Tangent Velocity Of Schwarzschild Geodesics

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Schwarzschild metric describes the geodesic of massless point in a manifold created by a point mass at the origin. The metric forbids any mass on the geodesic. The speed of light along the geodesic is a function of its distance to the origin. The time is also a function of the distance. Consequently, light accelerates through the geodesic unless the geodesic follows a circular orbit around the origin which Schwarzschild characterized with Kepler's law.

I. INTRODUCTION

In 1916, Karl Schwarzschild published a paper[1], "ber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie" in response to Albert Einstein's paper[2], "Erklrung Perihelbewegung des Merkur aus". The response presented an exact solution in comparison with Einstein's approximation solution. Schwarzschild considered the exact solution superior by stating:

"Es ist immer angenehm, ber strenge Lsungen einfacher Form zu verfgen. Wichtiger ist, da die Rechnung zugleich die eindeutige Bestimmtheit der Lsung ergibt, ber die Hrn. Einsteins Behandlung noch Zweifel lie, und die nach der Art, wie sie sich unten einstellt, wohl auch nur schwer durch ein solches Annherungsverfahren erwiesen werden knnte. "

Translated in English as: "It is always pleasant to have strict, simple form solutions. It is more important that the calculation also gives the unambiguous certainty of the solution, about which Mr. Einstein's treatment still left doubts, and which, according to the way in which it appears below, could hardly be proved by such an approximation procedure.".

The solution is a metric for a static mass at the origin with isotropic symmetry. Schwarzschild described how a massless point should propagate through an empty space. With the exact solution, Schwarzschild confirmed Einstein's approximation solution that light propagates along a specific path under the influence of a nearby mass.

The path coincides with the geodesic for a manifold equipped with such metric. Schwarzschild illustrated the geodesic with Kepler's law for the circular orbit. Unknown to Schwarzschild, new type of geodesic exists. In this geodesic, light travels in the radial direction. However, the speed along the geodesic is not the subject of Schwarzschild's original intention.

The speed is a crucial information for light. The speed in each type of geodesics is presented in the same exact style as Schwarzschild had preferred.

Readers with difficulty reading Schwarzschild's paper will find an excellent English translation[3] handy.

II. PROOF

A. Schwarzschild Metric

Schwarzschild derived the metric based on conditions from Einstein[2]:

1. All the components are independent of the time x_4.
2. The equations $g_{\rho 4} = g_{4\rho} = 0$ hold exactly for $\rho =$ 1,2,3.
3. The solution is spatially symmetric with respect to the origin of the coordinate system in the sense that one finds again the same solution when x_1,x_2,x_3 are subjected to an orthogonal transformation (rotation).
4. The $g_{\nu\mu}$ vanish at infinity, with the exception of the following four limit values different from zero:

$$g_{44} = 1 \tag{1}$$

$$g_{11} = g_{22} = g_{33} = -1 \tag{2}$$

He emphasized the metric is exclusively for massless object by stating[1]: "Dies ist nun nach der Einsteinschen Theorie dann die Bewegung eines masselosen Punktes in dem Gravitationsfeld einer im Punkt $x_1 = x_2 = x_3 = 0$ befindlichen Masse, wenn die Komponenten des Gravitationsfeldes Γ berall, mit Ausnahme des Punktes $x_1 = x_2 = x_3 = 0$"

Translated in English: "this is, after Einstein's theory, the motion of a massless point in the gravitational field of a mass at the point $x_1 = x_2 = x_3 = 0$, if the 'components of the gravitational field' Γ fulfill everywhere, with exception of the point $x_1 = x_2 = x_3 = 0$."

The metric in equation 14 of the Schwarzschild's paper[1] is reproduced as

$$ds^2 = (1 - \frac{\alpha}{R})dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{3}$$

$$R = (r^3 + \alpha^3)^{1/3} \tag{4}$$

The metric requires all mass at the origin. Any presence of mass on the geodesic will violate the conditions from Einstein and requires a new metric.

B. Equatorial Geodesics

Schwarzschild applied the rotational symmetry to the metric by setting

$$\theta = \frac{\pi}{2} \tag{5}$$

and remarked: "If one also restricts himself to the motion in the equatorial plane ($\theta = 90^o$,$d\theta = 0$)".

He applied the variation principle to the line element to obtain three equations of motion equivalent to geodesic equations.

Equation 15 of Schwarzschild's paper[1] is reproduced as

$$(1-\frac{\alpha}{R})(\frac{dt}{ds})^2-\frac{1}{1-\frac{\alpha}{R}}(\frac{dR}{ds})^2-R^2(\frac{d\phi}{ds})^2 = const. = h \tag{6}$$

Equation 16 of Schwarzschild's paper[1] is reproduced as

$$R^2\frac{d\phi}{ds} = const. = c \tag{7}$$

Equation 17 of Schwarzschild's paper[1] is reproduced as

$$(1-\frac{\alpha}{R})\frac{dt}{ds} = const. = 1 \tag{8}$$

Equations (6,7,8) are valid only if $ds \neq 0$. An assumption made by Schwarzschild. From equations (6,7,8),

$$(1-\frac{\alpha}{R})^{-1} - (1-\frac{\alpha}{R})(\frac{dR}{dt})^2 - \frac{c^2}{R^2} = h \tag{9}$$

$$\frac{dR}{dt} = \sqrt{(1-\frac{\alpha}{R})^{-1}((1-\frac{\alpha}{R})^{-1} - h - \frac{c^2}{R^2})} \tag{10}$$

The speed of light in the geodesic is

$$V = \sqrt{V_r^2 + V_\phi^2} \tag{11}$$

From equations (4,10), the radial speed is

$$V_r = \frac{dr}{dt} = \frac{\partial r}{\partial R}\frac{dR}{dt} = \frac{R^2}{r^2}\frac{dR}{dt} \tag{12}$$

From equations (7,8), the angular speed is

$$V_\phi = r\frac{d\phi}{dt} = (1-\frac{\alpha}{R})\frac{rc}{R^2} \tag{13}$$

From equations (10,11,12,13),

$$\frac{dV}{dt} = \frac{\partial V}{\partial R}\frac{dR}{dt} \tag{14}$$

The speed of light is a function of the radius. Light accelerates unless $dR = 0$.

C. Circular Geodesics

The geodesic can be circular if

$$dR = 0 \tag{15}$$

Define x as

$$x = \frac{1}{R} \tag{16}$$

Equation 18 of Schwarzschild's paper[1] is reproduced as

$$(\frac{dx}{d\phi})^2 = \frac{1-h}{c^2} + \frac{h\alpha}{c^2}x - x^2 + \alpha x^3 \tag{17}$$

Differentiate equation (17) with respect to $d\phi$ to get

$$2\frac{d^2x}{d\phi^2} = \frac{h\alpha}{c^2} - 2x + 3\alpha x^2 \tag{18}$$

Schwarzschild stated[1]: "For circular orbits both $\frac{dx}{d\phi}$ and $\frac{d^2x}{d\phi^2}$ must vanish."

$$\frac{dx}{d\phi} = 0 = \frac{d^2x}{d\phi^2} \tag{19}$$

From equations (17,18,19),

$$x = \frac{h\alpha}{c^2} + \sqrt{(\frac{h\alpha}{c^2})^2 + 3\frac{1-h}{c^2}} \tag{20}$$

From equations (4,16,20), the possible radius is

$$r = (x^{-3} - \alpha^3)^{1/3} \tag{21}$$

From equations (13,16,21), the speed of light is

$$V_\phi = (x^{-3} - \alpha^3)^{1/3}(1 - x\alpha)cx^2 \tag{22}$$

The speed of light depends on the value of α which is a function of the mass at the origin.

D. Radial Geodesics

The geodesics can be restricted to the radial direction if

$$d\phi = 0 = d\theta \tag{23}$$

From equations (6,8,23),

$$\frac{1}{1-\frac{\alpha}{R}} - \frac{1}{1-\frac{\alpha}{R}}(\frac{dR}{ds})^2 = h \tag{24}$$

$$\frac{dR}{ds} = \sqrt{1 - h(1-\frac{\alpha}{R})} \tag{25}$$

From equations (8,25),

$$\frac{dR}{dt} = (1-\frac{\alpha}{R})\sqrt{1-h(1-\frac{\alpha}{R})} \tag{26}$$

The speed of light is a function of the radius and the mass at the origin

E. Null Geodesic

The null geodesic is defined with

$$ds = 0 \tag{27}$$

Adjustment to equation (6) is required by replacing ds with affine parameter $d\lambda$.

From equations (6,27),

$$(1-\frac{\alpha}{R})(\frac{dt}{d\lambda})^2 - \frac{1}{1-\frac{\alpha}{R}}(\frac{dR}{d\lambda})^2 - R^2(\frac{d\phi}{d\lambda})^2 = 0 \tag{28}$$

The radial geodesic from equation (28) with $d\phi = 0$ is represented by

$$(1-\frac{\alpha}{R})(\frac{dt}{d\lambda})^2 - \frac{1}{1-\frac{\alpha}{R}}(\frac{dR}{d\lambda})^2 = 0 \tag{29}$$

$$\frac{dR}{dt} = 1 - \frac{\alpha}{R} \tag{30}$$

The radial speed for a null geodesics restricted in the radial direction is a function of the radius.

The circular orbit from equation (28) with $dR = 0$ is described by

$$(1-\frac{\alpha}{R})(\frac{dt}{d\lambda})^2 - R^2(\frac{d\phi}{d\lambda})^2 = 0 \tag{31}$$

$$\frac{d\phi}{dt} = \frac{1}{R}\sqrt{1-\frac{\alpha}{R}} \tag{32}$$

The speed for a null geodesics restricted in a circular orbit is

$$V_\phi = r\frac{d\phi}{dt} = \frac{r}{R}\sqrt{1-\frac{\alpha}{R}} \tag{33}$$

III. CONCLUSION

The speed of light in Schwarzschild geodesic depends on the distance to the origin where the mass resides. Light accelerates and decelerates except in a circular geodesic.

Schwarzschild metric represents the manifold of a point mass at the origin. It describes massless object around the point mass. The metric is exclusively for light. It requires all mass to be at the origin. No mass is allowed on the geodesic. A new metric is required if there is any mass on the geodesic. Any attempt to apply the metric to massive object on the geodesic will result in approximation which is exctly what Schwarzschild intended to avoid.

Schwarzschild repeated against approximation[1]: "Die Eindeutigkeit der Lsung hat sich durch die vorstehende Rechnung von selbst ergeben. Da es schwer wre, aus einem Annherungsverfahren nach Hrn. Einsteins Art die Eindeutigkeit zu erkennen, sieht man an folgendem"

Translated as: "The uniqueness of the solution resulted spontaneously through the present calculation. From what follows we can see that it would have been difficult to ascertain the uniqueness from an approximation procedure in the manner of Mr. Einstein."[3]

The speed is determined with coordinate time instead of proper time. The proper time is a coordinate time in the rest frame of a moving object along geodesic. It should not be used in the rest frame of the obserber. The physics of one reference frame should not be mixed with the physics of another reference frame. However, the elapsed time is conserved in all reference frames[4,5,6]. The elapsed proper time is indeed identical to the elapsed coordinate time.

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