Nested Radicals and Gray Code
Simpler methods of converting integers to Gray code

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Abstract

There is a characteristic form of nested radicals that contains many signs (+, -) and is expressed in a closed trigonometric form. When the value of the trigonometric expression is known, we are called to find the values of the signs of the radical representation (the reverse is easier). The solution is provided directly by the Gray code of an integer contained in the known member of this equation of signs. However, the method of converting integers to Gray code differs from the standard method and is easier. The reason is that I found the equivalent method without knowing the Gray code. At the same time, a model emerged that interprets and facilitates the conversion of a group of integers into Gray code.

This is a simplified description of how to deal with such equations.

We consider the following form

$$\underbrace{\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}}_{n\text{ roots}}$$

This expression is divided into two parts

$$\underbrace{\sqrt{2+\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}}}_{n\text{ roots}}=2\cos\left(\frac{90^\circ(2a+1)}{2^n}\right)$$

$$\underbrace{\sqrt{2-\sqrt{2\pm\sqrt{2\pm\sqrt{2\pm...\pm\sqrt{2}}}}}}_{n\text{ roots}}=2\sin\left(\frac{90^\circ(2a+1)}{2^n}\right)$$

where

$$a\in\{0,1,2,...,2^{n-2}-1\}$$

If the values of n and 'a' are known then we can easily find the unknown signs (±) of the radical representation by the following way.

My method for converting an integer to Gray code (binary) uses successive divisions by powers of 2 and looks at the parity of the rounded quotient.

Example with n=8, a=29

$$29/2 = 14.5 \approx 15 \Rightarrow 1$$

$$29/4 = 7.25 \approx 7 \Rightarrow 1$$

$$29/8 = 3.625 \approx 4 \Rightarrow 0$$

$$29/16 = 1.8125 \approx 2 \Rightarrow 0$$

$$29/32 = 0.90625 \approx 1 \Rightarrow 1$$

$$29/64 = 29/2^{n-2} = 0.453... \approx 0 \Rightarrow 0$$

The decimal value 29 has the binary value 10011 in Gray code.

However, since we have to divide 29 by the power 2^{n-2}, we have to include in Gray the code of 29 and the digit 0 of the last division, ie we get the result 010011.

Now we correspond these digits to the signs by setting 0 = +, 1 = –. We get this:

$+-++--$

This string is the solution for constructing the radical representation - here let's choose the expression sin():

$$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2}}}}}}}} = 2\sin\left(\frac{90^\circ\,(2\cdot 29+1)}{2^8}\right)$$

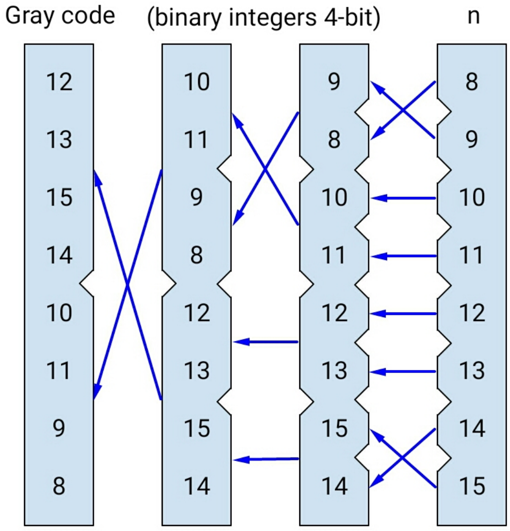
If the radical representation is known, then by definition the value of n will also be known. Then the value of 'a' can be obtained from the relationship

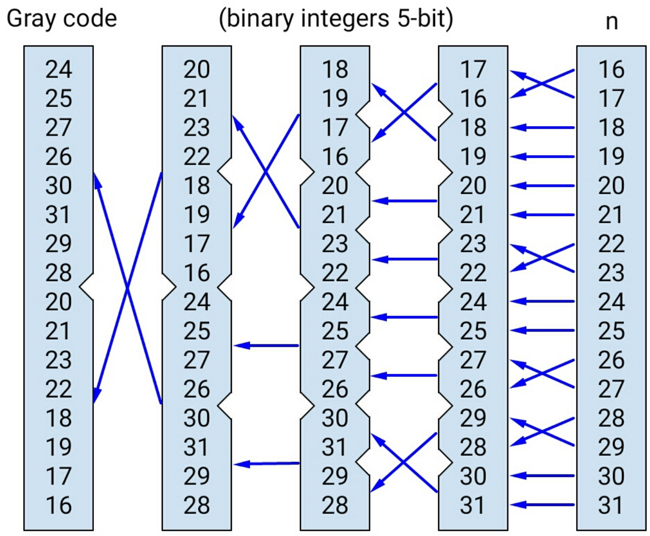
$$a = \frac{2^8\sin^{-1}\left(0.5\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2}}}}}}}}\right)}{180^\circ} - 0.5 = 29$$

We can also find the value of "a" by decoding the value of the gray code formed by the signs.

The following tables show the characteristic matching pattern in the area of four-digit binary numbers (8-15) and five-digit binary numbers (16-31). The numbers that are rearranged in Gray code are arranged in the last column. There are two ways to transfer groups of numbers from column to column, crosswise (×) and parallel (=).

Eg if n=14 then Gray code = 9 (and vice versa).





The arrangement of the arrows is determined by the Thue Morse sequence:

0110100110010110...

We omit the first digit and divide the sequence into groups of digits that double in each subsequent term:

1, 10, 1001, 10010110, ...

Each term in this sequence is created by copying the previous term, which we repeat on the right side of the term, after first reversing its digits. Then we make the substitutions by setting 0 = "=" and 1 = ".". This is how we get:

$\times, \times =, \times = = \times, \times = = \times = \times \times =, \ldots$

These terms represent arrow pairs and are read from top to bottom in the consecutive columns of the array.

Some observations that can help in further research, especially when 'n' tends to infinity

When the signs of the radical representation form a repetitive pattern, then 'a' will be of the form int(2n/m), where m is a constant number for each value of n (eg m=6, 9.5, 7.25, 9.333...). In place of m, other numbers can be placed, m2, m3, ... for which the same pattern formed by the signs starts from a different point in the radical representation. Often, the numbers m2, m3, ... are quotient of fractions m/1, m/2, m/3, m/4, ... In all cases, the pattern of the binary representation of 'a' is also periodic.

Examples

$$++++\ldots \quad a = 0$$
$$----\ldots \quad a = \mathrm{int}\left(\frac{2^{\mathrm{n}}}{6}\right) = 1010\ldots_2$$
$$+-+-\ldots \quad a = \mathrm{int}\left(\frac{2^{\mathrm{n}}}{10/1}\right) = 110000110000\ldots_2$$
$$-+-+\ldots \quad a = \mathrm{int}\left(\frac{2^{\mathrm{n}}}{10/2}\right) = 110000110000\ldots_2$$
$$+--+--\ldots \quad a = \mathrm{int}\left(\frac{2^{\mathrm{n}}}{14/1}\right) = 100100\ldots_2$$
$$--+--+\ldots \quad a = \mathrm{int}\left(\frac{2^{\mathrm{n}}}{14/2}\right) = 100100\ldots_2$$
$$-+--+-\ldots \quad a = \mathrm{int}\left(\frac{2^{\mathrm{n}}}{14/3}\right) = 110000110000\ldots_2$$
$$\textit{General form}: \; a = \mathrm{int}\left(\frac{2^{\mathrm{n}}}{m/q}\right), \; m, q \in \mathbb{Z}$$

Seth Zimmerman explores the same radical expression where n tends to infinity.

It is fair to clarify a few things.

I have known all this since 2004. Until 2015, I had nothing to do with the internet. That year, in April 2015, I published the first relevant article at the following address (the official website of the Hellenic Mathematical Society).

https://www.mathematica.gr/forum/viewtopic.php?t=49175
File name (in Greek): ΡιζΤριγΣυν-I.doc (free)
Work title: "Ριζικές Τριγωνομετρικές Συναρτήσεις" (Radical Trigonometric Functions).

Several mathematicians worked on the same problem. I mention here those who have achieved substantial results, with respectfully to all of those who I do not mention.

Pierluigi Vellucci came up with a similar solution based on an observation linking the nested radicals to the Gray code. Apparently, he matched the signs in the binary digits and sorted the resulting values according to the size of the values of the radical representation. He then examined the relevant sequence in OEIS, finding that this is the sequence of the Gray code. Pierluigi provides proofs of his findings.

Jayantha Senadheera has also come up with a solution, but his method is difficult and complicated.

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References

Pierluigi Vellucci and Alberto Maria Bersani, "Ordering of nested square roots of 2 according to Gray code" (2016).

Pierluigi Vellucci and Alberto Maria Bersani, "New formulas for π involving infinite nested square roots and Gray code" (2018).

Jayantha Senadheera, "On the periodic continued radicals of 2 and generalization for Vieta's product" (2013).

Seth Zimmerman & Chungwu Ho, "On Infinitely Nested Radicals" (2008).

George Plousos, "Nested Radicals and Trigonometry"