Revisiting and Extending Kepler's Laws

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Kepler summarised Brahe and others record of planetary trajectories into 3 simple laws. Newton further reduced them into 1 little equation, $\vec{F} = m\frac{d^2\vec{r}}{dt^2} = -\frac{GM_s m}{|\vec{r}|^3}\vec{r}$, where $\vec{F}$ = Force on a planet due to Sun, m = Mass of the planet, $\vec{r}$ = Position vector of the planet, G = Universal Gravitational Constant and M_s = Mass of Sun. Newton henceforth setup the idea that the entity $m\frac{d^2\vec{r}}{dt^2}$ defined as Force($\vec{F}$) is somehow more fundamental than all other quantities and a physical interaction essentially involves a law governing the evolution of $m\frac{d^2\vec{r}}{dt^2}$ term.
Newton successfully applied the equation inspired by Kepler's celestial laws to explain terrestrial phenomena such as the parabolic trajectory of cannonballs, the period of a pendulum, ... Yet he was not successfull in extending his equation to explain the precessions in Moon's orbit. Further, it was found that Newton's equation produce unstable solutions when extended to 3 or more interacting objects. And his method leaves us with ill-posed equations in 3D and N-body cases because Kepler's datasets were limited to flat 2D space with 2-body type interactions where one of the interacting mass is extremely heavy. Newton built his theories based upon Kepler's observational analysis. We show here that Newton's law of Gravity is not universal. And his method of framing problems of motion in terms of force balance equations only captures one small subset of all possible processes in which Energy is conserved.
For example the uniform gravitational acceleration(i.e. $\frac{d^2\vec{r}}{dt^2}$ = Constant $= -g\hat{j}$, here $\hat{j}$ is the unit vector along vertical axis pointing upwards) condition observed near Earth's surface is only true when the object is not interacting with any other object. If the object is interacting with an inclined plane, where inclination is represented by angle θ_0 then $\frac{d^2\vec{r}}{dt^2} = -gSin(\theta_0)[Cos(\theta_0)\hat{i} + Sin(\theta_0)\hat{j}]$. If the object is interacting with a Brachistochrone then $\frac{d^2\vec{r}}{dt^2} = -g[Sin(\omega t)\hat{i} + Cos(\omega t)\hat{j}]$. In contrast the form of graviational potential energy (near Earth surface) remains fixed, $P = -mgy$ or $P = P_0 - mgy$. Perhaps this hints at an Universal Law of Gravitational Potential(not Force).
Infact starting from Kepler's laws we can not only derive Newton's Force balance equation but also the Energy(E) conservation equation. We can derive that, $E = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}\right) + P$, where P = Potential Energy $= -\frac{GMm}{r}$ and E = Constant. Note that, $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$, $\vec{F}_g = -\frac{GMm}{r^3}\vec{r}$ and $\frac{dP}{dt} = -\vec{F}_g\bullet\frac{d\vec{r}}{dt}$. Thus, we get $\frac{dE}{dt} = \left[m\frac{d^2\vec{r}}{dt^2} - \vec{F}_g\right]\bullet\frac{d\vec{r}}{dt}$ this is true even when E $\neq$ Constant. Since E = Constant, $\frac{dE}{dt} = \left[\vec{F} - \vec{F}_g\right]\bullet\frac{d\vec{r}}{dt} = 0$. This in general means, $\vec{F} - \vec{F}_g \perp \frac{d\vec{r}}{dt}$. Not always $m\frac{d^2\vec{r}}{dt^2} - \vec{F}_g = 0$ as Newton's Universal law of Gravity is stated. Hence Newton's equation encompass only a small subset of all the phenomena covered by the equation $\frac{dE}{dt} = 0$. The equation $\vec{F} = \vec{F}_g$ in that form is not even applicable for all 2-body problems in 2D. In general, $\vec{F}$ = Some component of $\vec{F}_g$. Since $\vec{F}_g = -\nabla P$ we also get that, in general, $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$ = Some component of $-\nabla P$. Thus assuming $\vec{F} = -\nabla P$ is not valid in general. Determining which component of $\vec{F}_g$ is causing the body to accelerate is non-trivial. The free-body diagrams are of limited use and the principle of least action - Lagrangian calculations employ energy terms but in a much more complicated manner. We can achieve better results directly using the Energy conservation equation.
Further we extend the analysis to include Lagrange type 3-body periodic orbit solutions with equilateral configuration and show that Lagrangian/Newtonian method gives some sporadic, apparently unstable solutions, where as the Energy method provides the entire set of stable elliptical orbit solutions including non-equilateral configurations. With Energy method we can also derive a condition which determines whether the 3-bodies end up in an orbit with 1 center of revolution(like in Lagrange type periodic orbits) or end up with 2 centers of revolution(like in the Sun-Earth-Moon system).
In case of restricted 3-body problem such as the Sun-Earth-Moon system or in case of $(J_1, J_2, ..)$perturbations of Artificial satellites due to Earths non-spherical shape, using Newton's method gives wrong results unless we tamper his equation with non-Newtonian terms as has been done historically. In the method used here, we show the solutions using just 3 variables, i.e. r = Distance, θ = Longitude, ϕ = Latitude of the Spherical-Polar coordinate system instead of the standard 6 Elements of Orbit approach. This reduces mathematical complexity and helps in clearly identifying the dynamical terms behind apsidal and nodal precession.
We also note that the term Inertia coined by Galileo to explain the height conserving property of balls rolling down inclined planes has to be properly interpreted as energy. That is, Inertia = Energy. And we point at the need to replace Newton's Laws of Motion(and Gravity) by the Energy conservation principle. And principle of angular

momentum conservation or angular velocity conservation and such.

I. KEPLER'S LAWS

The main dataset analysed by Johannes Kepler for over 30 years was collected by Tycho Brahe meticulously for over 40 years using Quadrants and Sextants before Telescopes were in vogue. Brahe was the first to properly record the Earth-Mars distances on a regular basis using the Parallax method which greatly contributed in Kepler working out his 3 laws (Dillon, 2016; Dreyer, 1906; Lankford and Rothenberg, 1997; Thoren, 1973).

Kepler's I Law *The Planetary Orbits are Elliptical*

$$r = \frac{r_0}{1+\varepsilon Cos(\theta)}, Or, Cos(\theta) = \frac{1}{\varepsilon}\left(\frac{r_0}{r} - 1\right) \tag{1}$$

Eqn(1) represents an elliptic curve. Here, r = Radial distance from the Center of Revolution, θ = Angular displacement in the plane of revolution, ε = Eccentricity and r_0 = Semi-latus rectum. Also note, Semi-major axis $= X_m = \frac{r_0}{1-\varepsilon^2}$ and Semi-minor axis $= Y_m = \frac{r_0}{\sqrt{1-\varepsilon^2}}$ (Bate *et al.*, 1971; Curtis, 2013; Goldstein *et al.*, 2001; Sinha, 2013; Vallado *et al.*, 1997). In Eqn(1) at $\varepsilon = 0$ the equation becomes $r = r_0 \implies$ a Circle.

Kepler's II Law *The Planets Sweep Equal Area in Equal time intervals Or Area swept by a Planet in 1 Unit Time is a Constant* Assume that at a particular radius r and angle θ we consider a short piece of trajectory covered in a time interval δt. Time in this case is measured on the basis of the rotation/spin of Earth on its own axis, assuming that rotation rate is constant. In the short time interval δt, r is nearly constant and the angle swept is $\delta\theta$. Then Area covered in time interval $\delta t = r^2.\delta\theta$, consider a quantity defined as, a = Area covered per Unit Time, then according to the Kepler's II Law, a = Constant. (Bate *et al.*, 1971; Curtis, 2013; Goldstein *et al.*, 2001; Sinha, 2013; Vallado *et al.*, 1997)

$$\begin{aligned} a &= r^2\frac{\delta\theta}{\delta t} = Constant \\ \delta t \to 0, a &= r^2\frac{d\theta}{dt} \end{aligned} \tag{2}$$

In Eqn(2) a = angular momentum per unit mass.

Kepler's III Law *The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.* Which is equivalent to saying that for the Planets in the Solar system, the constant r_0 used in First law(Eqn(1)) and the constant a used in Second law(Eqn(2)) are connected by another constant, which was argued to be related to the mass of Sun by Newton inspired by Galileo's experiments. (Bate *et al.*, 1971; Curtis, 2013; Goldstein *et al.*, 2001; Sinha, 2013; Vallado *et al.*, 1997).

Suppose T = Period, then

$$\begin{aligned} &\frac{4\pi^2 X_m^3}{T^2} = \frac{a^2}{r_0} = b = Constant, Kepler \\ &\frac{a^2}{r_0} = b = G.M = Constant, Newton \end{aligned} \tag{3}$$

G = Gravitational Constant, M = Mass of Sun.

Kepler's Laws already specify the trajectory of planets. Newton surmised that we ought to be able to derive the trajectories from some deeper elementary laws which determine the instantaneous local behaviour of moving objects. Based on Galileo's and his own experiments (Dreyer, 1906; Goodstein, 1985; Lankford and Rothenberg, 1997) Newton had axiomatized that an object with constant velocity moving in a straight line does not need any cause for its motion. It is only the changes from this uniform straight line motion that needs any explanation. Thus, Newton considered that a formula specifying the evolution of acceleration terms might be an elementary law. Instead of acceleration terms, Newton could have as well assumed that laws governing the Energy terms are the deeper principles. Because from Kepler's I and II Laws, Eqns(1,2) we can derive both the Force Balance and Energy Conservation Equations. While Eqn(3) determines the value of constant G*M.

There are atleast 2 possible ways in which we can arrive at the shape and nature of planetary orbits by using elementary laws. Method $\mathcal{A}$ is the Newton's method of framing Force balance equation And Method $\mathcal{B}$ is the method of framing Energy conservation equation using the Potential energy term. We can derive the Energy equation directly from Kepler's laws(without resorting to Work-Energy Theorem as done in Newtonian paradigm) as shown in Eqn(14). Below we describe both the methods in detail.

II. FORCE AND ENERGY EQUATIONS

A. Method $\mathcal{A}$: Force Balance

Consider $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$ of Eqn(1)

$$\begin{aligned} &\frac{dr}{dt} = \frac{r_0\varepsilon Sin(\theta)}{(1+\varepsilon Cos(\theta))^2}\frac{d\theta}{dt} = \frac{r^2}{r_0}\frac{d\theta}{dt}\varepsilon Sin(\theta) \\ &Using, Eqn(2), Then, Eqn(1) \\ &\frac{dr}{dt} = \frac{a}{r_0}\varepsilon Sin(\theta) = \frac{a}{r_0}\sqrt{\varepsilon^2 - \left(\frac{r_0}{r} - 1\right)^2} \end{aligned} \tag{4}$$

Again using Eqn(1) and Eqn(2)

$$\begin{aligned} &\frac{d^2r}{dt^2} = \frac{a}{r_0}\varepsilon Cos(\theta)\frac{d\theta}{dt} = \frac{a^2}{r^3} - \frac{a^2}{r_0 r^2} \\ &Using, Eqn(3), \frac{d^2r}{dt^2} = \frac{a^2}{r^3} - \frac{b}{r^2} \end{aligned} \tag{5}$$

Using Newton's form of Kepler's III Law, from Eqn(3) we can rewrite Eqn(5) as,

$$\frac{d^2r}{dt^2} = \frac{a^2}{r^3} - \frac{GM}{r^2}, also$$
$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \quad (6)$$

Consider the first time derivative$\left(\frac{d}{dt}\right)$ of Eqn(2)

$$\frac{da}{dt} = r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} = 0$$
$$So, r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0 \quad (7)$$

Eqn(6) and Eqn(7) can be elegantly combined together into one vector equation.

Let, $\vec{r}$ = r[Cos(θ)$\hat{i}$ + Sin(θ)$\hat{j}$] = 2D Position Vector
$\hat{r}$ = Cos(θ)$\hat{i}$ + Sin(θ)$\hat{j}$ = Unit Vector,
$\hat{q}$ = − Sin(θ)$\hat{i}$ + Cos(θ)$\hat{j}$ = Unit Vector, Then

$$\frac{d^2\vec{r}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{q}$$
$$Therefore, \frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^2}\hat{r} \quad (8)$$

We can also rewrite Eqn(8) as,

$$Force = \vec{F} = m\frac{d^2\vec{r}}{dt^2} = -\frac{GMm}{r^2}\hat{r} \quad (9)$$

We see that we can reduce the 3 laws of Kepler into one little vector equation(Eqn(9)) valid for all planets in the solar system. Newton considered Eqn(9) as the deeper principle governing all objects in the universe and called it the **Universal Law of Gravity**. We show here that Newton's formulation is not the Universal law.

Eqn(9) is the Newton's method of framing the Equation of Motion. The beauty of this method is its brevity. One equation, Eqn(9), encompasses all the 3 Kepler's Laws. But the method has very limited applicability. It is applicable only for problems involving 2 masses with angular momentum conservation and Energy conservation. Eqn(9) is certainly not universal for it can not be readily applied in that form to solve problems involving inclined planes (Goldstein *et al.*, 2001) and brachistochrones (Radhakrishnamurty, 2019) where angular momentum is not conserved.

N-body case: In general if there are N-masses interacting and m_i, m_j are the i^{th} and j^{th} mass located at positions $\vec{r}_i$, $\vec{r}_j$ respectively then, Newtonian Gravitational Force on m_i due to m_j can be written as $\vec{F}_{ij}$,

$$\vec{F}_{ij} = -\frac{Gm_im_j}{|\vec{r}_i - \vec{r}_j|^3}(\vec{r}_i - \vec{r}_j) = -\vec{F}_{ji}$$
$$\vec{F}_i = \sum_{j=1,j\neq i}^{j=N}\vec{F}_{ij} = -\sum_{j=1,j\neq i}^{j=N}\vec{F}_{ji}$$

$\vec{F}_i$ is the sum of all the Newtonian (gravitational) force balance terms on mass m_i. Therefore using Newton's Law,

$$m_i\frac{d^2\vec{r}_i}{dt^2} - \vec{F}_i = m_i\frac{d^2\vec{r}_i}{dt^2} + \sum_{j=1,j\neq i}^{j=n}\frac{Gm_im_j}{|\vec{r}_i - \vec{r}_j|^3}(\vec{r}_i - \vec{r}_j) = 0 \quad (10)$$

This(Eqn(10)) is how we frame equations of motion in Method $\mathcal{A}$ for an N-body system. (Bate *et al.*, 1971; Goldstein *et al.*, 2001; Sinha, 2013; Vallado *et al.*, 1997)

B. Method $\mathcal{B}$: Energy Conservation

Continuing with the $\frac{dr}{dt}$ term from Eqn(4),

$$\left(\frac{dr}{dt}\right)^2 = \frac{a^2}{r_0^2}\left[\varepsilon^2 - \left(\frac{r_0}{r} - 1\right)^2\right]$$
$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{a^2}{r^2} = \frac{b}{r} - \frac{a^2(1-\varepsilon^2)}{2r_0^2}$$
$$Apply, Eqn(2), on, LHS$$
$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}r^2\left(\frac{d\theta}{dt}\right)^2 = \frac{b}{r} - \frac{b(1-\varepsilon^2)}{2r_0} \quad (11)$$

LHS is the total kinetic energy per unit mass of the planet. Eqn(11) indicates a conserved quantity,

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}r^2\left(\frac{d\theta}{dt}\right)^2 - \frac{b}{r} = -\frac{b(1-\varepsilon^2)}{2r_0} = Constant \quad (12)$$

So along with the total kinetic energy, the planet is associated with an energy term $-\frac{b}{r}$. This can be interpreted as the potential energy of the Planet. Adding it to the Kinetic Energy term gives a constant of motion.

Following Newton we replace b by GM in Eqn(12).

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}r^2\left(\frac{d\theta}{dt}\right)^2 - \frac{GM}{r} = -\frac{GM(1-\varepsilon^2)}{2r_0} \quad (13)$$

Eqn(13)looks alright mathematically. But if we interpret that as actually representing the Energy, then physically we have problem because it has net negative energy(in RHS) when $\varepsilon < 1$. And has zero net energy at $\varepsilon = 1$ even when the total kinetic energy is non-zero. So, in order to interpret the energy correctly we need to make a minute correction.

Using the intuition obtained from projectiles and escape velocities on Earth surface we can cook up and add a constant quantity $\frac{GM}{R}$ to both LHS and RHS. Let E/m = Net Energy per unit mass of m. Then,

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2 + \frac{GMm}{R} - \frac{GMm}{r}$$
$$E = \frac{GMm}{R} - \frac{GMm(1-\varepsilon^2)}{2r_0} \geq 0 \quad (14)$$

Where R is a representative radius of Sun. Then similar to the case of rocket escape velocity on Earth we get an escape velocity value from Sun to be $\sqrt{\frac{2GM}{R}}$. And a binding energy per unit mass of $\frac{GM}{R}$. Let us assume the Gravitational Potential Energy(P) of mass m under the influence of another mass M is given by Eqn(15)

$$P = \frac{GMm}{R} - \frac{GMm}{r} \tag{15}$$

Thus, $P_{r=R} = 0 =$ Zero Potential Energy at the Surface of M. Assume $r \geq R$. At a great distance $(r \to \infty)$ there is a finite non-zero amount of potential energy, $P_{r=\infty} = \frac{GMm}{R}$. This can be interpreted as the binding energy, i.e. this is the amount of energy that will be lost when the smaller object falls from a great distance and onto the surface of the larger mass. Consequently this is the minimum amount of energy an object must possess in order to escape the gravitational influence of M. In some problems the binding energy term is important but for problems solved in this article, it is not important.

Eqn(15) demonstrates Method $\mathcal{B}$. i.e. Energy Method for framing the equation of motion. Eqn(15) shows the potential energy term that we should add to the kinetic energy term in order to frame the energy equation.

In general if there are N-masses interacting instead of 2 and m_i, m_j are the i^{th} and j^{th} mass located at positions $\vec{r}_i$, $\vec{r}_j$ respectively then,

N-body case: If E is the net energy of the N-body system and the potential energy shared between m_i, m_j can be written as P_{ij} then,

$$P_{ij} = -\frac{Gm_im_j}{|\vec{r}_i - \vec{r}_j|} = P_{ji}, \frac{dP_{ij}}{dt} = -\vec{F}_{ij} \bullet \frac{d(\vec{r}_i - \vec{r}_j)}{dt}$$
$$E = \sum_{i=1}^{i=n} \frac{1}{2} m_i \left(\frac{d\vec{r}_i}{dt} \bullet \frac{d\vec{r}_i}{dt} \right) + \sum_{i=1}^{i=n} \frac{1}{2} \sum_{j=1, j \neq i}^{j=n} P_{ij}$$
$$E = \sum_{i=1}^{i=n} \frac{1}{2} m_i \left(\frac{d\vec{r}_i}{dt} \bullet \frac{d\vec{r}_i}{dt} \right) + \sum_{i=1}^{i=n} \sum_{j=i+1}^{j=n} P_{ij} \tag{16}$$

This(Eqn(16)) is how we frame equations in Method $\mathcal{B}$ for an N-body system.

C. Relationship between Force and Energy

Let K represent the Kinetic Energy, P represent the Potential Energy and E represent the Net Energy in the Kepler-1-Body approximation scenario.

From Eqn(15) we have $P = \frac{GMm}{R} - \frac{GMm}{r}$ and from Eqn(9) we have $\vec{F} = -\frac{GMm}{r^3}\vec{r}$.
Note that, $\vec{r} \bullet \frac{d\vec{r}}{dt} = r\frac{dr}{dt}$

$$\frac{dP}{dt} = \frac{GMm}{r^3} r \frac{dr}{dt} = \frac{GMm}{r^3} \vec{r} \bullet \frac{d\vec{r}}{dt} = -\vec{F} \bullet \frac{d\vec{r}}{dt} \tag{17}$$

Since in Kepler-1-Body form $E = \frac{1}{2}m\left(\frac{d\vec{r}}{dt} \bullet \frac{d\vec{r}}{dt}\right) + P$

$$\frac{dE}{dt} = m\frac{d\vec{r}}{dt} \bullet \frac{d^2\vec{r}}{dt^2} + \frac{dP}{dt}$$
$$\frac{dE}{dt} = \left[m\frac{d^2\vec{r}}{dt^2} - \vec{F} \right] \bullet \frac{d\vec{r}}{dt} \tag{18}$$

If E is conserved,

$$\frac{dE}{dt} = \left[m\frac{d^2\vec{r}}{dt^2} - \vec{F} \right] \bullet \frac{d\vec{r}}{dt} = 0 \tag{19}$$

Eqn(19) implies 2 scenarios
1) Newtonian: $m\frac{d^2\vec{r}}{dt^2} - \vec{F} = 0 \; \forall$ i = 1,2,3,...,n
2) Perpendicular: $m\frac{d^2\vec{r}}{dt^2} - \vec{F} \perp \frac{d\vec{r}}{dt} \; \forall$ i = 1,2,3,...,n

Thus, Newton's method only deals with a subset of all possible energy conserving processes even in 1-body approximation scenario. That is Energy conservation phenomena is the superset of phenomena described by Newton's force balance equation.

Suppose there are n masses labelled $m_1, m_2, m_3, ..., m_n$ located at $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$. Let E represent the Net energy of the n-nody system.

$$E = \sum_{i=1}^{i=n} \frac{1}{2} m_i \left(\frac{d\vec{r}_i}{dt} \bullet \frac{d\vec{r}_i}{dt} \right) + \sum_{i=1}^{i=n} \frac{1}{2} \sum_{j=1, j \neq i}^{j=n} \frac{Gm_im_j}{|\vec{r}_i - \vec{r}_j|} \tag{20}$$

For example when n = 3

$$\begin{aligned} E = &\frac{1}{2}m_1\left(\frac{d\vec{r}_1}{dt} \bullet \frac{d\vec{r}_1}{dt}\right) + \frac{1}{2}\frac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{1}{2}\frac{Gm_1m_3}{|\vec{r}_1 - \vec{r}_3|} \\ &\frac{1}{2}m_2\left(\frac{d\vec{r}_2}{dt} \bullet \frac{d\vec{r}_2}{dt}\right) + \frac{1}{2}\frac{Gm_2m_1}{|\vec{r}_2 - \vec{r}_1|} + \frac{1}{2}\frac{Gm_2m_3}{|\vec{r}_2 - \vec{r}_3|} \\ &\frac{1}{2}m_3\left(\frac{d\vec{r}_3}{dt} \bullet \frac{d\vec{r}_3}{dt}\right) + \frac{1}{2}\frac{Gm_3m_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{1}{2}\frac{Gm_3m_2}{|\vec{r}_3 - \vec{r}_2|} \end{aligned}$$

$$\begin{aligned} \frac{dE}{dt} = &\sum_{i=1}^{i=n} m_i \frac{d^2\vec{r}_i}{dt^2} \bullet \frac{d\vec{r}_i}{dt} \\ &+ \sum_{i=1}^{i=n} \frac{1}{2} \sum_{j=1, j \neq i}^{j=n} \frac{Gm_im_j}{|\vec{r}_i - \vec{r}_j|^3} (\vec{r}_i - \vec{r}_j) \bullet \frac{d(\vec{r}_i - \vec{r}_j)}{dt} \end{aligned}$$

Just consider the potential energy portion,

$$\begin{aligned} &\sum_{i=1}^{i=n} \frac{1}{2} \sum_{j=1, j \neq i}^{j=n} \frac{Gm_im_j}{|\vec{r}_i - \vec{r}_j|^3} (\vec{r}_i - \vec{r}_j) \bullet \frac{d(\vec{r}_i - \vec{r}_j)}{dt} \\ &= \sum_{i=1}^{i=n} \frac{1}{2} \sum_{j=1, j \neq i}^{j=n} \frac{Gm_im_j}{|\vec{r}_i - \vec{r}_j|^3} (\vec{r}_i - \vec{r}_j) \bullet \frac{d\vec{r}_i}{dt} \\ &+ \sum_{i=1}^{i=n} \frac{1}{2} \sum_{j=1, j \neq i}^{j=n} \frac{Gm_jm_i}{|\vec{r}_j - \vec{r}_i|^3} (\vec{r}_j - \vec{r}_i) \bullet \frac{d\vec{r}_j}{dt} \end{aligned} \tag{21}$$

There is repetition of pairs in the summation above. That is suppose (i=e,j=f) then on RHS we get 2 entries (i)$(\vec{r}_e - \vec{r}_f)\bullet\frac{d\vec{r}_e}{dt}$ and (ii)$(\vec{r}_f - \vec{r}_e)\bullet\frac{d\vec{r}_f}{dt}$ and when (i=f,j=e) we get 2 entires (iii)$(\vec{r}_f - \vec{r}_e)\bullet\frac{d\vec{r}_f}{dt}$ and (iv)$(\vec{r}_e - \vec{r}_f)\bullet\frac{d\vec{r}_e}{dt}$. This implies,

$$\sum_{i=1}^{i=n}\frac{1}{2}\sum_{j=1,j\neq i}^{j=n}\frac{Gm_im_j}{|\vec{r}_i-\vec{r}_j|^3}(\vec{r}_i-\vec{r}_j)\bullet\frac{d\vec{r}_i}{dt}$$
$$=\sum_{i=1}^{i=n}\frac{1}{2}\sum_{j=1,j\neq i}^{j=n}\frac{Gm_jm_i}{|\vec{r}_j-\vec{r}_i|^3}(\vec{r}_j-\vec{r}_i)\bullet\frac{d\vec{r}_j}{dt}$$

Therefore we can rewrite Eqn(21) as,

$$\sum_{i=1}^{i=n}\frac{1}{2}\sum_{j=1,j\neq i}^{j=n}\frac{Gm_im_j}{|\vec{r}_i-\vec{r}_j|^3}(\vec{r}_i-\vec{r}_j)\bullet\frac{d(\vec{r}_i-\vec{r}_j)}{dt}$$
$$=\sum_{i=1}^{i=n}\sum_{j=1,j\neq i}^{j=n}\frac{Gm_im_j}{|\vec{r}_i-\vec{r}_j|^3}(\vec{r}_i-\vec{r}_j)\bullet\frac{d\vec{r}_i}{dt}$$

$\vec{F}_i$ is the of sum of all the Newtonian gravitational force balance terms on m_i

$$\vec{F}_i = \sum_{j=1,j\neq i}^{j=n}\vec{F}_{ij} = -\sum_{j=1,j\neq i}^{j=n}\frac{Gm_im_j}{|\vec{r}_i-\vec{r}_j|^3}(\vec{r}_i-\vec{r}_j)$$

If E is the net Energy of the system

$$\frac{dE}{dt} = \sum_{i=1}^{i=n}\left[m_i\frac{d^2\vec{r}_i}{dt^2}-\vec{F}_i\right]\bullet\frac{d\vec{r}_i}{dt}$$

If E is conserved then,

$$\frac{dE}{dt} = \sum_{i=1}^{i=n}\left[m_i\frac{d^2\vec{r}_i}{dt^2}-\vec{F}_i\right]\bullet\frac{d\vec{r}_i}{dt} = 0 \qquad (22)$$

Eqn(22) implies several scenarios
1) Newtonian: $m_i\frac{d^2\vec{r}_i}{dt^2}-\vec{F}_i = 0\ \forall$ i = 1,2,3,...,n
2) Perpendicular: $m_i\frac{d^2\vec{r}_i}{dt^2}-\vec{F}_i \perp \frac{d\vec{r}_i}{dt}\ \forall$ i = 1,2,3,...,n
3) Mixed: Mixture of 1 and 2
4) Irregular: $\left[m_i\frac{d^2\vec{r}_i}{dt^2}-\vec{F}_i\right]\bullet\frac{d\vec{r}_i}{dt}\neq 0\ \forall$ i.

We can easily verify that in problems involving inclined plane near Earth surface and in the Brachistochrone problem $m\frac{d^2\vec{r}}{dt^2}-\vec{F}\neq 0$. Thus, Method($\mathcal{A}$) is not Universal.

D. Partial Spatial Derivatives of Energy

It is usually assumed that Force($\vec{F}$) is equal to -ve Potential energy gradient($-\nabla P$), that is $\vec{F} = -\nabla P$. This is in-general not true. By definition Force = $\vec{F} = m\frac{d^2\vec{r}}{dt^2}$ = mass × acceleration. Potential Energy = P = $-\frac{GMm}{r}$. Let, Force due to Gravity be = $\vec{F}_g = -\frac{GMm}{r^2}\hat{r}$. Thus $\vec{F}_g = -\nabla P$. So only whenever $\vec{F} = \vec{F}_g$ we can write $\vec{F} = -\nabla P$. But as noted in earlier sections, Newton's method is a subset of all Energy conserving phenomena, what is in general true is $\vec{F}$ = Some component of $-\nabla P$. Even though this equation($\vec{F} = -\nabla P$) can be written down easily by utilising the quirks of partial derivatives definition, $\nabla = \frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$, taking gradients($\nabla$) of Kinetic energy($K_i's$) of the individual particles and the gradient of the Net energy(E) of the system may not make much sense and may lead to inconsistent results.

With full(not partial) derivatives,

$$K = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]$$
$$\frac{dK}{dx} = m\left(\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}\frac{dy}{dx}+\frac{d^2z}{dt^2}\frac{dz}{dx}\right)$$
$$So, m\frac{d^2\vec{r}}{dt^2}\neq\frac{dK}{dx}\hat{i}+\frac{dK}{dy}\hat{j}+\frac{dK}{dz}\hat{k}$$

With partial derivatives,

$$K = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]$$
$$\frac{\partial K}{\partial x} = m\left(\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}\frac{\partial y}{\partial x}+\frac{d^2z}{dt^2}\frac{\partial z}{\partial x}\right)$$
$$Assume, \frac{\partial y}{\partial x} = \frac{\partial z}{\partial x} = 0, So, \frac{\partial K}{\partial x} = m\frac{d^2x}{dt^2}$$

Similarly assume $\frac{\partial x}{\partial y} = \frac{\partial z}{\partial y} = 0$ and $\frac{\partial x}{\partial z} = \frac{\partial y}{\partial z} = 0$
Thus we get,

$$\frac{\partial K}{\partial x} = m\frac{d^2x}{dt^2}, \frac{\partial K}{\partial y} = m\frac{d^2y}{dt^2}, \frac{\partial K}{\partial z} = m\frac{d^2z}{dt^2}$$
$$And, m\frac{d^2\vec{r}}{dt^2} = \nabla K = \frac{\partial K}{\partial x}\hat{i}+\frac{\partial K}{\partial y}\hat{j}+\frac{\partial K}{\partial z}\hat{k}$$

Note in the above equation we simultaneously assume all these mutually contradictory criteria $\frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0$, $\frac{\partial y}{\partial z} = \frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial x} = \frac{\partial x}{\partial z} = 0$ in one equation.

The contradiction becomes apparent when we try to evaluate ∇K from the data or formulaically,

Consider for simplicity an uniform circular motion (we know the final solution trajectory), $\vec{r} = r[Cos(\theta)\hat{i} + Sin(\theta)\hat{j}]$ with r = Constant. If we want to map it to the rectangular co-ordinate system we get, $x = rCos(\theta)$ and $y = rSin(\theta)$. Therefore kinetic energy = K = $\frac{1}{2}mr^2\omega^2 = \frac{1}{2}m(x^2+y^2)\omega^2$, where $\omega = \frac{d\theta}{dt}$ = Constant and circular orbit implies $x^2+y^2 = r^2$ = Constant.

Now let us consider the full derivatives of K.

$$K = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}m(x^2+y^2)\omega^2 = Constant$$
$$\frac{dK}{dx} = 0 = m\left(x + y\frac{dy}{dx}\right)\omega^2, and, \frac{dy}{dx} = -\frac{x}{y}$$
$$\frac{dK}{dy} = 0 = m\left(x\frac{dx}{dy} + y\right)\omega^2, and, \frac{dx}{dy} = -\frac{y}{x}$$

This makes sense since the object is confined to a circle and its kinetic energy is constant at every point on that circle hence when we take deriatives $\frac{dK}{dx}$ and $\frac{dK}{dy}$ we are still confined to that circle so we evaluate those derivatives to be zero.

Now let us consider the partial derivatives of K.

In one form where we evaluate it from the data and do not formulaically resolve the two variables x,y we get the same result as we got in the case of full derivative above.

$$K = \frac{1}{2}mr^2\omega^2 = Constant$$
$$\frac{\partial K}{\partial x} = 0, \frac{\partial K}{\partial y} = 0$$
$$Thus, m\frac{d^2\vec{r}}{dt^2} \neq \nabla K = \frac{\partial K}{\partial x}\hat{i} + \frac{\partial K}{\partial y}\hat{j}$$

In another form where we formulaically resolve the two variables x,y we get a different result. In this form we also drop the 'Constant' from the equation because we are not confined to the circular path anymore, we are assuming that K is defined everywhere in space by the given formula.

$$K = \frac{1}{2}m(x^2+y^2)\omega^2$$
$$\frac{\partial K}{\partial x} = mx\omega^2, as, \frac{\partial y}{\partial x} = 0$$
$$\frac{\partial K}{\partial y} = my\omega^2, as, \frac{\partial x}{\partial y} = 0$$
$$m\frac{d^2\vec{r}}{dt^2} = \nabla K = \frac{\partial K}{\partial x}\hat{i} + \frac{\partial K}{\partial y}\hat{j} = -\nabla P$$
$$If, E = Constant, \nabla E = 0$$
$$\nabla E = \nabla K + \nabla P = 0, 1 - body - approx$$

Note, partial derivatives of particle trajectories is something physically impossible to measure. It is a manoeuvre that can be done only mathematically. When we do the partial derivative along some x axis, we assume that the planet is moving only along x axis and not along any other direction. But if the basic trajectory is elliptic/circular or some such curve then we can not make this assumption of motion being confined along only 1 axis. Partial derivatives make sense on field variables extending in space, for example we can consider Temperature variation only along some x axis but it can not be done with an individual particle trajectory. Hence in these problems ∇ is not defined.

The analysis done above is applicable for 1-body approximation. We can already see that the relationship($\vec{F} = -\nabla P$) is broken by considering the solution to full 2-body problem in Eqn(33). We can write the net energy E as the sum of energies of object 1(E_1) and object 2(E_2), $E = E_1 + E2$. We can write the separate energy equations of the two objects as,

$$E_1 = \frac{E}{1+\frac{m_1}{m_2}} = \frac{1}{2}m_1\left(\frac{d\vec{r}_1}{dt}\bullet\frac{d\vec{r}_1}{dt}\right) - \frac{Gm_1m_2}{(r_1+r_2)\left(1+\frac{m_1}{m_2}\right)}$$
$$E_2 = \frac{E}{1+\frac{m_2}{m_1}} = \frac{1}{2}m_2\left(\frac{d\vec{r}_2}{dt}\bullet\frac{d\vec{r}_2}{dt}\right) - \frac{Gm_1m_2}{(r_1+r_2)\left(1+\frac{m_2}{m_1}\right)}$$
$$E = \frac{1}{2}m_1\left(\frac{d\vec{r}_1}{dt}\bullet\frac{d\vec{r}_1}{dt}\right) + \frac{1}{2}m_2\left(\frac{d\vec{r}_2}{dt}\bullet\frac{d\vec{r}_2}{dt}\right) - \frac{Gm_1m_2}{(r_1+r_2)}$$

If taking the gradients is valid, assume Potential $= P = -\frac{Gm_1m_2}{r_1+r_2}$, evaluate $\nabla_1 E_1$ and $\nabla_2 E_2$,

$$\nabla_1 E_1 = m_1\frac{d^2\vec{r}_1}{dt^2} + \frac{\nabla_1 P}{1+\frac{m_1}{m_2}} = 0$$
$$\nabla_2 E_2 = m_2\frac{d^2\vec{r}_2}{dt^2} + \frac{\nabla_2 P}{1+\frac{m_2}{m_1}} = 0$$

Here consider,
$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$,
$\nabla_1 = \frac{\partial}{\partial x_1}\hat{i} + \frac{\partial}{\partial y_1}\hat{j} + \frac{\partial}{\partial z_1}\hat{k}$ and $\nabla_2 = \frac{\partial}{\partial x_2}\hat{i} + \frac{\partial}{\partial y_2}\hat{j} + \frac{\partial}{\partial z_2}\hat{k}$.

Thus, Force on mass m_1 is $\vec{F}_1 = m_1\frac{d^2\vec{r}_1}{dt^2} = -\frac{\nabla_1 P}{1+\frac{m_1}{m_2}}$ and Force on mass m_2 is $\vec{F}_2 = m_2\frac{d^2\vec{r}_2}{dt^2} = -\frac{\nabla_2 P}{1+\frac{m_2}{m_1}}$. The relationship $\vec{F} = -\nabla P$ derived from 1-body approximation is not valid for multi-body configurations.

III. 2-BODY SYSTEM IN 2D

A. 1-Body Approximation

Let us modify Eqn(2) to define a new constant A dependent on the mass of the object as well. i.e. $A = mr^2\frac{d\theta}{dt}$ = Constant. This quantity(A) is called the Angular momentum. Applying it on the rotational energy $\left(\frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2\right)$ term in Eqn(14) we get $\frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2}\frac{A^2}{mr^2}$ therefore,

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A^2}{mr^2} + \frac{GMm}{R} - \frac{GMm}{r}$$

Rearranging,

$$\frac{dr}{dt} = \pm\frac{1}{r}\sqrt{2\left(\frac{E}{m} - \frac{GM}{R}\right)r^2 - \frac{A^2}{m^2} + 2GMr}$$
$$\frac{dr}{dt} = \pm\frac{mr}{A}\frac{d\theta}{dt}\sqrt{2\left(\frac{E}{m} - \frac{GM}{R}\right)r^2 - \frac{A^2}{m^2} + 2GMr} \tag{23}$$

At this point we should recognize that Equation(23) gives elliptic solutions when $E < \frac{GMm}{R}$.
If $r = \frac{r_0}{1 \pm \varepsilon Cos(\theta)}$ then,

$$\frac{dr}{dt} = \pm\frac{r}{r_0}\frac{d\theta}{dt}\sqrt{(\varepsilon^2-1)r^2 - r_0^2 + 2rr_0}$$
$$Equate, with, Eqn(23)$$
$$\sqrt{(\varepsilon^2-1)\frac{r^2}{r_0^2} - 1 + 2\frac{r}{r_0}}$$
$$= \sqrt{2\left(\frac{Em}{A^2} - \frac{GMm^2}{RA^2}\right)r^2 - 1 + \frac{2GMm^2}{A^2}r} \tag{24}$$

So, $r_0 = \frac{A^2}{GMm^2} = \frac{a^2}{b}$ and $E = \frac{GMm}{R} - (1-\varepsilon^2)\frac{GMm}{2r_0}$

At $\varepsilon = 0$ we get circular orbit solutions since $r = r_0$ = Constant. At $\varepsilon = 1$ we get escape velocity trajectories because $r \to \infty$ for certain values of θ. Therefore bounded orbits exist as long as $0 \leq \varepsilon < 1$ or when $\frac{GMm}{R} - \frac{GMm}{2r_0} \leq E < \frac{GMm}{R}$ we can also write the Energy limits as $\frac{GMm}{R} - \frac{G^2M^2m^3}{2A^2} \leq E < \frac{GMm}{R}$ which implies that for a given angular momentum A $E_{min} = \frac{GMm}{R} - \frac{G^2M^2m^3}{2A^2}$ is the least possible energy in the Kepler-1-Body approximation. If the mass has to lose further energy (below E_{min}), it has to lose angular momentum also. However there is no such restriction on acquiring more energy at the same angular momentum. At the lowest energy state (for a given Angular momentum) i.e. $E = E_{min}$ we get circular orbit solution. Energy added above E_{min} at constant A goes into increasing the eccentricity of the elliptical orbit. Conversely circular orbits are also states of highest angular momentum for a given Energy. If we have to increase A above this we also have to add energy.

When $\varepsilon \geq 1$ or when $E \geq \frac{GMm}{R}$ we get a different solution other than $r = \frac{r_0}{1 \pm \varepsilon Cos(\theta)}$ from the integral equation in Eqn(23). However, since we are interested in the bounded orbits, we shall work in the limit $0 \leq \varepsilon < 1$.

B. 1-Body Apsidal Precession in 2D

With a little modification of Eqn(14) we can get precession. Let us write the modified equation as Eqn(25),

$$A = mr^2\left(\frac{d\theta}{dt}\right)$$
$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A^2\kappa^2}{mr^2} + \frac{GMm}{R} - \frac{GMm}{r} \tag{25}$$

The solution to Eqn(25) is,

$$r = \frac{r_0}{1 + \varepsilon Cos(\kappa\theta)}; \kappa \neq 1 \implies Precession$$
$$r_0 = \frac{A^2\kappa^2}{GMm^2}, E = \frac{GMm}{R} - (1-\varepsilon^2)\frac{GMm}{2r_0} \tag{26}$$

Eqns(25,26) are what Newton called as Revolving orbits. This occurs when a disturbance alters rotational energy term $\left(\frac{A^2}{GMm^2} \to \frac{A^2\kappa^2}{GMm^2}\right)$ in the Energy equation without altering Angular momentum per unit mass.

C. 2-body problem, in 2D

Consider 2 bodies of mass m_1 and m_2 kg situated at (r_1, θ_1) and (r_2, θ_2) respectively in the circular-polar coordinate system. We can write the net energy(E) as,

$$E = \frac{1}{2}m_1\left(\frac{d\vec{r}_1}{dt}\bullet\frac{d\vec{r}_1}{dt}\right) + \frac{1}{2}m_2\left(\frac{d\vec{r}_2}{dt}\bullet\frac{d\vec{r}_2}{dt}\right) - \frac{Gm_1m_2}{\sqrt{{r_1}^2 + {r_2}^2 - 2r_1r_2Cos(\theta_1-\theta_2)}} \tag{27}$$

Note that we have dropped the binding energy portion from Eqn(27) just for some mathematical simplicity. If R_1 is the radius of mass m_1 and R_2 is the radius of mass m_2 and if x is the distance between the masses excluding the radial lengths then we can define $P = \frac{GMm}{R_1+R_2} - \frac{GMm}{R_1+R_2+x} \equiv \frac{GMm}{R} - \frac{GMm}{r}$.

In scalar form Eqn(27) is,

$$E = \frac{1}{2}m_1\left(\frac{dr_1}{dt}\right)^2 + \frac{1}{2}m_2\left(\frac{dr_2}{dt}\right)^2 + \frac{1}{2}m_1{r_1}^2\left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}m_2{r_2}^2\left(\frac{d\theta_2}{dt}\right)^2 - \frac{Gm_1m_2}{\sqrt{{r_1}^2 + {r_2}^2 - 2r_1r_2Cos(\theta_1-\theta_2)}} \tag{28}$$

The total angular momentum(A) of the system is,

$$A = m_1{r_1}^2\left(\frac{d\theta_1}{dt}\right) + m_2{r_2}^2\left(\frac{d\theta_2}{dt}\right) \tag{29}$$

The relative angle between the 2 masses remains π, i.e. they are always on a straight line

$$\theta_2 - \theta_1 = \pi$$
$$So, \frac{d\theta_1}{dt} = \frac{d\theta_2}{dt} = \frac{d\theta}{dt} \tag{30}$$

Eqn(30) implies that the 2 masses have the same(but not constant) angular velocity $\frac{d\theta}{dt}$.

Assume the center of mass is stationary at the origin. Origin = Center of Revolution.

$$\begin{aligned}&\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}=0\hat{i}+0\hat{j}, So, m_1r_1=m_2r_2\\&m_1r_1Cos(\theta_1)=-m_2r_2Cos(\theta_2)\\&m_1r_1Sin(\theta_1)=-m_2r_2Sin(\theta_2)\end{aligned} \quad (31)$$

Use Eqns(30,31) in Eqn(29) to reduce variables,

$$A=m_1\left(1+\frac{m_1}{m_2}\right)r_1^2\frac{d\theta}{dt}=m_2\left(1+\frac{m_2}{m_1}\right)r_2^2\frac{d\theta}{dt} \quad (32)$$

Let us define $\zeta=1+\frac{m_1}{m_2}$ and $\zeta'=1+\frac{m_2}{m_1}$. We can express energy(E) in two different forms, one with (m_1,r_1) and another with (m_2,r_2) variables. Because we can reduce the 2-body problem to 1-body format.

Using Eqns(30-32) in Eqn(28)

$$\begin{aligned}E&=\frac{1}{2}m_1\zeta\left(\frac{dr_1}{dt}\right)^2+\frac{1}{2}\frac{A^2}{m_1\zeta{r_1}^2}-\frac{Gm_1m_2}{r_1\zeta}\\E&=\frac{1}{2}m_2\zeta'\left(\frac{dr_2}{dt}\right)^2+\frac{1}{2}\frac{A^2}{m_2\zeta'{r_2}^2}-\frac{Gm_1m_2}{r_2\zeta'}\end{aligned} \quad (33)$$

Apply Eqn(30) on Eqn(27),

$$\begin{aligned}E=&\frac{1}{2}m_1\left(\frac{d\vec{r}_1}{dt}\bullet\frac{d\vec{r}_1}{dt}\right)+\frac{1}{2}m_2\left(\frac{d\vec{r}_2}{dt}\bullet\frac{d\vec{r}_2}{dt}\right)\\&-\frac{Gm_1m_2}{r_1+r_2}\end{aligned} \quad (34)$$

Taking time derivative of Eqn(34)

$$\begin{aligned}\frac{dE}{dt}=&m_1\frac{d^2\vec{r}_1}{dt^2}\bullet\frac{d\vec{r}_1}{dt}+m_2\frac{d^2\vec{r}_2}{dt^2}\bullet\frac{d\vec{r}_2}{dt}\\&+\frac{Gm_1m_2}{(r_1+r_2)^2}\frac{dr_1}{dt}+\frac{Gm_1m_2}{(r_1+r_2)^2}\frac{dr_2}{dt}\end{aligned} \quad (35)$$

Noting that $m_1\frac{d\vec{r}_1}{dt}=-m_2\frac{d\vec{r}_2}{dt}$ and $m_1\frac{d^2\vec{r}_1}{dt^2}=-m_2\frac{d^2\vec{r}_2}{dt^2}$

$$\begin{aligned}\frac{dE}{dt}=&m_1\left(1+\frac{m_1}{m_2}\right)\left[\frac{d^2\vec{r}_1}{dt^2}+\frac{Gm_2\hat{r}_1}{(r_1+r_2)^2}\right]\bullet\frac{dr_1}{dt}=0\\\frac{dE}{dt}=&m_2\left(1+\frac{m_2}{m_1}\right)\left[\frac{d^2\vec{r}_2}{dt^2}+\frac{Gm_1\hat{r}_2}{(r_1+r_2)^2}\right]\bullet\frac{dr_2}{dt}=0\end{aligned} \quad (36)$$

The 2 terms inside the square brackets in Eqn(36) $\frac{d^2\vec{r}_1}{dt^2}+\frac{Gm_2\hat{r}_1}{(r_1+r_2)^2}$ and $\frac{d^2\vec{r}_2}{dt^2}+\frac{Gm_1\hat{r}_2}{(r_1+r_2)^2}$ are nothing but the 2 Force balance expressions that we would get by applying Newton's method on the 2-body system in 2D. As noted wrt Eqn(19) here again we note that Newton's method of equating the 2 expressions to zero i.e. $\frac{d^2\vec{r}_1}{dt^2}+\frac{Gm_2\hat{r}_1}{(r_1+r_2)^2}=0$ and $\frac{d^2\vec{r}_2}{dt^2}+\frac{Gm_1\hat{r}_2}{(r_1+r_2)^2}=0$ is a subset of all possibilities of Eqn(36). We could also get $\frac{d^2\vec{r}_1}{dt^2}+\frac{Gm_2\hat{r}_1}{(r_1+r_2)^2}\perp\frac{d\vec{r}_1}{dt}$ and/or $\frac{d^2\vec{r}_2}{dt^2}+\frac{Gm_1\hat{r}_2}{(r_1+r_2)^2}\perp\frac{d\vec{r}_2}{dt}$.

Below we briefly derive the Solution based on equivalence with Eqns(23,24).

Define, $m=m_1\zeta$ and $M=\frac{m_2}{\zeta^2}$.
Use it on Eqns(32,33) to get Eqn(37)

$$E=\frac{1}{2}m\left(\frac{dr_1}{dt}\right)^2+\frac{1}{2}\frac{A^2}{mr_1^2}-\frac{GmM}{r_1}, A=mr_1^2\frac{d\theta}{dt} \quad (37)$$

Eqn(37) has elliptic orbit solutions such that,

$$\begin{aligned}&r_{10}=\frac{A^2}{GMm^2}=\frac{A^2}{Gm_2m_1^2}, r_{20}=\frac{A^2}{Gm_1m_2^2}\\&r_{10}^3\omega_0^2=GM=Gm_2^3/(m_1+m_2)^2\\&r_{20}^3\omega_0^2=Gm_1^3/(m_1+m_2)^2\\&r_1=\frac{r_{10}}{1+\varepsilon Cos(\theta)}, r_2=\frac{r_{20}}{1+\varepsilon Cos(\theta)}\end{aligned}$$

r_{10} = Semi-latus rectum of m_1
r_{20} = Semi-latus rectum of m_2.

We get bounded orbits when, $-\frac{G^2m_2^3m_1^3}{2(m_1+m_2)A^2}\leq E<0$. Or in other words when $0\leq\varepsilon<1$. If we had retained the constant binding energy term E_b in Eqn(27) then $E_b-\frac{G^2m_2^3m_1^3}{2(m_1+m_2)A^2}\leq E<E_b$.

r_1 and r_2 show individual trajectories of masses m_1 and m_2 respectively. In case of the generalized 2-body system in 2D r_1 and r_2 are confocal ellipses sharing a common focus and having the same eccentricity. But size and orientation of the ellipses may be different.

IV. 3-BODY PROBLEM IN 2D

A. Lagrange type Periodic Orbits

Energy Method, Method $\mathcal{B}$
Consider 3 bodies of mass m_1, m_2 and m_3 Kg situated at (r_1,θ_1), (r_2,θ_2) and (r_3,θ_3) respectively in the circular-polar coordinate system. Then there exists 3 potential energy components due to ${}^3C_2$ combinations of masses.

$$\begin{aligned}P_{12}&=-\frac{Gm_1m_2}{\sqrt{{r_1}^2+{r_2}^2-2r_1r_2Cos(\theta_1-\theta_2)}}\\P_{23}&=-\frac{Gm_2m_3}{\sqrt{{r_2}^2+{r_3}^2-2r_2r_3Cos(\theta_2-\theta_3)}}\\P_{31}&=-\frac{Gm_3m_1}{\sqrt{{r_3}^2+{r_1}^2-2r_3r_1Cos(\theta_3-\theta_1)}}\end{aligned}$$

Then we can write the energy(E) conservation equation of the system as,

$$\begin{aligned}E=&\frac{1}{2}m_1\left(\frac{d\vec{r}_1}{dt}\bullet\frac{d\vec{r}_1}{dt}\right)+\frac{1}{2}m_2\left(\frac{d\vec{r}_2}{dt}\bullet\frac{d\vec{r}_2}{dt}\right)\\&+\frac{1}{2}m_3\left(\frac{d\vec{r}_3}{dt}\bullet\frac{d\vec{r}_3}{dt}\right)+P_{12}+P_{23}+P_{31}\end{aligned}$$

$$\begin{aligned} E =& \frac{1}{2}m_1\left(\frac{dr_1}{dt}\right)^2 + \frac{1}{2}m_2\left(\frac{dr_2}{dt}\right)^2 + \frac{1}{2}m_3\left(\frac{dr_3}{dt}\right)^2 \\ &+\frac{1}{2}m_1{r_1}^2\left(\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}m_2{r_2}^2\left(\frac{d\theta_2}{dt}\right)^2 \\ &+\frac{1}{2}m_3{r_3}^2\left(\frac{d\theta_3}{dt}\right)^2 + P_{12} + P_{23} + P_{31} \end{aligned} \quad (38)$$

The net angular momentum(A) of the system is,

$$A = m_1{r_1}^2\frac{d\theta_1}{dt} + m_2{r_2}^2\frac{d\theta_2}{dt} + m_3{r_3}^2\frac{d\theta_3}{dt} \quad (39)$$

Assume, Center of Mass = Origin.

$$\begin{aligned} &\frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1+m_2+m_3} = 0\hat{i} + 0\hat{j} \\ &m_1r_1Cos(\theta_1) + m_2r_2Cos(\theta_2) = -m_3r_3Cos(\theta_3) \\ &m_1r_1Sin(\theta_1) + m_2r_2Sin(\theta_2) = -m_3r_3Sin(\theta_3) \end{aligned} \quad (40)$$

$$r_2 = \frac{m_1r_1Sin(\theta_1-\theta_3)}{m_2Sin(\theta_3-\theta_2)}, r_3 = \frac{m_1r_1Sin(\theta_2-\theta_1)}{m_3Sin(\theta_3-\theta_2)} \quad (41)$$

Rearranging Eqn(41)

$$\frac{m_1r_1}{Sin(\theta_3-\theta_2)} = \frac{m_2r_2}{Sin(\theta_1-\theta_3)} = \frac{m_3r_3}{Sin(\theta_2-\theta_1)}$$

Like in the 2 body case where we see that the angle between the 2 bodies remains constant (= π Radians) here also we can assume that the relative angles between the objects remains constant. That is,

$$\begin{aligned} &Cos(\theta_2-\theta_1) = Cos(\alpha) = Constant \\ &Cos(\theta_3-\theta_2) = Cos(\beta) = Constant \\ &Cos(\theta_1-\theta_3) = Cos(\gamma) = Constant \\ &\alpha+\beta+\gamma = 2\pi \\ &Therefore, \frac{d\theta_1}{dt} = \frac{d\theta_2}{dt} = \frac{d\theta_3}{dt} \end{aligned} \quad (42)$$

Thus we get that, the three masses have the same angular velocity(not constant, but same).

Using Eqn(40,42) we can express r_2, r_3 in terms of r_1

$$r_2 = \frac{m_1Sin(\gamma)}{m_2Sin(\beta)}r_1, r_3 = \frac{m_1Sin(\alpha)}{m_3Sin(\beta)}r_1 \quad (43)$$

Let $r_2 = \mu_2r_1$ and $r_3 = \mu_3r_1$. Where μ's are Constants.

Using Eqns(40-43) in Eqns(38,39) we can reduce the 3-body problem to Kepler 1-body format.

Eqn(39) becomes,

$$A = m_1r_1^2\frac{d\theta_1}{dt}\left(1 + \frac{m_2}{m_1}\mu_2^2 + \frac{m_3}{m_1}\mu_3^2\right) \quad (44)$$

Let us define, $m = m_1\left(1+\frac{m_2}{m_1}\mu_2^2+\frac{m_3}{m_1}\mu_3^2\right)$ and,

$$\begin{aligned} M =& \frac{m_1m_2/m}{\sqrt{1+\mu_2^2-2\mu_2Cos(\alpha)}} + \frac{m_2m_3/m}{\sqrt{\mu_2^2+\mu_3^2-2\mu_2\mu_3Cos(\beta)}} \\ &+\frac{m_3m_1/m}{\sqrt{\mu_3^2+1-2\mu_3Cos(\gamma)}} \end{aligned}$$

So Eqn(44) gets simplified as $A = mr_1^2\frac{d\theta_1}{dt}$.

Use Eqns(40-43) and the above definitions in Eqn(38),

$$\begin{aligned} E &= \frac{1}{2}m\left(\frac{dr_1}{dt}\right)^2 + \frac{1}{2}\frac{A^2}{mr_1^2} - \frac{GMm}{r_1} \\ \frac{dE}{dr_1} &= \frac{d^2r_1}{dt^2} - \frac{A_1^2}{m_1r_1^3} + \frac{GM}{r_1^2} = 0 \end{aligned} \quad (45)$$

$\frac{dE}{dr_1} = 0 \implies$ E = Constant

Eqns(44,45) have elliptical solution such that,

$$\begin{aligned} r_1 &= \frac{r_{10}}{1+\varepsilon Cos(\theta_1)}, r_{10} = \frac{A^2}{GMm^2} \\ r_2 &= \frac{r_{10}\zeta_2}{1+\varepsilon Cos(\theta_2-\alpha)}, r_3 = \frac{r_{10}\zeta_3}{1+\varepsilon Cos(\theta_3+\gamma)} \end{aligned} \quad (46)$$

That is we get 3 ellipses with the same eccentricity and sharing a common focus but with different sizes/orientations. But from the work of Lagrange we know that Newtonian equations give a very different solution than Eqn(46).

Force Method, Method $\mathcal{A}$

Let us find the scalar equation for the 3-body case formed by using Newtons method i.e. $m_i\frac{d^2\vec{r}_i}{dt^2} - \vec{F}_i = 0$. From Eqn(9) applying angular momentum conservation and using, $\vec{r}_1 = r_1[Cos(\theta_1)\hat{i} + Sin(\theta_1)\hat{j}]$,
$\vec{r}_2 = r_2[Cos(\theta_2)\hat{i} + Sin(\theta_2)\hat{j}]$,
$\vec{r}_3 = r_3[Cos(\theta_3)\hat{i} + Sin(\theta_3)\hat{j}]$ we get,

$$\begin{aligned} &m_1\left[\frac{d^2r_1}{dt^2} - \frac{A_1^2}{m_1r_1^3}\right]\hat{r}_1 \\ &= -\frac{Gm_1m_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2) - \frac{Gm_1m_3}{|\vec{r}_1-\vec{r}_3|^3}(\vec{r}_1-\vec{r}_3) \end{aligned} \quad (47)$$

Comparing the $\hat{i}$ component and rearranging,

$$\begin{aligned} &m_1\left[\frac{d^2r_1}{dt^2} - \frac{A_1^2}{m_1r_1^3}\right] \\ &= -\frac{Gm_1m_2[r_1 - r_2\frac{Cos(\theta_2)}{Cos(\theta_1)}]}{[r_1^2+r_2^2-2r_1r_2Cos(\theta_2-\theta_1)]^{3/2}} \\ &-\frac{Gm_1m_3[r_1 - r_3\frac{Cos(\theta_3)}{Cos(\theta_1)}]}{[r_1^2+r_3^2-2r_1r_3Cos(\theta_1-\theta_3)]^{3/2}} \end{aligned} \quad (48)$$

Comparing the $\hat{j}$ component and rearranging,

$$m_1\left[\frac{d^2r_1}{dt^2}-\frac{A_1^2}{m_1r_1^3}\right]$$
$$=-\frac{Gm_1m_2[r_1-r_2\frac{Sin(\theta_2)}{Sin(\theta_1)}]}{[r_1^2+r_2^2-2r_1r_2Cos(\theta_2-\theta_1)]^{3/2}}$$
$$-\frac{Gm_1m_3[r_1-r_3\frac{Sin(\theta_3)}{Sin(\theta_1)}]}{[r_1^2+r_3^2-2r_1r_3Cos(\theta_1-\theta_3)]^{3/2}} \quad (49)$$

Taking the difference of Eqns(48,49) and rearranging,

$$0=\frac{Gm_1m_2r_2[\frac{Sin(\theta_1-\theta_2)}{Sin(\theta_1)Cos(\theta_1)}]}{[r_1^2+r_2^2-2r_1r_2Cos(\theta_2-\theta_1)]^{3/2}}$$
$$+\frac{Gm_1m_3r_3[\frac{Sin(\theta_1-\theta_3)}{Sin(\theta_1)Cos(\theta_1)}]}{[r_1^2+r_3^2-2r_1r_3Cos(\theta_1-\theta_3)]^{3/2}}$$

Using Eqn(41) in the above equation gives Equilateral triangle solutions,

$$|\vec{r}_1-\vec{r}_2|=|\vec{r}_2-\vec{r}_3|=|\vec{r}_3-\vec{r}_1|=r_C$$

Using this in Eqn(47) reduces it to,

$$\left[\frac{d^2r_1}{dt^2}-\frac{A_1^2}{m_1r_1^3}\right]\hat{r}=-\frac{G(m_1+m_2+m_3)}{r_C^3}\vec{r}_1 \quad (50)$$

Eqn(46) gives a family of solutions for a given set of m_1, m_2, m_3 values depending on α, β values whereas Eqn(50) gives just one possible solution for a given set of m_1, m_2, m_3 values because it also determines unique values for α, β. Suppose $m_1 = m_2 = m_3$ then from Eqn(43).

$$\frac{r_1}{Sin(\beta)}=\frac{r_2}{Sin(\gamma)}=\frac{r_3}{Sin(\alpha)}$$

the Equilateral triangle constraint implies

$$r_1^2+r_2^2-2r_1r_2Cos(\alpha)=r_2^2+r_3^2-2r_2r_3Cos(\beta)$$
$$\frac{r_1^2}{r_2^2}-\frac{2r_1}{r_2}Cos(\alpha)=\frac{r_3^2}{r_2^2}-\frac{2r_3}{r_2}Cos(\beta)$$
$$\frac{Sin^2(\beta)}{Sin^2(\gamma)}-\frac{2Sin(\beta)}{Sin(\gamma)}Cos(\alpha)=\frac{Sin^2(\alpha)}{Sin^2(\gamma)}-\frac{2Sin(\alpha)}{Sin(\gamma)}Cos(\beta)$$

$$Sin^2(\beta)-Sin^2(\alpha)=2Sin(\beta-\alpha)Sin(\gamma)$$
$$Sin^2(\beta)-Sin^2(\alpha)=-2Sin(\beta-\alpha)Sin(\beta+\alpha)$$
$$Cos^2(\alpha)=Cos^2(\beta)$$
$$\alpha=\beta=\gamma, r_1=r_2=r_3$$

Method $\mathcal{A}$: In Eqn(47) when $m_1 = m_2 = m_3$ the only possible solution is $r_1 = r_2 = r_2$, $\alpha = \beta = \gamma$ such that the 3 masses are placed at the 3 vertices of an equilateral triangle and are revolving about the center of mass. The size of the triangle can be either static in time or pulsating (imploding - exploding). If we slightly alter the conditions such that $r_1 \to r_1 + \delta, \delta << r_1$ and integrate Eqn(47) we get (wrong)unstable trajectories which do not conserve energy. And do not resemble the Equilateral triangle configuration.

Method $\mathcal{B}$: In Eqn(46) When $m_1 = m_2 = m_3$ we get a family of solutions such that $\frac{r_1}{Sin(\beta)} = \frac{r_2}{Sin(\gamma)} = \frac{r_3}{Sin(\alpha)}$. The configuration can be any general static/pulsating triangle with the masses revolving around the center of mass. If we slightly alter the conditions such that $r_1 \to r_1+\delta, \delta << r_1$ then from Eqn(46) we get slightly altered but stable orbits. But beyond a certain limit on the perturbation, the solution shifts from the Lagrange type periodic orbits with 1 center of revolution common to all 3 mass to a state like the Sun-Moon-Earth restricted 3-body system with 2 centers of revolution. Method $\mathcal{B}$ covers the entire possible solution space whereas Method $\mathcal{A}$ catches only a few special cases. Most often Method $\mathcal{A}$ gives wrong results.

From the Energy method we can derive elliptical orbits with general N-gon configurations with N-masses. In this solution all the masses are in one phase, moving in unison around a single center of mass. But in case of restricted N-body problems there are multiple centers of rotation and the bodies are not moving in unison.

B. Switchover from 1 Center to 2 Centers of Revolution

In the condition inspired by Eqn(41),

$$\frac{m_1r_1}{Sin(\beta)}=\frac{m_2r_2}{Sin(\gamma)}=\frac{m_3r_3}{Sin(\alpha)}=Constant=\frac{1}{\sqrt{\mathcal{Q}}}$$

Of the 9 variables $m_1, m_2, m_3, r_1, r_2, r_3$ and α, β, γ specifying only 6 of them determines the constant $\sqrt{\mathcal{Q}}$ and also the remaining 3 variables.

$$Sin(\beta)=\sqrt{\mathcal{Q}}m_1r_1, Sin(\gamma)=\sqrt{\mathcal{Q}}m_2r_2, Sin(\alpha)=\sqrt{\mathcal{Q}}m_3r_3$$

Assume we specify $m_1, m_2, m_3, r_1, r_2, r_3$ and we have to determine, α, β, γ

$$Sin(\gamma)=\sqrt{\mathcal{Q}}m_2r_2=-Sin(\alpha+\beta)$$
$$=-\sqrt{\mathcal{Q}}m_3r_3Cos(\beta)-\sqrt{\mathcal{Q}}m_1r_1Cos(\alpha)$$

$$m_2r_2=\pm m_3r_3\sqrt{1-\mathcal{Q}m_1^2r_1^2}\pm m_1r_1\sqrt{1-\mathcal{Q}m_3^2r_3^2}$$

$$\left[\frac{m_2^2r_2^2-m_3^2r_3^2-m_1^2r_1^2}{2m_1m_3r_1r_3}\right]^2+[m_2^2r_2^2]\mathcal{Q}=1$$

If $\mathcal{Q} > 0$ then we get Lagrange type periodic orbits with just one common center of revolution. If $\mathcal{Q} \leq 0$ then we get restricted 3-body type orbits with 2 centers of revolution.

Suppose $m_1 = m_2 = m_3 = m =$ Constant and $r_1 = r_3 = r =$ Constant, we vary r_2 from 0 to ∞ then $\mathcal{Q}$ becomes 0 at $r_2 = 2r$ and $\mathcal{Q} < 0$ when $r_2 > 2r$.

$$\left[\frac{m^2r_2^2 - m^2r^2 - m^2r^2}{2m^2r^2}\right]^2 + [m^2r_2^2]\mathcal{Q} = 1$$

$$\left[\frac{r_2^2}{2r^2} - 1\right]^2 + [m^2r_2^2]\mathcal{Q} = 1$$

V. 2-BODY IN 3D, IDEALIZED PRECESSION

A. Unperturbed Orbit in 3D

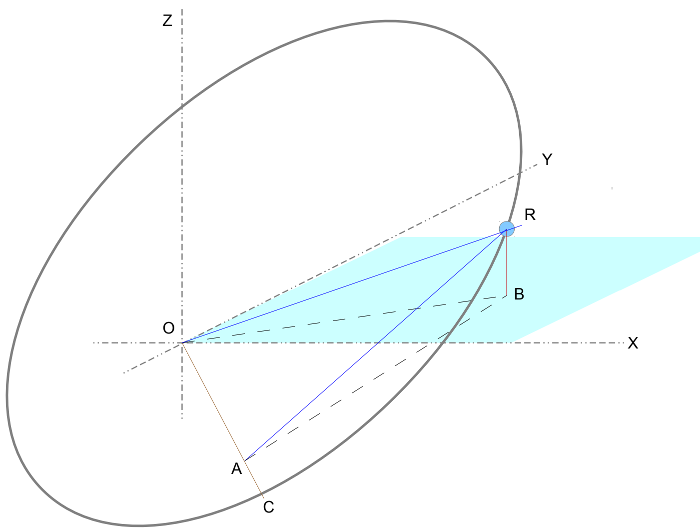

FIG. 1 Motion on an inclined elliptic orbit. Lines OX, OY, OZ represent the 3 geometric axis. OX and OY are in the horizontal plane OZ is the vertical axis. The ellipse is inclined to the horizontal plane at some angle. O is one of the foci of the ellipse. OC is the semi-latus rectum in the horizontal plane. R is the instantaneous position of the moving object. BR is parallel to OZ. Point B is the projection of R on the horizontal plane. Lines OB and AB are in the horizontal plane. Line AB is perpendicular to Line OC. Angle RAB is the Inclination of the ellipse and Angle AOR is the angular displacement(λ) in the plane of revolution.

We observe Nodal and Apsidal Precession in case of both Artificial Satellites in orbits around Earth and in case of Moons orbit around Earth. In the Spherical-Polar Co-ordinate system (r,θ,ϕ) where $\vec{r}$ is the position vector
θ = longitude angle and
ϕ = latitude angle such that the position vector $\vec{r}$ is,
$\vec{r} = r[Cos(\phi)Cos(\theta)\hat{i} + Cos(\phi)Sin(\theta)\hat{j} + Sin(\phi)\hat{k}]$.

Let,
$\vec{u}_1 = Cos(\theta)\hat{i} + Sin(\theta)\hat{j}$, $\vec{u}_2 = -Sin(\theta)\hat{i} + Cos(\theta)\hat{j}$

Let us find $\frac{d\vec{r}}{dt}$,

$$\begin{aligned}\frac{d\vec{r}}{dt} =& \frac{dr}{dt}\hat{r} - rSin(\phi)\frac{d\phi}{dt}\vec{u}_1 \\ &+ rCos(\phi)\frac{d\theta}{dt}\vec{u}_2 + rCos(\phi)\frac{d\phi}{dt}\hat{k}\end{aligned} \tag{51}$$

Let us also find $\frac{d^2\vec{r}}{dt^2}$,

$$\begin{aligned}\frac{d^2\vec{r}}{dt^2} =& \left[\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2\right]\hat{r} \\ &- \left[2\frac{dr}{dt}\frac{d\phi}{dt} + rCot(\phi)\left(\frac{d\theta}{dt}\right)^2 + r\frac{d^2\phi}{dt^2}\right]Sin(\phi)\vec{u}_1 \\ &+ \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} - 2Tan(\phi)r\frac{d\phi}{dt}\frac{d\theta}{dt}\right]Cos(\phi)\vec{u}_2 \\ &+ \left[2\frac{dr}{dt}\frac{d\phi}{dt} + r\frac{d^2\phi}{dt^2}\right]Cos(\phi)\hat{k}\end{aligned} \tag{52}$$

Let us also find, $\vec{r} \times \frac{d\vec{r}}{dt}$

$$\begin{aligned}m\vec{r} \times \frac{d\vec{r}}{dt} =& mr^2Cos^2(\phi)\frac{d\theta}{dt}\hat{k} \\ &- mr^2\frac{d\phi}{dt}\vec{u}_2 - \frac{1}{2}mr^2Sin(2\phi)\frac{d\theta}{dt}\vec{u}_1\end{aligned} \tag{53}$$

From Eqn(53) we can define the $\hat{i}, \hat{j}, \hat{k}$ components of angular momentum $\vec{A} = A_i\hat{i} + A_j\hat{j} + A_k\hat{k}$ where $\vec{A} = \vec{r} \times m\frac{d\vec{r}}{dt}$. Note that,

$$A^2 = A_i^2 + A_j^2 + A_k^2 = \frac{A_k^2}{Cos^2(\phi)} + m^2r^4\left(\frac{d\phi}{dt}\right)^2 \tag{54}$$

Consider the energy equation,

$$E = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}\right) + \frac{GMm}{R} - \frac{GMm}{r}$$

Assume the k^{th} component $A_k = mr^2Cos^2(\phi)\frac{d\theta}{dt}$ = Constant.

$$\begin{aligned}E = & \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A_k^2}{mr^2Cos^2(\phi)} + \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 \\ &+ \frac{GMm}{R} - \frac{GMm}{r}\end{aligned} \tag{55}$$

If we assume A = Constant then we can reframe Eqn(54) as,

$$\frac{1}{2}\frac{A_k^2}{mr^2Cos^2(\phi)} + \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 = \frac{1}{2}\frac{A_k^2\eta^2}{mr^2} \tag{56}$$

In Eqn(54) $A = A_k\eta = \sqrt{A_i^2 + A_j^2 + A_k^2}$

Using Eqn(56) in Eqn(55)

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A_k^2\eta^2}{mr^2} + \frac{GMm}{R} - \frac{GMm}{r} \tag{57}$$

Let λ be the angle wrt semi-minor axis in the plane of revolution then

$$z = rSin(\lambda)Sin(\phi_m) = rSin(\phi) \tag{58}$$

From Eqn(56) we get,

$$\frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 = \frac{1}{2}\frac{A_k^2}{mr^2}\left[\eta^2 - \frac{1}{Cos^2(\phi)}\right]$$
$$A_k = mr^2Cos^2(\phi)\frac{d\theta}{dt}$$
$$\left(\frac{d\phi}{dt}\right)^2 = Cos^4(\phi)\left(\frac{d\theta}{dt}\right)^2\left[\eta^2 - \frac{1}{Cos^2(\phi)}\right] \quad (59)$$

Note, $\frac{1}{Cos^2(\phi)} = Sec^2(\phi) = 1 + Tan^2(\phi)$

$$\frac{d\phi}{dt} = \pm Cos^2(\phi)\frac{d\theta}{dt}\sqrt{\eta^2 - 1 - Tan^2(\phi)}$$
$$Tan(\phi) = \sqrt{\eta^2-1}Sin(\theta_c \pm \theta)$$
$$Tan(\phi) = Tan(\phi_m)Sin(\theta_c \pm \theta) \quad (60)$$

θ_c = Constant of integration. $\eta = Sec(\phi_m)$
Let $\frac{d\nu}{dt} = \eta Cos^2(\phi)\frac{d\theta}{dt}$, So, $A_k\eta = mr^2\frac{d\nu}{dt}$
Differentiating Eqn(60)

$$Sec^2(\phi)d\phi = Tan(\phi_m)Cos(\theta_c \pm \theta)(\pm d\theta)$$
$$d\phi = Tan(\phi_m)Cos(\theta_c \pm \theta)(\pm Cos^2(\phi)d\theta)$$
$$\frac{Cos(\phi)}{\sqrt{Sin^2(\phi_m) - Sin^2(\phi)}}d\phi = \pm d\nu \quad (61)$$

Integrating Eqn(61)

$$Sin(\phi) = Sin(\phi_m)Sin\left(\nu_c \pm \nu\right) \quad (62)$$

Equate Eqn(58) and Eqn(62) to get,

$$Sin(\lambda) = Sin\left(\nu_c \pm \nu\right)$$
$$\lambda = \nu_c \pm \nu$$
$$\frac{d\lambda}{dt} = \pm\frac{d\nu}{dt} = \pm\eta Cos^2(\phi)\frac{d\theta}{dt} \quad (63)$$

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A_k^2\eta^2}{mr^2} + \frac{GMm}{R} - \frac{GMm}{r}$$
$$A_k\eta = \eta mr^2Cos^2(\phi)\frac{d\theta}{dt} = mr^2\frac{d\nu}{dt} = \pm mr^2\frac{d\lambda}{dt} \quad (64)$$

The solution to Eqn(64) is,

$$r = \frac{r_0}{1+\varepsilon Cos(\lambda_c \pm \lambda)}, r_0 = \frac{A_k^2\eta^2}{GMm^2} \quad (65)$$

B. Ideal Perturbation

The easiest way to induce nodal precession in Eqn(55-56) is to introduce a disturbance of the same form as the energy term due to (A_k)Angular Momentum$\left(i.e. \frac{1}{2}\frac{A_k^2}{mr^2Cos^2(\phi)}\right)$. The Energy term $\frac{1}{2}\frac{B}{mr^2Cos^2(\phi)}$ in Eqn(66) is the added disturbance. B can be both +ve or -ve.

And the easiest way to induce apsidal precession in Eqn(55-56) is to introduce a disturbance of the form $\frac{1}{2}\frac{C}{mr^2}$ (in Eqn(66)). C can be both +ve or -ve.

$$E = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}\right) + \frac{1}{2}\frac{B}{mr^2Cos^2(\phi)} + \frac{1}{2}\frac{C}{mr^2} + \frac{GMm}{R} - \frac{GMm}{r}$$
$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A_k^2}{mr^2Cos^2(\phi)} + \frac{1}{2}\frac{B}{mr^2Cos^2(\phi)} + \frac{1}{2}\frac{C}{mr^2} + \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 + \frac{GMm}{R} - \frac{GMm}{r} \quad (66)$$

Using $\vec{r}\bullet\frac{d\vec{r}}{dt} = r\frac{dr}{dt}$ and $(Cos(\theta)\hat{i} + Sin(\theta)\hat{j})\bullet\frac{d\vec{r}}{dt} = rSin(\phi)\frac{d\phi}{dt} - Cos(\phi)\frac{dr}{dt}$

$$\frac{dE}{dt} = \left[m\frac{d^2\vec{r}}{dt^2} - \vec{F}\right]\bullet\frac{d\vec{r}}{dt} = 0 \quad (67)$$

Where,
$\vec{F} = -\frac{GMm}{r^3}\vec{r} + \frac{C}{mr^4}\vec{r} + \frac{B(Cos(\theta)\hat{i}+Sin(\theta)\hat{j})}{mr^3Cos^2(\phi)}$.

Let us extend the observation made about rotational energy terms in Eqn(56),

$$\frac{1}{2}\frac{(A_k^2+B)}{mr^2Cos^2(\phi)} + \frac{1}{2}\frac{C}{mr^2} + \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 = \frac{1}{2}\frac{A_k^2\eta_1^2}{mr^2} \quad (68)$$

Note $A^2 = A_i^2 + A_j^2 + A_k^2 = A_k^2\eta_1^2$. A and A_κ remain constant, A_i, A_j get modulated by the additional B and C terms in Eqn(66). From Eqn(68) we get the solution,

$$Tan(\phi) = \sqrt{\frac{\eta_1^2 - 1 - \frac{(B+C)}{A_k^2}}{1+\frac{B}{A_k^2}}}Sin\left(\sqrt{1+\frac{B}{A_k^2}}[\theta_c \pm \theta]\right)$$
$$Tan(\phi) = Tan(\phi_m).Sin\left(\sqrt{1+\frac{B}{A_k^2}}[\theta_c \pm \theta]\right) \quad (69)$$

Eqn(69) is similar to Eqn(60) but with modifications induced by the disturbance term B. The term $\sqrt{1+\frac{B}{A_k^2}}$ modulating θ indicates Nodal Precession. When B = 0, there is no Nodal Precession and we get back the same solution as in Eqn(60). η_1 is some constant depending on A_k^2, B, C and Sec(ϕ_m). From Eqn(69) we get $\eta_1 = \pm\sqrt{\left[1+\frac{B}{A_k^2}\right]Sec^2(\phi_m) + \frac{C}{A_k^2}}$.

Note that $\hat{i}$ and $\hat{j}$ components of Ang. Mom are not conserved separately but $A_i^2 + A_j^2$ = Constant.

Using Eqn(68) in Eqn(66)

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A_k^2\eta_1^2}{mr^2} + \frac{GMm}{R} - \frac{GMm}{r}$$
$$A_k\eta = \eta_1 mr^2Cos^2(\phi)\frac{d\theta}{dt} = mr^2\frac{d\nu}{dt} \quad (70)$$

Differentiating the $Tan(\phi)$ Eqn in Eqn(69) and using $A_k = mr^2Cos^2(\phi)\frac{d\theta}{dt}$ we get,

$$\frac{1}{\sqrt{Tan^2(\phi_m) - Tan^2(\phi)}}d\phi = \pm\sqrt{1 + \frac{B}{A_k^2}}\frac{1}{\eta_1}d\nu \quad (71)$$

Integrating Eqn(71) we get,

$$Sin(\phi) = Sin(\phi_m)Sin\left(\lambda_c \pm \sqrt{1 + \frac{B}{A_k^2}}\frac{Sec(\phi_m)}{\eta_1}\nu\right) \quad (72)$$

Eqn(58) is still valid, if λ is the angle wrt semi-minor axis in the plane of revolution then
$Sin(\phi) = Sin(\phi_m)Sin(\lambda)$

$$\lambda = \lambda_c \pm \sqrt{1 + \frac{B}{A_k^2}}\frac{Sec(\phi_m)}{\eta_1}\nu$$
$$\nu = \pm\sqrt{1 + \frac{C}{(A_k^2 + B)Sec^2(\phi_m)}}(\lambda - \lambda_c) \quad (73)$$

Solving Eqn(70) we get,

$$r = \frac{r_0}{1 + \varepsilon Cos(\nu)}, r_0 = \frac{A_k^2\eta_1^2}{GMm^2}$$
$$r = \frac{r_0}{1 + \varepsilon Cos\left(\sqrt{1 + \frac{C}{(A_k^2+B)Sec^2(\phi_m)}}(\lambda - \lambda_c)\right)} \quad (74)$$

Eqn(74) is similar to Eqn(65) but with modifications induced by the disturbance terms B,C. The term $\sqrt{1 + \frac{C}{(A_k^2+B)Sec^2(\phi_m)}}$ modulating λ indicates Apsidal Precession. When C = 0, there is no Apsidal Precession and we get back the same solution as in Eqn(65).

In case of ideal perturbation we can verify that in Eqn(67) $m\frac{d^2\vec{r}}{dt^2} = \vec{F} = -\frac{GMm}{r^3}\vec{r} + \frac{C}{mr^4}\vec{r} + \frac{B(Cos(\theta)\hat{i}+Sin(\theta)\hat{j})}{mr^3Cos^2(\phi)}$ or $m\frac{d^2\vec{r}}{dt^2} - \vec{F} = 0$. Hence Newtonian method gives the same precession rates as the Energy method. Possibly because Eqn(66) is a 2-body system which can be expressed entirely in terms or one variable r using Eqn(68). Also both the net Angular momentum(A) and k^{th} component A_k are conserved like in the 1-body approximation case. But this is not true in case of precession of Moon or Artificial satellites around the Earth. The perturbations are not ideal there.

VI. SUN-MOON-EARTH 3 BODY SYSTEM

A. Unperturbed Orbit

3 bodies with 2 different centers of revolution. Let us denote Earth as mass m_1 and Moon as mass m_2. Assume that the Sun is stationary at the center of revolution(Origin) around which m_1, m_2 revolve together. Let $\vec{x}_1$, $\vec{x}_2$ be the position vectors of m_1, m_2 respectively.

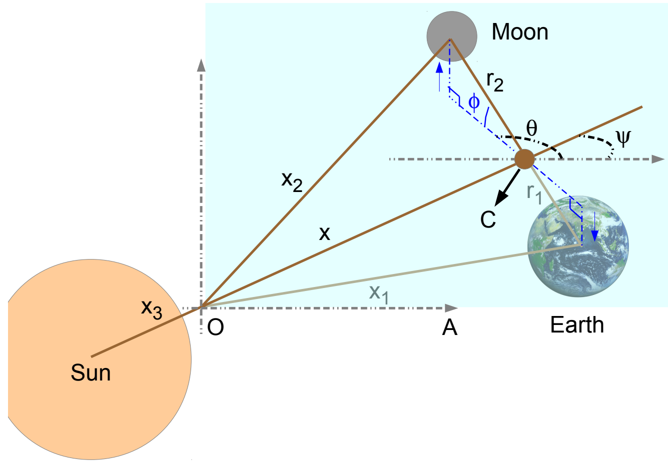

FIG. 2 Sun-Moon-Earth 3-body System. O is the Origin of 3-axis, Vector X_3(Sun) and Vector X lie on the horizontal plane(as the page). Vertical axis is out-of-the-page. C is a point along Vector X(Center of Earth Moon System) in the horizontal plane. Moon is elevated up at an angle(ϕ) wrt page(horizontal plane) and the Earth is below the horizontal plane at the same angle(ϕ). Vector r_1 connects C to Earth and Vector r_2 connects C to Moon. Note again vector r_2 is inclined up, vector r_1 is inclined down. θ and ψ are angles wrt a line parallel to OA.

Let $\vec{x}$ be the Center of Mass(CoM) of Earth+Moon (m_1+m_2) system

$$(m_1 + m_2)\vec{x} = m_1\vec{x}_1 + m_2\vec{x}_2$$
$$\vec{x} + \vec{r}_1 = \vec{x}_1, \vec{x} + \vec{r}_2 = \vec{x}_2$$
$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

The vectors $\vec{r}_1$, $\vec{r}_2$, $\vec{x}$ and $\vec{x}_3$ are described as below
The vectors $\vec{x}$ and $\vec{x}_3$ are co-linear

$$\vec{r}_1 = -r_1[Cos(\phi)Cos(\theta)\hat{i} + Cos(\phi)Sin(\theta)\hat{j} + Sin(\phi)\hat{k}]$$
$$\vec{r}_2 = r_2[Cos(\phi)Cos(\theta)\hat{i} + Cos(\phi)Sin(\theta)\hat{j} + Sin(\phi)\hat{k}]$$
$$\vec{x} = x[Cos(\psi)\hat{i} + Sin(\psi)\hat{i}], \vec{x}_3 = -x_3[Cos(\psi)\hat{i} + Sin(\psi)\hat{i}]$$

The center of mass of Sun-Earth-Moon (m_1, m_2, m_3) system is stationary.

$$m_1\vec{x}_1 + m_2\vec{x}_2 + m_3\vec{x}_3 = (m_1 + m_2)\vec{x} + m_3\vec{x}_3 = 0$$

Let us find $\frac{d\vec{x}_1}{dt}$ and $\frac{d\vec{x}_2}{dt}$,
$\frac{d\vec{x}_1}{dt} = \frac{d\vec{x}}{dt} + \frac{d\vec{r}_1}{dt}$, $\frac{d\vec{x}_2}{dt} = \frac{d\vec{x}}{dt} + \frac{d\vec{r}_2}{dt}$

$$m_1\frac{d\vec{x}_1}{dt}\bullet\frac{d\vec{x}_1}{dt} = m_1\frac{d\vec{x}}{dt}\bullet\frac{d\vec{x}}{dt} + m_1\frac{d\vec{r}_1}{dt}\bullet\frac{d\vec{r}_1}{dt} + 2m_1\frac{d\vec{x}}{dt}\bullet\frac{d\vec{r}_1}{dt}$$

$$m_2\frac{d\vec{x}_2}{dt}\bullet\frac{d\vec{x}_2}{dt} = m_2\frac{d\vec{x}}{dt}\bullet\frac{d\vec{x}}{dt} + m_2\frac{d\vec{r}_2}{dt}\bullet\frac{d\vec{r}_2}{dt} + 2m_2\frac{d\vec{x}}{dt}\bullet\frac{d\vec{r}_2}{dt}$$

The net Energy(E) equation is,

$$E = \frac{1}{2}m_1\left(\frac{d\vec{x}_1}{dt}\bullet\frac{d\vec{x}_1}{dt}\right) + \frac{1}{2}m_2\left(\frac{d\vec{x}_2}{dt}\bullet\frac{d\vec{x}_2}{dt}\right)$$
$$+\frac{1}{2}M_s\left(\frac{d\vec{x}_3}{dt}\bullet\frac{d\vec{x}_3}{dt}\right) - \frac{GM_sm_1}{|\vec{x}_1-\vec{x}_3|} - \frac{GM_sm_2}{|\vec{x}_2-\vec{x}_3|} - \frac{Gm_1m_2}{|\vec{x}_2-\vec{x}_1|} \quad (75)$$

$$E = \frac{1}{2}(m_1+m_2)\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}(m_1+m_2)x^2\left(\frac{d\psi}{dt}\right)^2$$
$$-\frac{GM_sm_1}{|\vec{x}_1-\vec{x}_3|} - \frac{GM_sm_2}{|\vec{x}_2-\vec{x}_3|} - \frac{Gm_1m_2}{r_1+r_2} + \frac{1}{2}m_1\left(\frac{dr_1}{dt}\right)^2$$
$$+\frac{1}{2}m_1r_1^2Cos^2(\phi)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}m_1r_1^2\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}m_2\left(\frac{dr_2}{dt}\right)^2$$
$$+\frac{1}{2}m_2r_2^2Cos^2(\phi)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}m_2r_2^2\left(\frac{d\phi}{dt}\right)^2$$
$$+\frac{1}{2}m_3\left(\frac{dx_3}{dt}\right)^2 + \frac{1}{2}m_3x_3^2\left(\frac{d\psi}{dt}\right)^2 \quad (76)$$

$(m_1+m_2)x = m_3x_3$

Let us find $\vec{x}_1 \times \frac{d\vec{x}_1}{dt}$ and $\vec{x}_2 \times \frac{d\vec{x}_2}{dt}$,

$m_1(\vec{x}+\vec{r}_1)\times\left(\frac{d\vec{x}}{dt}+\frac{d\vec{r}_1}{dt}\right)$

$m_1\vec{x}\times\frac{d\vec{x}}{dt} + m_1\vec{x}\times\frac{d\vec{r}_1}{dt} + m_1\vec{r}_1\times\frac{d\vec{x}}{dt} + m_1\vec{r}_1\times\frac{d\vec{r}_1}{dt}$

$m_2\vec{x}\times\frac{d\vec{x}}{dt} + m_2\vec{x}\times\frac{d\vec{r}_2}{dt} + m_2\vec{r}_2\times\frac{d\vec{x}}{dt} + m_2\vec{r}_2\times\frac{d\vec{r}_2}{dt}$

$$A_k = (m_1+m_2)x^2\frac{d\psi}{dt} + m_3x_3^2\frac{d\psi}{dt} + \left[m_1r_1^2+m_2r_2^2\right]Cos^2(\phi)\frac{d\theta}{dt} \quad (77)$$

Let $\gamma = \frac{m_1+m_2+m_3}{m_3} \approx 1$, so $x + x_3 = \gamma x$, and

$$\frac{GM_sm_1}{|\vec{x}_1-\vec{x}_3|} = \frac{GM_sm_1}{|\vec{x}-\vec{x}_3+\vec{r}_1|} = \frac{GM_sm_1}{|\gamma\vec{x}+\vec{r}_1|}$$
$$\frac{GM_sm_2}{|\vec{x}_2-\vec{x}_3|} = \frac{GM_sm_2}{|\vec{x}-\vec{x}_3+\vec{r}_2|} = \frac{GM_sm_2}{|\gamma\vec{x}+\vec{r}_2|} \quad (78)$$

$$\frac{GM_sm_1}{|\gamma\vec{x}+\vec{r}_1|} = \frac{GM_sm_1}{\sqrt{\gamma^2x^2+r_1^2-2\gamma xr_1Cos(\phi)Cos(\theta-\psi)}}$$
$$\frac{GM_sm_2}{|\gamma\vec{x}+\vec{r}_2|} = \frac{GM_sm_2}{\sqrt{\gamma^2x^2+r_2^2+2\gamma xr_2Cos(\phi)Cos(\theta-\psi)}}$$

Using Taylor Expansion upto 2^{nd} order. No $\frac{r_1}{\gamma x}$ terms will be left in $\frac{m_1}{x_{13}} + \frac{m_2}{x_{23}}$. Ignore the 3^{rd} and higher order terms. So the perturbation Δ is given by,

$$\Delta = \frac{GM_s(m_1+m_2)}{\gamma x} - \frac{GM_sm_1}{x_{13}} - \frac{GM_sm_2}{x_{23}}$$
$$\Delta \approx \frac{GM_sm}{2\gamma x}\left(\frac{r_1}{\gamma x}\right)^2\left[1-3Cos^2(\phi)Cos^2(\theta-\psi)\right] \quad (79)$$

Where, $m = (m_1+m_2)\frac{m_1}{m_2} = m_1\zeta$

In Eqn(79) we can express $Cos^2(\theta-\psi) = (1+Cos(2[\theta-\psi]))/2$.
The averaged out value of $Cos(2[\theta-\psi]))$ is zero.
So the secular effect of the disturbance is Δ_{av},

$$\Delta_{av} = \frac{GM_sm}{2x\gamma^3}\left(\frac{r_1}{x}\right)^2\left[1-\frac{3}{2}Cos^2(\phi)\right] \quad (80)$$

Using Eqn(80), Eqn(76) becomes

$$E = \frac{1}{2}(m_1+m_2)\gamma\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}(m_1+m_2)\gamma x^2\left(\frac{d\psi}{dt}\right)^2$$
$$-\frac{GM_s(m_1+m_2)}{\gamma x} + \Delta_{av} - \frac{Gm_1m_2}{r_1+r_2} + \frac{1}{2}m\left(\frac{dr_1}{dt}\right)^2$$
$$+\frac{1}{2}mr_1^2Cos^2(\phi)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}mr_1^2\left(\frac{d\phi}{dt}\right)^2 \quad (81)$$

Assume A_k is not altered by Δ_{av}, From Eqn(77)

$$A_k = (m_1+m_2)\gamma x^2\frac{d\psi}{dt} + mr_1^2Cos^2(\phi)\frac{d\theta}{dt} \quad (82)$$

ψ and θ are angles in the XY plane the CoM of m_1+m_2 system is revolving in the XY plane. ϕ is the angle of elevation wrt the XY plane. θ is calculated wrt to the line joining CoM to Sun. And ψ is the angle of the line joining CoM to Sun wrt to some reference axis. The angles θ and ϕ of m_1 and m_2 are offset by π Radians such that they both have the same $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$ values.

In Eqns(81,82) let us assume x = Constant and $\frac{d\psi}{dt}$ = Constant. Therefore we can define a new constant E_θ

From Eqn(81),

$$E_\theta = \frac{1}{2}m\left(\frac{dr_1}{dt}\right)^2 + \frac{1}{2}mr_1^2Cos^2(\phi)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}mr_1^2\left(\frac{d\phi}{dt}\right)^2 - \frac{Gm_1m_2}{r_1+r_2} + \Delta_{av} \quad (83)$$

In Eqn(83) if $\Delta_{av} = 0$ then we recover the 2-body configuration between Earth and Moon without the effect of Sun.

Similarly from Eqn(82),

$$A_\theta = mr_1^2Cos^2(\phi)\frac{d\theta}{dt} \quad (84)$$

The pair of Eqns $[A_\theta, E_\theta]$ describe the altered elliptical orbit of Moon including the effect of Sun to some degree of approximation.

B. Solar Gravitational Perturbation on Moon

Thus the E_θ eqn would be,
With $\zeta = 1 + \frac{m_1}{m_2}$, $m = (m_1+m_2)\frac{m_1}{m_2}$ and $M = \frac{m_2}{\zeta^2}$

$$E_\theta = \frac{1}{2}m\left(\frac{dr_1}{dt}\right)^2 + \frac{1}{2}\frac{A_\theta^2}{mr_1^2Cos^2(\phi)} + \frac{1}{2}mr_1^2\left(\frac{d\phi}{dt}\right)^2 - \frac{GMm}{r_1} + \frac{GM_smr_1^2}{2\gamma^3x^3}\left[1-\frac{3}{2}Cos^2(\phi)\right] \quad (85)$$

In Eqn(85) we can not make a similar assumption as in Eqn(68) because it gives contradictory results.

In vector form of kinetic energy term

$$E_\theta = \frac{1}{2}m\left(\frac{d\vec{r}_1}{dt}\bullet\frac{d\vec{r}_1}{dt}\right) - \frac{GMm}{r_1} + \frac{GM_smr_1^2}{2\gamma^3x^3}\left[1-\frac{3}{2}Cos^2(\phi)\right]$$

Find $\frac{dE_\theta}{dt}$ assuming x is constant

$$\frac{dE_\theta}{dt} = m\frac{d^2\vec{r}_1}{dt^2}\bullet\frac{d\vec{r}_1}{dt} + \frac{GMm}{r_1^3}\vec{r}_1\bullet\frac{d\vec{r}_1}{dt} + \frac{GM_sm}{\gamma^3x^3}\vec{r}_1\bullet\frac{d\vec{r}_1}{dt} - \frac{3GM_smr_1}{2\gamma^3x^3}Cos(\phi)\left[Cos(\phi)\frac{dr_1}{dt} - r_1Sin(\phi)\frac{d\phi}{dt}\right]$$

$$(-Cos(\theta)\hat{i} - Sin(\theta)\hat{j})\bullet\frac{d\vec{r}_1}{dt} = \frac{dr_1}{dt}Cos(\phi) - r_1Sin(\phi)\frac{d\phi}{dt}$$
$$(Cos(\theta)\hat{i} + Sin(\theta)\hat{j})\bullet\frac{d\vec{r}_2}{dt} = \frac{dr_2}{dt}Cos(\phi) - r_2Sin(\phi)\frac{d\phi}{dt}$$

$$\frac{dE_\theta}{dt} = m\frac{d^2\vec{r}_1}{dt^2}\bullet\frac{d\vec{r}_1}{dt} + \frac{GMm}{r_1^3}\vec{r}_1\bullet\frac{d\vec{r}_1}{dt} + \frac{GM_sm}{\gamma^3x^3}\vec{r}_1\bullet\frac{d\vec{r}_1}{dt} - \frac{3GM_smr_1}{2\gamma^3x^3}Cos(\phi)\left(-Cos(\theta)\hat{i} - Sin(\theta)\hat{j}\right)\bullet\frac{d\vec{r}_1}{dt} \quad (86)$$

If energy is conserved then $\left[m\frac{d^2\vec{r}_1}{dt^2} - \vec{F}_1\right]\bullet\frac{d\vec{r}_1}{dt} = 0$. So $\vec{F}_1$ can be written as,

$$\vec{F}_1 = -\frac{GMm}{r_1^3}\vec{r}_1 - \frac{GM_sm}{\gamma^3x^3}\vec{r}_1 + \frac{3GM_smr_1}{2\gamma^3x^3}Cos(\phi)\left(-Cos(\theta)\hat{i} - Sin(\theta)\hat{j}\right) \quad (87)$$

$$\vec{F}_1 = -\frac{Gm_1m_2}{r_1^3}\frac{m_2}{(m_1+m_2)}\vec{r}_1 - \frac{GM_sm_1}{x^3}\frac{m_2^2}{(m_1+m_2)^2}\vec{r}_1 + \frac{3GM_sm_1r_1}{2x^3}\frac{m_2^2}{(m_1+m_2)^2}Cos(\phi)\left(-Cos(\theta)\hat{i} - Sin(\theta)\hat{j}\right)$$

C. Newton's Method

We can write down the forces on mass m_1 and m_2 using Newton's Law of Gravity. The reference frame in a genuinely 3-body Sun-Earth-Moon problem must be the center of mass of the 3-body system which will be located close to the center of mass of Sun itself. Hence for simplicity we can consider Sun as stationary at the center(Fig-2). So the force balance equation on Moon with reference to Fig-2 becomes (Brown, 2016; Curtis, 2010)

$$m_1\frac{d^2\vec{x}_1}{dt^2} = -\frac{GM_sm_1}{|\vec{x}_1-\vec{x}_3|^3}(\vec{x}_1-\vec{x}_3) - \frac{Gm_1m_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2)$$
$$m_2\frac{d^2\vec{x}_2}{dt^2} = -\frac{GM_sm_2}{|\vec{x}_2-\vec{x}_3|^3}(\vec{x}_2-\vec{x}_3) - \frac{Gm_1m_2}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1)$$

Using $\vec{x}_1 = \vec{x}+\vec{r}_1$, $\vec{x}_2 = \vec{x}+\vec{r}_2$, $m_1\vec{r}_1 + m_2\vec{r}_2 = 0$ and $\vec{r} = \vec{r}_1 + \vec{r}_2$, we can rewrite the above eqn as,

$$m_1\frac{d^2\vec{x}}{dt^2} + m_1\frac{d^2\vec{r}_1}{dt^2} = -\frac{GM_sm_1}{|\gamma\vec{x}+\vec{r}_1|^3}(\gamma\vec{x}+\vec{r}_1) - \frac{Gm_1m_2}{r^2}\hat{r}$$
$$m_2\frac{d^2\vec{x}}{dt^2} + m_2\frac{d^2\vec{r}_2}{dt^2} = -\frac{GM_sm_2}{|\gamma\vec{x}+\vec{r}_2|^3}(\gamma\vec{x}+\vec{r}_2) + \frac{Gm_1m_2}{r^2}\hat{r} \quad (88)$$

Note $\gamma = \frac{m_1+m_2+m_3}{m_3} \approx 1$

By manipulating the 2 equations in Eqn(88) we get,

$$\frac{d^2\vec{r}_1}{dt^2} = \frac{GM_sm_2}{m_1+m_2}\left[\frac{1}{|\gamma\vec{x}+\vec{r}_2|^3} - \frac{1}{|\gamma\vec{x}+\vec{r}_1|^3}\right]\gamma\vec{x} - \frac{Gm_2}{r^2}\hat{r} - \frac{GM_s\vec{r}_1}{m_1+m_2}\left[\frac{m_1}{|\gamma\vec{x}+\vec{r}_1|^3} + \frac{m_2}{|\gamma\vec{x}+\vec{r}_2|^3}\right]$$
$$\frac{d^2\vec{r}_2}{dt^2} = -\frac{GM_sm_1}{m_1+m_2}\left[\frac{1}{|\gamma\vec{x}+\vec{r}_2|^3} - \frac{1}{|\gamma\vec{x}+\vec{r}_1|^3}\right]\gamma\vec{x} - \frac{Gm_1}{r^2}\hat{r} - \frac{GM_s\vec{r}_2}{m_1+m_2}\left[\frac{m_1}{|\gamma\vec{x}+\vec{r}_1|^3} + \frac{m_2}{|\gamma\vec{x}+\vec{r}_2|^3}\right] \quad (89)$$

Using Taylor Expansion upto 2^{nd} order,

$$\frac{d^2\vec{r}_1}{dt^2} \approx -\frac{3GM_sr_1}{\gamma^3x^3}Cos(\phi)Cos(\theta-\psi)\hat{x} - \frac{Gm_2}{r^2}\hat{r} - \frac{GM_sr_1}{\gamma^3x^3}\hat{r}_1$$
$$\frac{d^2\vec{r}_2}{dt^2} \approx \frac{3GM_sr_2}{\gamma^3x^3}Cos(\phi)Cos(\theta-\psi)\hat{x} - \frac{Gm_1}{r^2}\hat{r} - \frac{GM_sr_2}{\gamma^3x^3}\hat{r}_2 \quad (90)$$

Let us define 2 unit vectors $\vec{v}_1, \vec{v}_2$,
$\vec{v}_1 = Cos(\theta)\hat{i} + Sin(\theta)\hat{j}$ and $\vec{v}_2 = Sin(\theta)\hat{i} - Cos(\theta)\hat{j}$
$\hat{x} = Cos(\theta-\psi)\vec{v}_1 + Sin(\theta-\psi)\vec{v}_2$

$$Cos(\phi)Cos(\theta-\psi)\hat{x} = Cos(\phi)Cos^2(\theta-\psi)\vec{v}_1 + Cos(\phi)Cos(\theta-\psi)Sin(\theta-\psi)\vec{v}_2$$

Averaged out (Secular) effect would be,
$Cos(\phi)Cos(\theta-\psi)\hat{x} = \frac{1}{2}Cos(\phi)\vec{v}_1$

$$\begin{aligned}\frac{d^2\vec{r}_1}{dt^2} \approx& \frac{3GM_s r_1}{2\gamma^3 x^3}Cos(\phi)[-Cos(\theta)\hat{i} - Sin(\theta)\hat{j}] \\ &-\frac{Gm_2}{r^2}\hat{r} - \frac{GM_s}{\gamma^3 x^3}\vec{r}_1 \\ \frac{d^2\vec{r}_2}{dt^2} \approx& \frac{3GM_s r_2}{2\gamma^3 x^3}Cos(\phi)[Cos(\theta)\hat{i} + Sin(\theta)\hat{j}] \\ &-\frac{Gm_1}{r^2}\hat{r} - \frac{GM_s}{\gamma^3 x^3}\vec{r}_2 \end{aligned} \tag{91}$$

We can note that RHS of $\frac{d^2\vec{r}_1}{dt^2}$ equation in Eqn(91) and RHS of $\vec{F}_1$ in Eqn(87) are the same, only $m_1\frac{d^2\vec{r}_1}{dt^2} - \vec{F}_1 \neq 0$.

By adding the m_1, m_2 equations in Eqn(88) we get,

$$\begin{aligned}(m_1+m_2)\frac{d^2\vec{x}}{dt^2} =& -GM_s\gamma\vec{x}\left[\frac{m_1}{|\gamma\vec{x}+\vec{r}_1|^3} + \frac{m_2}{|\gamma\vec{x}+\vec{r}_2|^3}\right] \\ &-GM_s m_1\vec{r}_1\left[\frac{1}{|\gamma\vec{x}+\vec{r}_2|^3} - \frac{1}{|\gamma\vec{x}+\vec{r}_1|^3}\right]\end{aligned}$$

Following the same procedure of Taylor approximation and averaging out we get,

$$\begin{aligned}(m_1+m_2)\frac{d^2\vec{x}}{dt^2} =& -\frac{GM_s(m_1+m_2)}{\gamma^2 x^2}\hat{x} \\ &-\frac{3}{2}\frac{GM_s m r_1^2}{\gamma^4 x^4}\left[1 - \frac{3}{2}Cos^2(\phi)\right]\hat{x}\end{aligned} \tag{92}$$

Eqn(92) implies that in Newtonian method there is an insignificant level of inherent Apsidal precession. In principle we can not get rid off this precession. Hence the similarity we noted between Eqn(91) and Eqn(87) may not be really valid because Eqn(91) has an inherent Apsidal Precession. And Eqn(87) was derived after assuming a constant circular orbit for the Earth-Moon system around the Sun.

Let us now find the scalar version of Eqn(91) to estimate the precession rates predicted by it, By comparing the $\hat{i}, \hat{j}, \hat{k}$ components on RHS and LHS of Eqn(91) and Eqn(52) we get the scalar $\frac{d^2 r_1}{dt^2}$ equation.

$$\begin{aligned}&\frac{d^2 r_1}{dt^2} - r_1\left(\frac{d\phi}{dt}\right)^2 + \frac{Gm_2}{(r_1+r_2)^2} \\ &= r_1 Cos^2(\phi)\left(\frac{d\theta}{dt}\right)^2 - \frac{GM_s r_1}{\gamma^3 x^3}\left[1 - \frac{3}{2}Cos^2(\phi)\right]\end{aligned} \tag{93}$$

Let us simplify the analysis of the 2 equations Eqns(85,93) by splitting them into two limiting cases
case (i) when $\phi = 0$(XY-plane orbit) and
case (ii) when $r = r_0$(circular orbit).

(i), $\phi = 0$, Apsidal Precession From Eqn(85) we get,

$$E_\theta = \frac{1}{2}m\left(\frac{dr_1}{dt}\right)^2 + \frac{1}{2}\frac{A_\theta^2}{mr_1^2} - \frac{GMm}{r_1} - \frac{1}{2}\frac{GM_s m}{2\gamma^3 x}\left(\frac{r_1}{x}\right)^2 \tag{94}$$

limit(i), $r_1 \approx r_{10}$ and $\phi = 0$

Use Eqn(94) the XY-plane orbit approximation to estimate the pure apsidal precession rate.
Let $r_1 = r_{10} + h, h << r_{10}$ and r_{10} = Constant. Let us use some approximate expansion of r_1^2 in Eqn(94)

$r_1^2 = r_{10}^2\frac{r_1^2}{r_{10}^2} = r_{10}^2\frac{r_1^2}{(r_1-h)^2} = r_{10}^2\frac{1}{\left(1-\frac{h}{r_1}\right)^2}$

$r_1^2 = r_{10}^2\left[1 + 2\frac{h}{r_1} + 3\frac{h^2}{r_1^2}\right]$, $r_1^2 = r_{10}^2\left[6 - 8\frac{r_{10}}{r_1} + 3\frac{r_{10}^2}{r_1^2}\right]$

Introducing higher order $\frac{1}{r_1^3}, \frac{1}{r_1^4}, ..$ terms into Eqn(94) is undesirable and if we approximate them back to forms with only $\frac{1}{r_1}$ and $\frac{1}{r_1^2}$ terms then the approximation more or less reduces to the 2^{nd} order approximation already shown. So that seems to be the best approximation.

$$\begin{aligned}E_\theta =& \frac{1}{2}m\left(\frac{dr_1}{dt}\right)^2 + \frac{1}{2}\frac{A^2}{mr_1^2}\left[1 - \frac{3}{2}\frac{M_s}{M}\frac{r_{10}^3}{x^3}\right] \\ &-\frac{GMm}{r_1}\left[1 - \frac{2M_s r_{10}^3}{Mx^3}\right] - \frac{3GM_s m r_{10}^2}{2x^3}\end{aligned} \tag{95}$$

Comparing Eqn(95) with the analysis of Apsidal precession in Eqns(25-26) and Eqns(66-74) we get,

$$r_1 \approx \frac{r_{10}}{1 + \varepsilon Cos\left(\theta_1\sqrt{1 - \frac{3}{2}\frac{M_s}{M}\frac{r_{10}^3}{x^3}}\right)} \tag{96}$$

At $\theta_1 = 0$, $r = \frac{r_{10}}{1+\varepsilon}$. If there was no Apsidal precession then At $\theta_1 = 2\pi$ we would get $r = \frac{r_{10}}{1+\varepsilon}$ again. But in Eqn(96) when $\theta_1\sqrt{1 - \frac{3}{2}\frac{M_s}{M}\frac{r_{10}^3}{x^3}} = 2\pi$ we get back $r = \frac{r_{10}}{1+\varepsilon}$. Or when $\theta_1 \approx 2\pi\left(1 + \frac{3}{4}\frac{M_s}{M}\frac{r_{10}^3}{x^3}\right)$. That is there builds up a phase difference of $2\pi\left(\frac{3}{4}\frac{M_s}{M}\frac{r_{10}^3}{x^3}\right)$ radians between the variation of θ_1 and r_1 in each cycle. That is r_1 gets delayed and the maxima(or minima) of r_1 keeps drifting from its previous position. It revolves in the same direction as θ_1. That is a prograde apsidal precession.

M_s = Mass of Sun = 1.989*10^{30} kg,
m_1 = Mass of Earth = 5.972*10^{24} kg
m_2 = Mass of Moon = 7.348*10^{22} kg,
$\zeta = 1 + \frac{m_1}{m_2} = 82.27$, $M = \frac{m_2}{\zeta^2} = 0.1086 * 10^{20}$ kg
x ≈ Distance between Sun and Earth = 147.1*10^9 m
r_{00} = Earth-Moon Distance = 0.3844*10^9 m
r_{10} = Distance between Earth and the Center of Mass of Earth-Moon system = $\frac{r_{00}}{\zeta}$ = 4.672*10^6 m

Therefore $1 - \frac{3}{4}\frac{M_s}{M}\frac{r_{10}^3}{x^3} = 0.9956$ Therefore,

$$r_1 \approx \frac{r_{10}}{1 + \varepsilon Cos(0.9956 * \theta_1)} \tag{97}$$

Phase difference in each cycle = $\Phi = 2\pi\left(\frac{3}{4}\frac{M_s}{M}\frac{r_{10}^3}{x^3}\right)$
$\Phi = 2\pi * 0.0044$ radians.

No. of cycles needed for a phase difference of $2\pi = \frac{2\pi}{\Phi} = \frac{1}{0.0044} = 227.3$ cycles.
Each Moon cycle (Synodic Month) = 29.5 Days
So, 227.3 cycles $\implies$ 18.36 years... But the observed Lunar apsidal precession rate is 8.85 years.

(i), $\phi = 0$, Apsidal Precession From Eqn(93) we get,

$$\frac{d^2r_1}{dt^2} + \frac{Gm_2}{(r_1+r_2)^2} = r_1\left(\frac{d\theta}{dt}\right)^2 + \frac{GM_s r_1}{2\gamma^3x^3} \quad (98)$$

Because it is in 1 variable we can get away with the following analysis, extending such equations through the potential gradient method is flawed and should not be used. So let us consider the $\frac{dE_\theta}{dr_1} = 0$ from Eqn(94).

$$\frac{1}{m}\frac{dE_\theta}{dr_1} = \frac{d^2r_1}{dt^2} - r_1\left(\frac{d\theta}{dt}\right)^2 + \frac{Gm_2}{(r_1+r_2)^2} - \frac{GM_s}{2\gamma^3x^3}r_1$$

$$i.e., \frac{d^2r_1}{dt^2} + \frac{Gm_2}{(r_1+r_2)^2} = r_1\left(\frac{d\theta}{dt}\right)^2 + \frac{GM_s r_1}{2\gamma^3x^3} \quad (99)$$

Both Method $\mathcal{A}$ and $\mathcal{B}$ predict the same(wrong) apsidal precession rate. Predicted precession rate 18.36 years. Observed Lunar apsidal precession 8.85 years.

(ii), $r_1 = r_{10}$, Nodal Precession From Eqn(85) we get

$$E_\theta = \frac{1}{2}\frac{A_\theta^2}{mr_{10}^2Cos^2(\phi)} + \frac{1}{2}mr_{10}^2\left(\frac{d\phi}{dt}\right)^2 - \frac{GMm}{r_{10}} + \frac{GM_smr_{10}^2}{2\gamma^3x^3}\left[1 - \frac{3}{2}Cos^2(\phi)\right]$$

limit(ii), $r_1 = r_{10}$ and $\phi \approx 0$ Using the approximation $1 - \frac{3}{2}Cos^2(\phi) \approx \frac{3}{2Cos^2(\phi)} - 2$ when $\phi \approx 0$

$$E_\theta + \frac{GM_sm}{\gamma^3x}\left(\frac{r_{10}}{x}\right)^2 + \frac{GMm}{r_{10}} = \frac{1}{2}mr_{10}^2\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}\frac{A_\theta^2}{mr_{10}^2Cos^2(\phi)}\left[1 + \frac{3}{2}\frac{M_s}{M}\frac{r_{10}^3}{x^3}\right] \quad (100)$$

Comparing Eqn(100) with the analysis of Apsidal precession in Eqns(25,26) and Eqns(66-74) we get,

$$\phi \approx \phi_{max}Sin\left(\theta_1\sqrt{1 + \frac{3}{2}\frac{M_s}{M}\frac{r_{10}^3}{x^3}}\right), \phi_{max} \approx 0$$

$$\phi \approx \phi_{max}Sin\left(1.0044 * \theta_1\right) \quad (101)$$

Thus energy method predicts a retrograde nodal precession of 18.36 years period. The observed Lunar nodal precession is retrograde with 18.6 years period.

limit(ii), $r_1 = r_{10}$ From Eqn(93) with Newton's method we get,

$$-r_{10}\left(\frac{d\phi}{dt}\right)^2 + \frac{GM}{r_{10}^2} = r_{10}Cos^2(\phi)\left(\frac{d\theta}{dt}\right)^2 - \frac{GM_sr_{10}}{\gamma^3x^3}\left[1 - \frac{3}{2}Cos^2(\phi)\right] \quad (102)$$

Multiply by $\times\frac{1}{2}mr_{10}$

$$-\frac{1}{2}mr_{10}^2\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}\frac{GMm}{r_{10}} = \frac{1}{2}\frac{A_\theta^2}{mr_{10}^2Cos^2(\phi)}\left(\frac{d\theta}{dt}\right)^2 - \frac{GM_smr_{10}^2}{2\gamma^3x^3}\left[1 - \frac{3}{2}Cos^2(\phi)\right]$$

limit(ii), $r_1 = r_{10}$ and $\phi \approx 0$, $E_\theta \approx \frac{1}{2}\frac{GMm}{r_{10}}$ Using the approximation $1 - \frac{3}{2}Cos^2(\phi) \approx \frac{3}{2Cos^2(\phi)} - 2$ when $\phi \approx 0$

$$-\frac{1}{2}mr_{10}^2\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}\frac{GMm}{r_{10}} = \frac{1}{2}\frac{A_\theta^2}{mr_{10}^2Cos^2(\phi)}\left(\frac{d\theta}{dt}\right)^2\left[1 - \frac{3}{2}\frac{M_s}{M}\frac{r_{10}^3}{x^3}\right] + \frac{GM_smr_{10}^2}{\gamma^3x^3}$$

Comparing Eqn(100) with the analysis of Apsidal precession in Eqns(25,26) and Eqns(66-74) we get,

$$\phi \approx \phi_{max}Sin\left(\theta_1\sqrt{1 - \frac{3}{2}\frac{M_s}{M}\frac{r_{10}^3}{x^3}}\right), \phi_{max} \approx 0$$

$$\phi \approx \phi_{max}Sin\left(0.9956 * \theta_1\right) \quad (103)$$

Newton's method predicts a **prograde** nodal precession of 18.36 years period. The direction of precession is opposite to that of the observed.

The value of apsidal precession rate predicted by both Newton's and Energy method is also 18.36 years instead of the observed 8.85 years. Perhaps we might need to reframe Eqn(75) itself as adding higher order approximations do not seem to improve the situation. Also we can not solve the full equation Eqn(85) because we have 2 variables r_1, ϕ and only 1 equation. We need to derive laws regarding ϕ evolution from 3D observations.

However if we can alter a term of the perturbation function in Eqn(80) to,

$$\Delta_{av} = \frac{GM_sm}{2\gamma^3x}\left(\frac{r_1}{x}\right)^2\left[\frac{1}{2} - \frac{3}{2}Cos^2(\phi)\right]$$

Then we get close to the observed apsidal precession rate without altering the calculated nodal precession rate.

VII. MOON ORBIT PERTURBED BY OBLATE EARTH

A. J_2 Perturbed Orbit

The effect due to oblateness(which is the supposed reason for Nodal precession of near Earth satellites) of the Earth is not as easily integrable as the ideal perturbation terms $\frac{1}{2}\frac{B}{mr^2Cos^2(\phi)}$ and $\frac{1}{2}\frac{C}{mr^2}$ in Eqn(66). The oblateness geopotential factor is represented by $J_2\frac{GMR^2}{r^3}(1 - \frac{3}{2}Cos^2(\phi))$ where $J_2 = 1.082 * 10^{-3}$. (Frick and Garber, 1962)

We can modify Eqn(55) to include the J_2 perturbation,

$$E= \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A^2}{mr^2Cos^2(\phi)} + \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 + \frac{GMm}{R} - \frac{GMm}{r} + J_2\frac{GMmR^2}{r^3}\left(1-\frac{3}{2}Cos^2(\phi)\right) \tag{104}$$

In Eqn(104) we can not make a similar assumption as in Eqn(68) because it gives contradictory results.

Eqn(104) in vector form of kinetic energy,

$$E = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}\right) + \frac{GMm}{R} - \frac{GMm}{r} + J_2\frac{GMmR^2}{r^3}\left(1-\frac{3}{2}Cos^2(\phi)\right)$$

$$\frac{dE}{dt}= m\frac{d^2\vec{r}}{dt^2}\bullet\frac{d\vec{r}}{dt} + \frac{GMm}{r^2}\hat{r}\bullet\frac{d\vec{r}}{dt} - 3J_2\frac{GMmR^2}{r^4}Cos(\phi)(Cos(\theta)\hat{i}+Sin(\theta)\hat{j})\bullet\frac{d\vec{r}}{dt} - 3J_2\frac{GMmR^2}{r^4}\left(1-\frac{5}{2}Cos^2(\phi)\right)\hat{r}\bullet\frac{d\vec{r}}{dt} \tag{105}$$

$(Cos(\theta)\hat{i}+Sin(\theta)\hat{j})\bullet\frac{d\vec{r}}{dt} = \frac{dr}{dt}Cos(\phi) - rSin(\phi)\frac{d\phi}{dt}$

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} + 3J_2\frac{GMmR^2}{r^4}\left(1-\frac{5}{2}Cos^2(\phi)\right)\hat{r} + 3J_2\frac{GMmR^2}{r^4}Cos(\phi)(Cos(\theta)\hat{i}+Sin(\theta)\hat{j}) \tag{106}$$

When E is conserved, Eqns(105, 106) imply

$$\frac{dE}{dt} = \left[m\frac{d^2\vec{r}}{dt^2} - \vec{F}\right]\bullet\frac{d\vec{r}}{dt} = 0$$

If $m\frac{d^2\vec{r}}{dt^2} - \vec{F} = 0$ then we get the scalar equation,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 + \frac{GM}{r^2} = rCos^2(\phi)\left(\frac{d\theta}{dt}\right)^2 + \frac{3J_2GMR^2}{r^4}\left[1-\frac{3}{2}Cos^2(\phi)\right] \tag{107}$$

Let us compare the 2 equations Eqn(104)(Energy) and Eqn(107)(Newton) in the 2 limits when $r = r_0$(circular orbit) and $\phi = 0$(XY-plane orbit).

<u>(i), $\phi = 0$</u> From Eqn(104) we get,

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A^2}{mr^2} + \frac{GMm}{R} - \frac{GMm}{r} - \frac{1}{2}J_2\frac{GMmR^2}{r^3} \tag{108}$$

From Eqn(107) we get,

$$\frac{d^2r}{dt^2} + \frac{GM}{r^2} = \frac{A^2}{m^2r^3} - \frac{3}{2}\frac{J_2GMR^2}{r^4} \tag{109}$$

Like in the case of restricted Sun-Earth-Moon 3-body system, we get the same equation from both Method $\mathcal{A}$, Eqn(107) and Method $\mathcal{B}$, Eqn(104) when $\phi = 0$ in case of J_2 perturbed apsidal precession.

<u>Calculating Apsidal Precession</u> Expand $\frac{1}{2}J_2\frac{GMmR^2}{r^3}$ in Eqn(108) in the limit $r = r_0 + h, h << r_0$, to get $\frac{1}{r^2}$ term like in the ideal perturbation term,

$$\begin{aligned}\frac{1}{2}J_2\frac{GMmR^2}{r^3} &\approx \frac{1}{2}J_2\frac{GMmR^2}{r^2(r_0+h)} \\ &\approx \frac{1}{2}J_2\frac{GMmR^2}{r^2r_0}\left(1-\frac{h}{r_0}+\frac{h^2}{r_0^2}\right) \\ &\approx \frac{1}{2}J_2\frac{GMmR^2}{r^2r_0}\left(1-\frac{r-r_0}{r_0}+\frac{(r-r_0)^2}{r_0^2}\right) \\ &\approx \frac{1}{2}J_2\frac{GMmR^2}{r^2r_0}\left(3-\frac{3r}{r_0}+\frac{r^2}{r_0^2}\right)\end{aligned}$$

$$\frac{1}{2}J_2\frac{GMmR^2}{r^3} \approx \frac{3}{2}J_2\frac{GMmR^2}{r^2r_0} - \frac{3}{2}J_2\frac{GMmR^2}{rr_0^2} + \frac{1}{2}J_2\frac{GMmR^2}{r_0^3} \tag{110}$$

Thus, the $\frac{1}{2}\frac{A^2}{mr^2}$ term in Eqn(108) is affected by $-\frac{3}{2}J_2\frac{GMmR^2}{r^2r_0}$ term in Eqn(110),

$$\frac{1}{2}\frac{A^2}{mr^2} - \frac{3}{2}J_2\frac{GMmR^2}{r^2r_0} = \frac{1}{2}\frac{A^2}{mr^2}\left[1-3J_2\frac{GMm^2R^2}{r_0A^2}\right]$$

Noting $A^2 \approx GMm^2r_0$ we get,

$$\frac{1}{2}\frac{A^2}{mr^2} - \frac{3}{2}J_2\frac{GMmR^2}{r^2r_0} \approx \frac{1}{2}\frac{A^2}{mr^2}\left[1-3J_2\frac{R^2}{r_0^2}\right] \tag{111}$$

The factor $1 - 3J_2\frac{R^2}{r_0^2}$ appearing with $\frac{1}{2}\frac{A^2}{mr^2}$ term in Eqn(111) deterimines the apsidal precession rate as we get a solution of the form,

$$r \approx \frac{r_0}{1+\varepsilon Cos\left(\theta\sqrt{1-3J_2\frac{R^2}{r_0^2}}\right)} \tag{112}$$

Assuming R = 6.378*10^6 m and r_0 = (0.75+6.378)*10^6 m. i.e. an orbit at ≈ 750 km altitude. we get,

$$r \approx \frac{r_0}{1+\varepsilon Cos(0.9987*\theta)}$$

In each cycle the phase difference is $2\pi*(1-0.9987) = 2\pi*0.0013$

Period of 1 cycle = T = $\sqrt{\frac{4\pi^2r_0^3}{Gm_2}}$

No. of cycles in 24 hours = C = $\frac{86400}{T}$

Total Phase Difference in 24 hours = $2\pi*0.0013*C$ = 6.75 Deg/Day. Observed is about 13.5 Deg/Day.

Both Method $\mathcal{A}$ and $\mathcal{B}$ predict only half the observed apsidal precession rate. Similar to the case of Sun-Earth-Moon apsidal precession noted after Eqn(99).

(ii), $r = r_0$ From Eqn(104) we get,

$$E = \frac{1}{2}\frac{A^2}{mr_0^2Cos^2(\phi)} + \frac{1}{2}mr_0^2\left(\frac{d\phi}{dt}\right)^2 + \frac{GMm}{R} - \frac{GMm}{r_0} + J_2\frac{GMmR^2}{r_0^3}\left(1 - \frac{3}{2}Cos^2(\phi)\right) \quad (113)$$

$E \approx \frac{GMm}{R} - \frac{GMm}{2r_0}$

$$r_0\left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r_0^2} - \frac{A^2}{m^2r_0^3Cos^2(\phi)} - \frac{2J_2GMR^2}{r_0^4}\left(1 - \frac{3}{2}Cos^2(\phi)\right) \quad (114)$$

From Eqn(107) we get,

$$r_0\left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r_0^2} - \frac{A^2}{m^2r_0^3Cos^2(\phi)} - \frac{3J_2GMR^2}{r_0^4}\left(1 - \frac{3}{2}Cos^2(\phi)\right) \quad (115)$$

We get different equations from Method $\mathcal{A}$, Eqn(115) and Method $\mathcal{B}$, Eqn(114) when $\phi \neq 0$ in case of J_2 perturbed orbits. Because in Method $\mathcal{A}$ the perturbation term is, $-\frac{3J_2GMR^2}{r_0^4}\left(1 - \frac{3}{2}Cos^2(\phi)\right)$ whereas in Method $\mathcal{B}$ the perturbation term is, $-\frac{2J_2GMR^2}{r_0^4}\left(1 - \frac{3}{2}Cos^2(\phi)\right)$, lowered by a factor of 1.5. But Method $\mathcal{B}$, Eqn(114) predicts the correct nodal precession rate while Newton's method underestimates the period of precession.

Calculating Nodal Precession Using $\phi \approx 0$ we can derive $1 - \frac{3}{2}Cos^2(\phi) \approx \frac{3}{2Cos^2(\phi)} - 2$

The term $\frac{1}{2}\frac{A^2}{mr_0^2Cos^2(\phi)}$ in Eqn(113) gets modulated by the term $J_2\frac{GMmR^2}{r_0^3}\left(\frac{3}{2Cos^2(\phi)} - 2\right)$ lets consider only the terms with $\frac{1}{Cos^2(\phi)}$ factor

$$\frac{1}{2}\frac{A^2}{mr_0^2Cos^2(\phi)} + J_2\frac{GMmR^2}{r_0^3}\frac{3}{2Cos^2(\phi)} = \frac{1}{2}\frac{A^2}{mr_0^2Cos^2(\phi)}\left[1 + 3J_2\frac{GMm^2R^2}{r_0A^2}\right] = \frac{1}{2}\frac{A^2}{mr_0^2Cos^2(\phi)}\left[1 + 3J_2\frac{R^2}{r_0^2}\right] \quad (116)$$

Eqn(116) has nodal precession solution given by,

$$\phi \approx \phi_{max}Sin\left(\theta\sqrt{1 + 3J_2\frac{R^2}{r_0^2}}\right), \phi_{max} \approx 0 \quad (117)$$

Total Phase Difference in 24 hours $= 2\pi * 0.0013 * C = 6.75$ Deg/Day. This is the correct rate of nodal precession.

Eqn(116) based on Energy method predicts the correct nodal precession rate, whereas Newton's method gives,

$$\phi \approx \phi_{max}Sin\left(\theta\sqrt{1 + \frac{9}{2}J_2\frac{R^2}{r_0^2}}\right), \phi_{max} \approx 0 \quad (118)$$

Total Phase Difference in 24 hours $= 2\pi * 0.0013 * C =$ 10.13 Deg/Day. The correct rate is 6.75 Deg/Day.

However if we alter a term from the J_2 perturbation function in Eqn(104),

$$J_2\frac{GMmR^2}{r^2}\left(1 - \frac{3}{2}Cos^2(\phi)\right) \rightarrow \rightarrow J_2\frac{GMmR^2}{r^2}\left(\frac{1}{2} - \frac{3}{2}Cos^2(\phi)\right)$$

Then we get the desired apsidal precession rate without any change to the calculated nodal precession rate in the Energy method. However the nodal prcession rate from Newton's method would still be wrong.

Thus, not only Newton's method of framing equations in terms of $\frac{d^2\vec{r}}{dt^2}$ is wrong(in general) even the idea of Inertia is problematic.

Definition of Inertia Let us look at the original experiment conducted by Galileo. He was rolling balls down an inclined plane and he noticed that if he connected another inclined plane rising up in height as the first inclined plane touches ground, the ball will rise up the new inclined plane. It will rise up to as much height as it originally had when it was rolled down the first plane. He called this height conserving property as Intertia (Goodstein, 1985). Today we know that it is the energy conservation that causes the ball to rise up the second inclined plane. So we should replace the term Inertia by its equivalent term Energy.

VIII. CONCLUSION

In the Kepler-1-Body approximation if E is the total energy and $\vec{F}$ is the Newtonian Gravitational Force on mass m then, $E = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\bullet\frac{d\vec{r}}{dt}\right) - \frac{GMm}{r}$ and $\vec{F} = -\frac{GMm}{r^3}\vec{r}$ therefore $\frac{dE}{dt} = \left[m\frac{d^2\vec{r}}{dt^2} - \vec{F}\right]\bullet\frac{d\vec{r}}{dt}$. Thus, Newton's Force equation $m\frac{d^2\vec{r}}{dt^2} - \vec{F} = 0$ is only a subset of all possible solutions of $\frac{dE}{dt} = 0$. In certain cases like the inclined plane and brachistochrone problems near Earth surface we find that $m\frac{d^2\vec{r}}{dt^2} - \vec{F} \neq 0$ but $m\frac{d^2\vec{r}}{dt^2} - \vec{F} \perp \frac{d\vec{r}}{dt}$. Because we have to make free body diagrams and determine the force components based on the particular problem setup, we have to appropriately modify Newton's law as, $m\frac{d^2\vec{r}}{dt^2}$ = some undermined component of $\vec{F}$ or $-\frac{GMm}{r^3}\vec{r}$.

In 1-body approximation problems in 2D, it is hard to miss the relationship between spatial derivatives of Energy and Force components. That is if $E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{A^2}{mr^2} - \frac{GMm}{r}$ then $\frac{dE}{dr} = m\frac{d^2r}{dt^2} - \frac{A^2}{mr^3} + \frac{GMm}{r^2} = 0$. Thus radial acceleration $= \frac{d^2r}{dt^2} = \frac{A^2}{mr^3} - \frac{GMm}{r^2}$. We could get swayed by this property and extend it to all other problem configurations. It is usually assumed that if net energy (E) is conserved then $m\frac{d^2\vec{r}}{dt^2} = -\nabla P$. But it is true only in some special cases, in general $m\frac{d^2\vec{r}}{dt^2} =$ Some component of $-\nabla P$. This is a relationship which can not be generalised to multi-body systems. Above all just defining the gradients of net Energy of an object(or the whole system) and defining the gradients of individual particle kinetic energies is a dubious operation. Best is to not perform them.

The correct relationship between Force and Energy (if needed for any calculation) is arrived at by extending equation of the type $\frac{dE}{dt} = \left[m\frac{d^2\vec{r}}{dt^2} - \vec{F}\right] \bullet \frac{d\vec{r}}{dt}$, through time derivative of Energy not its gradients.

In the Kepler-N-Body (N > 2) problem if $\vec{F}_i$ is the sum of all the Newtonian gravitational force balance terms on mass m_i then $\vec{F}_i = -\sum_{j=1,j\neq i}^{j=N} \frac{Gm_im_j}{|\vec{r}_i - \vec{r}_j|^3}(\vec{r}_i - \vec{r}_j)$. If E is the net Energy of the N-body system then,
$\frac{dE}{dt} = \sum_{i=1}^{i=N} \left[m_i\frac{d^2\vec{r}_i}{dt^2} - \vec{F}_i\right] \bullet \frac{d\vec{r}_i}{dt}$. Even when $\frac{dE}{dt} = 0$ it is not necessary that $m_i\frac{d^2\vec{r}_i}{dt^2} - \vec{F}_i = 0 \; \forall \; i$. For example in the case of Lagrange type periodic orbits solution of the 3-body problem we see that Newtons method does not cover the entire solution(incomplete) space.

1-body approximation is an extreme case of general 2-body systems. In 2-body systems in 2D if the net Energy of the system and individual Angular Momenta are conserved and if the Energy equation can be written down entirely in terms of one variable then both Newton's Method and Energy method give the same equations. Because both the time derivative of Energy and its gradients give same force equations.

In case of both Lunar orbit and orbits of artificial satellites around Earth we get the same Apsidal precession rates from both Newton's method(Method $\mathcal{A}$) and Energy method(Method $\mathcal{B}$) and both methods underestimate the rate of apsidal precession by half. However, we get different results for Nodal precession rates from Newton's method and Energy method. Energy method gives results closely corresponding to the observed nodal precession. In case of Lunar nodal precession Newton's method gets the nodal precession rate correct but in the wrong direction and in case of near Earth satellite Nodal precession, Newton's method gets the direction of precession right but the rate of precession has a large(33%) error.

Thus we ought to accept that the form of the gravitational potential energy term is more universal than the form of Newton's Law of Gravity yet those energy terms may not still explain the observed precession rates. Hence we need to replace Newton's Laws of Motion by an Energy conservation principle. And we need to find more observational laws governing the 3D and N-body trajectories to properly frame and solve the equations of motion.

REFERENCES

Bate, R. R., Mueller, D. D. and White, J. E., (1971), *Fundamentals of Astrodynamics*, (Dover)

Brown, E. W., (2016), *An Introductory Treatise on the Lunar Theory*, (Wentworth Press)

Curtis, H. D., (2013), *Orbital Mechanics for Engineering Students 3rd Edition*, (Butterworth-Heinemann)

Curtis, W., (2010), *The Hill-Brown Theory of the Moons Motion*, (Springer)

Dillon, J., (2016), *Before Telescopes 7 Tycho Brahe & Joannes Kepler*, (Robert Ferguson Observatory on youtube.com)

Dreyer, J. L. E., (1906), *History of the planetary systems from Thales to Kepler*, (Cambridge University Press, archive.org)

Frick, R. H. and Garber, T. B., (1962), "Perturbations of a Synchronous Satellite," The Rand Corp. **R-399-NASA**, –

Goldstein, H., Poole Jr, C. P. and Safco, J. L., (2001), *Classical Mechanics 3rd Edition*, (Pearson)

Goodstein, D., (1985), *Episode 4: Inertia - The Mechanical Universe*, (Caltech, intelecom.org and The Annenberg/CPB Project)

Lankford, J. and Rothenberg, M., (1997), *History of Astronomy: An Encyclopedia*, (Taylor & Francis)

Radhakrishnamurty, P., (2019), "Brachistochrone - The Path of Quickest Descent," Resonance, **24**, –

Sinha, M., (2013), *Space Flight Mechanics*, (nptelhrd on infocobuild.com and youtube.com)

Thoren, V. E., (1973), *Tycho Brahe: Past and Future Research*, (Cambridge University Press, archive.org)

Vallado, D. A., McClain, W. D. and Larson, W. J., (1997), *Fundamentals of Astrodynamics and Applications*, (McGraw-Hill, Space Tech. Series)