Comments on: A new additive decomposition of velocity gradient, by B. Sun [Phys. Fluids 31, 061702 (2019)]

Abhijit Mitra[1]

[1]Department of Aerospace Engineering, Indian Institute of Science, Bangalore

June 16, 2020

Abstract

Comments on "A new additive decomposition of velocity gradient [Phys. Fluids 31, 061702 (2019)]" is presented.

1 Introduction

The Cauchy-Stokes decomposition of the velocity gradient tensor into a symmetric strain rate tensor $\mathbf{D}$ and an anti-symmetric spin tensor $\mathbf{W}$ is well-known (Kundu and Cohen 2002).

$$\nabla \mathbf{u} = \mathbf{D} + \mathbf{W}$$

The spin tensor $\mathbf{W}$ is the tensor representation of the vorticity $2\boldsymbol{\omega}$, where $\mathbf{W}_{ij} = -\varepsilon_{ijk}\omega_k$ (ε_{ijk} is the permutation tensor). In the quest of finding characteristics that define a vortex, the vorticity field has been found lacking due to various reasons (Epps 2017). An interesting, alternative proposition of a novel decomposition of the velocity gradient tensor is presented by Sun (2019) based on the Lie algebra of the special orthonormal Lie group $SO(3)$. This decomposes the velocity gradient tensor into a component which is a rotation tensor instead of the usual spin tensor. As noted in Sun (2019), the deeper significance of this decomposition is not yet clear and further investigations are necessary in that direction. The comments here are intended to interpret and rectify some of the aspects of Sun (2019).

Sun (2019) decomposes the velocity gradient tensor as,

$$\nabla \mathbf{u} = \mathbf{K} + \mathbf{Q} \tag{1}$$

where $\mathbf{Q} \in SO(3)$ is a rotation tensor and $\mathbf{K}$ is the residual. One of questions raised by Sun (2019) is under what condition(s) $\mathbf{K}$ be symmetric? It will be shown here that it is impossible for $\mathbf{K}$ to be symmetric in a flow with vorticity.

Anti-symmetric tensors like $\mathbf{W}$ belong to the Lie algebra ($so(3)$) of the Lie group $SO(3)$. There exists an exponential map from $so(3) \rightarrow SO(3)$. Exploiting this, Sun (2019) expresses a rotation matrix $\mathbf{Q} \in SO(3)$ as,

$$\mathbf{Q} = e^{\mathbf{W}} \tag{2}$$

Sun (2019), however, does not address the issue of dimensional inconsistency in eq.(2). Physical dimension of $\mathbf{W}$ is sec^{-1} - there are obvious problems to exponentiate a dimensional quantity. Granting that $\nabla \mathbf{u}$ and $\mathbf{W}$ are non-dimensional right from the outset, there is a more fundamental problem with eq.(2). $\mathbf{Q}$ is not represented as a one-parameter subgroup in the neighborhood of $\mathbf{I}$ (Hall 2015; Fegan 1991) in eq.(2), where $\mathbf{I}$ is identity element of $SO(3)$. The exponential map in eq.(2) needs rectification for consistency.

2 Discussion and Conclusion

A Lie group, such as the $SO(3)$, has the structure of a differentiable manifold in the vector space of real matrices. On any integral curve induced by the tangent tensor field like $\mathbf{W}$ on $SO(3)$, the following hold (Hall 2015) in the neighborhood of $\mathbf{I}$,

$$\frac{d\sigma(\tau)}{d\tau} = \mathbf{W} \tag{3}$$

where τ is the parameter in the map $\sigma : I \rightarrow SO(3)$, with $\tau \in I = [a,b] \in \mathbb{R}$ and I contains 0 ($a, b \in \mathbb{R}$). τ might be interpreted as the time increment/decrement, $t - t_0$, where $\sigma(0) = \mathbf{I}$. Along the integral curve $\sigma(\tau) \in SO(3)$, eq.(3) demands,

$$\sigma(\tau) = \mathbf{Q}(\tau) = e^{\mathbf{W}\tau} \tag{4}$$

would describe a family of rotations parameterised by τ: a one parameter subgroup of $SO(3)$. Equation 4 can also be derived by a much simpler consideration of the rotation matrix $\mathbf{Q}(\tau)$. Time derivative of $\mathbf{Q}\mathbf{Q}^T$ ($= \mathbf{I}$) is

$$\frac{d(\mathbf{Q}\mathbf{Q}^T)}{d\tau} = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = \dot{\mathbf{I}} = \mathbf{0} \tag{5}$$

where $\dot{\mathbf{Q}}, \dot{\mathbf{I}}$ denotes time derivative of $\mathbf{Q}$ and $\mathbf{I}$ respectively. From eq.(5), it is obvious that $\dot{\mathbf{Q}}\mathbf{Q}^T$ is anti-symmetric. Thus for any $\mathbf{Q}$, there always exists a $\dot{\mathbf{Q}}$ such that,

$$\dot{\mathbf{Q}} = \mathbf{W}\mathbf{Q} \tag{6}$$

This tensorial differential equation is equivalent to eq.(3), and the following satifies eq.(6),

$$\mathbf{Q}(\tau) = e^{\mathbf{W}\tau}\mathbf{Q}_0 \tag{7}$$

Consider the integral curve through the identity with $\mathbf{Q}_0 = \mathbf{I}$ and eq.(7) reduces to eq.(4), reiterating the fact that $\mathbf{Q}$ as the one-parameter sub-group of $SO(3)$ near $\mathbf{I}$. This is mathematically and dimensionally a more consistent exponential map from $so(3) \rightarrow SO(3)$ than eq.(2). If $\mathbf{W}$ is independent of time, there is no restriction on τ in eq.(4), and eq.(2) is recovered for $\tau = 1$. But, in a generic fluid flow field, $\mathbf{W}$ must be a function of time for a material fluid. Therefore, this limits the validity of eq.(4) to $|\tau| \rightarrow 0$.

Based on this exponential map in eq.(7), the Rodrigues' formula used by Sun (2019) is valid only for $\tau = 1$. The complete Rodrigues' formula for $\mathbf{Q}$ is,

$$\mathbf{Q} = \mathbf{I} + \frac{\sin\omega\tau}{\omega}\mathbf{W} + \frac{1-\cos\omega\tau}{\omega^2}\mathbf{W}^2$$

And, if decomposition in eq.(1) for a non-dimensional $\nabla\mathbf{u}$ is demanded with a requirement that $\mathbf{K}$ be symmetric, then the following has to be true,

$$\left(1 - \frac{\sin\omega\tau}{\omega}\right)\mathbf{W} = \mathbf{0} \tag{8}$$

Equation 8 can be satisfied for any allowable τ ($|\tau| \to 0$), if and only if $\mathbf{W} = 0$ identically. Thus, $\mathbf{K}$ can never be symmetric in a vortical flow. This is in distinction to the possibility of $\mathbf{K}$ to be symmetric from Sun's exposition, where symmetric $\mathbf{K}$ is allowable for vortical flows with $\omega \to 0$.

For a turbulent flow, as mentioned earlier, $\mathbf{W}$ would have erratic dependence on time, and the exponentiation map would be valid for infinitesimal time duration, i.e., $|\tau| \to 0$. In that limit, $\mathbf{Q} = \mathbf{I}$ and $\mathbf{K} = \mathbf{D} + \mathbf{W} - \mathbf{I}$, severely restricting the applicability of this new decomposition of the velocity gradient tensor.

References

[1] Brenden Epps. "Review of Vortex Identification Methods". In: *55th AIAA Aerospace Sciences Meeting*. 2017. DOI: 10.2514/6.2017-0989. eprint: https://arc.aiaa.org/doi/pdf/10.2514/6.2017-0989. URL: https://arc.aiaa.org/doi/abs/10.2514/6.2017-0989 (cit. on p. 1).

[2] Howard D Fegan. *Introduction to Compact Lie Groups*. WORLD SCIENTIFIC, 1991. DOI: 10.1142/1436. eprint: https://www.worldscientific.com/doi/pdf/10.1142/1436. URL: https://www.worldscientific.com/doi/abs/10.1142/1436 (cit. on p. 2).

[3] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations*. Second. Graduate Texts in Mathematics. Springer International Publishing, 2015. DOI: 10.1007/978-3-319-13467-3 (cit. on p. 2).

[4] P. K. Kundu and I. M. Cohen. *Fluid Mechanics*. Second. Academic Press, U.S.A., 2002 (cit. on p. 1).

[5] Bohua Sun. "A new additive decomposition of velocity gradient". In: *Physics of Fluids* 31.6 (2019), p. 061702. DOI: 10.1063/1.5100872. eprint: https://doi.org/10.1063/1.5100872. URL: https://doi.org/10.1063/1.5100872 (cit. on pp. 1–3).