{P+Q}{2}\right)-n\left(\frac{P+Q}{2}\right)+n\left(\frac{P-Q}{2}\right)) \\ &= \sum_{k=0}^{n}\binom{n}{k}Cos((2k)\left(\frac{P+Q}{2}\right)+n\left(\frac{-p-Q+P-Q}{2}\right)) \\ 2^{n}Cos^{n}\left(\frac{P+Q}{2}\right)Cos\left(\frac{n(P-Q)}{2}\right) &= \sum_{k=0}^{n}\binom{n}{k}Cos((P+Q)k-nQ) \end{aligned}$$

This completes the proof of (2.4).

4. SOME OTHER NEW IDENTITIES

$$2^n Cosh^n(a)xSinh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} Sinh(2ak+m)x$$

$$2^n Cosh^n(a)xCosh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} Cosh(2ak+m)x$$

$$2^n(-1)^{\frac{n}{2}} Sin^n axSin(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sin(2ak+m)x \qquad (n\ is\ even)$$

$$2^n(-1)^{\frac{n-1}{2}} Sin^n axSin(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cos(2ak+m)x \qquad (n\ is\ odd)$$

$$2^n(-1)^{\frac{n+1}{2}} Sin^n axCos(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sin(2ak+m)x \qquad (n\ is\ odd)$$

$$2^n(-1)^{\frac{n}{2}} Sin^n axCos(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cos(2ak+m)x \qquad (n\ is\ even)$$

$$2^n Sinh^n axSinh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sinh(2ak+m)x \qquad (n\ is\ even)$$

$$-2^n Sinh^n axSinh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cosh(2ak+m)x \qquad (n\ is\ odd)$$

$$-2^n Sinh^n axCosh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Sinh(2ak+m)x \qquad (n\ is\ odd)$$

$$2^n Sinh^n axCosh(an+m)x = \sum_{k=0}^{n} \binom{n}{k} (-1)^k Cosh(2ak+m)x \qquad (n\ is\ even)$$

ACKNOWLEDGEMENTS

The author will like to thank Dr. A.M. Gbolagade for the inspiration his lectures on Algebra and Trigonometry gave me to be able to develop enough knowledge to make these discoveries.

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