The Riemann Hypothesis Proof

isaac mor
isaac.mor@hotmail.com

Abstract

I am using eta function because it extends the zeta function from Re(s) > 1 to the larger domain Re(s) > 0. I am going to use eta function spiral and its behavior of convergent points on the complex plane to get two functions f(x) and g(x). Then I am going to show when those two functions are eqaul to zero the spiral is converging to zero as well. I will then show that non trival zeros appears only when h(x) eqaul to zero. And also when function q(x) is eqauls to zeta(2a) then eta(a+ix) is a non trival zero. Then I am going to do convergence tests on the critical strip to show that there are no zeros on the strip other then the criticl line.

Contents

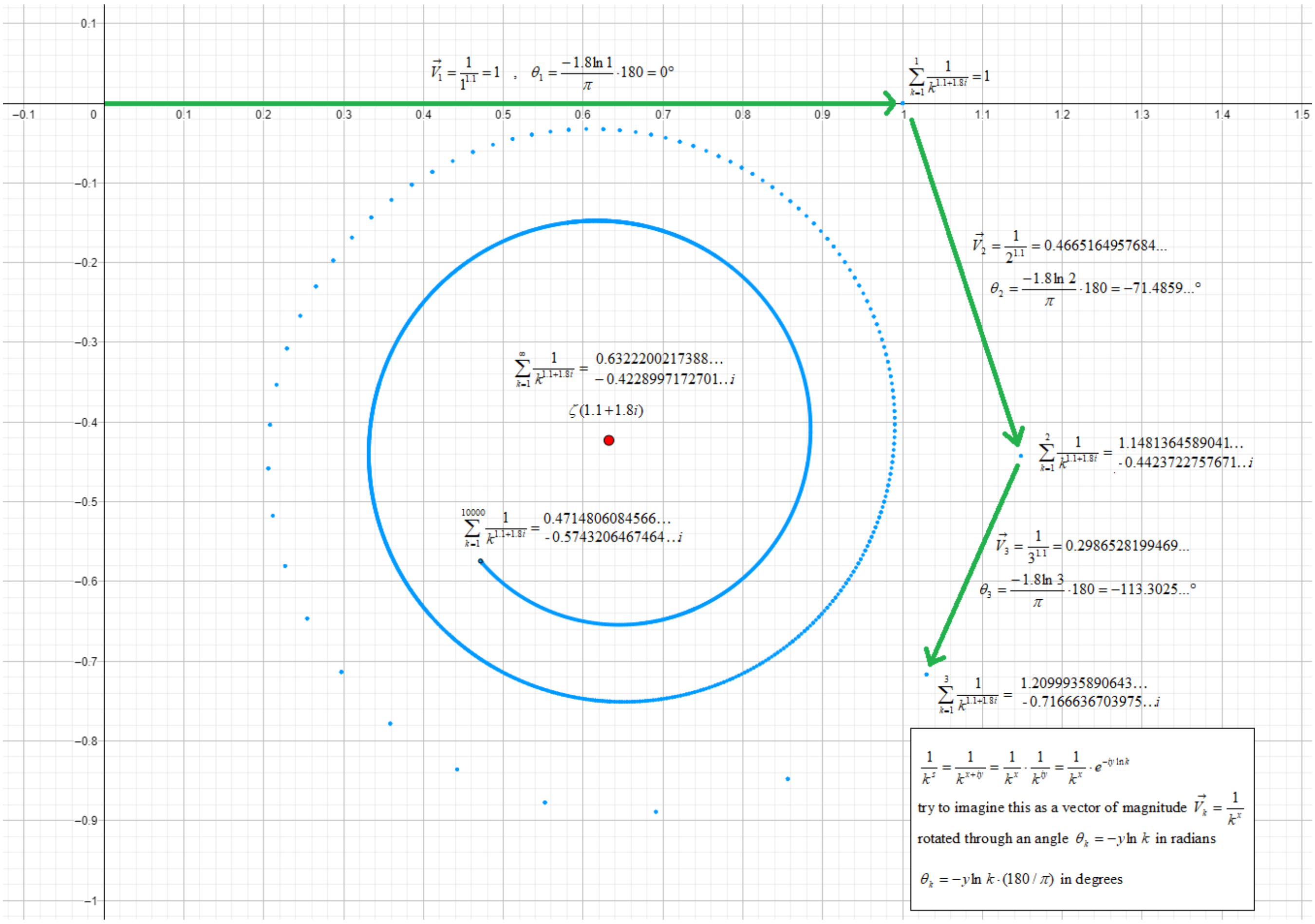
Picturing The Zeta Function

For me when I am looking at the zeta function I see “spirals” all around the grid.

$$\zeta(a+ib)=\left[\vec{V}_1\cdot\cos(\theta_1)+\vec{V}_2\cdot\cos(\theta_2)+\vec{V}_3\cdot\cos(\theta_3)+\ldots\right]+i\cdot\left[\vec{V}_1\cdot\sin(\theta_1)+\vec{V}_2\cdot\sin(\theta_2)+\vec{V}_3\cdot\sin(\theta_3)+\ldots\right]$$

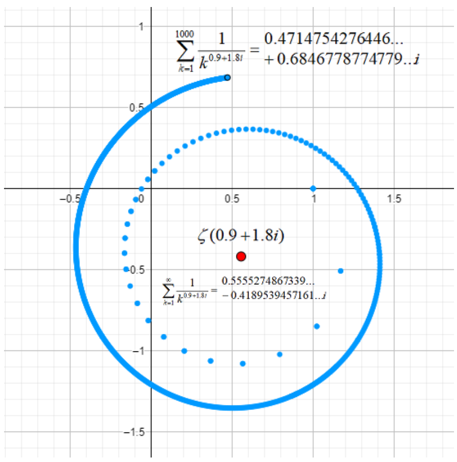
The simplest way is to first look at the behavior of convergent points on the complex plane $\zeta(a+ib)$ when $a>1$
The spiral swirls around inwards to an unique point which the series converges

$$\zeta(1.1+1.8i)=\left[\frac{1}{1^{1.1}}\cdot\cos(-1.8\ln 1)+\frac{1}{2^{1.1}}\cdot\cos(-1.8\ln 2)+\frac{1}{3^{1.1}}\cdot\cos(-1.8\ln 3)+\ldots\right]$$
$$+i\cdot\left[\frac{1}{1^{1.1}}\cdot\sin(-1.8\ln 1)+\frac{1}{2^{1.1}}\cdot\sin(-1.8\ln 2)+\frac{1}{3^{1.1}}\cdot\sin(-1.8\ln 3)+\ldots\right]=0.6322\ldots-0.4228\ldots i$$



when $a>1$ The spiral swirls around inwards to an unique point which the series converges but the Same goes for the other way around!

When I am looking at the Complex plane $\zeta(a+ib)$ where $a<1$ the series diverges the spiral swirls around outwards but if you look closely you will notice that the spiral has a “center point” or an “origin” and that “origin” is the “Assigned Values” or “Analytic Continuation” everyone is talking about.

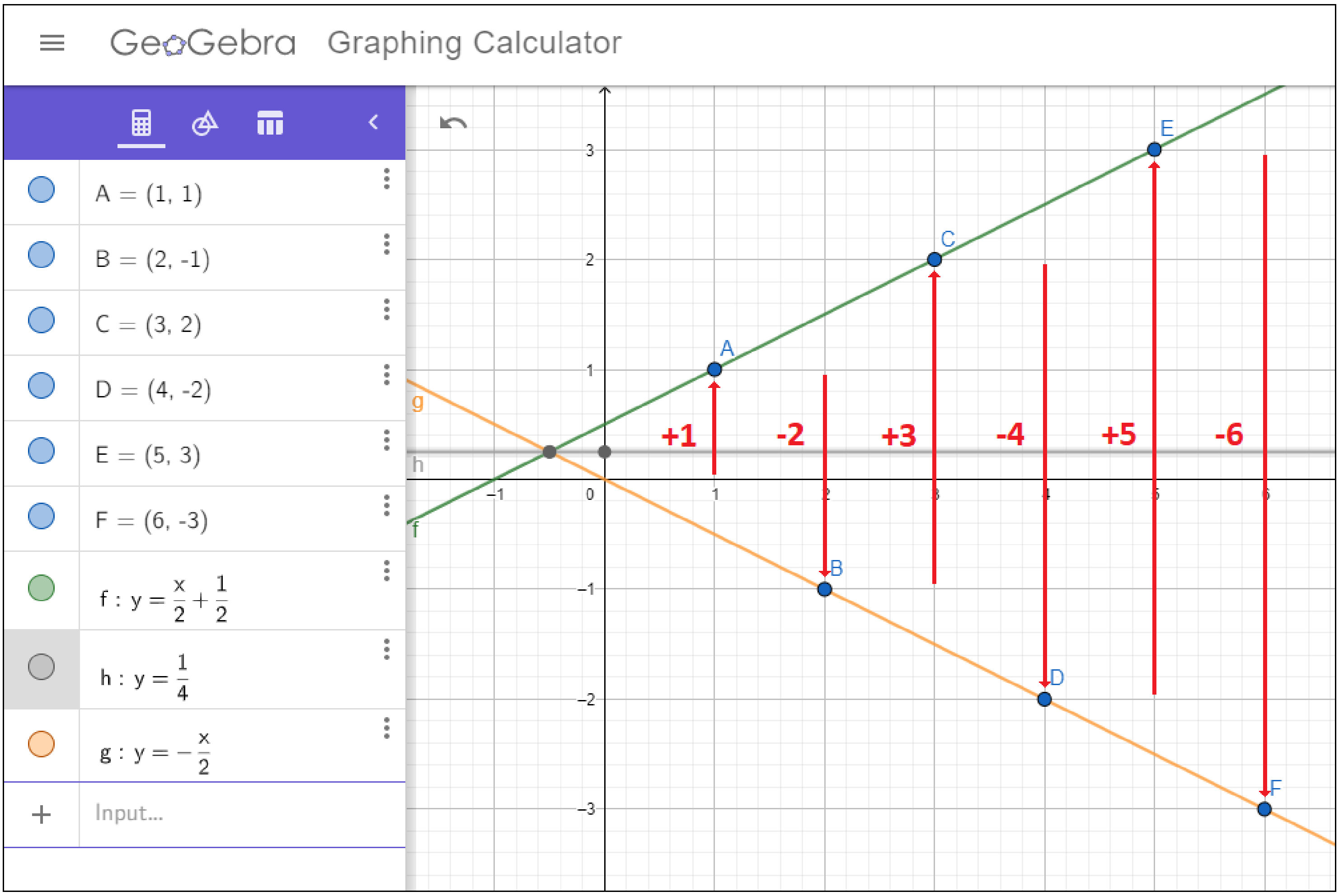


$$\zeta(0.9+1.8i)=\left[\frac{1}{1^{0.9}}\cdot\cos(-1.8\ln 1)+\frac{1}{2^{0.9}}\cdot\cos(-1.8\ln 2)+\frac{1}{3^{0.9}}\cdot\cos(-1.8\ln 3)+...\right]$$
$$+i\cdot\left[\frac{1}{1^{0.9}}\cdot\sin(-1.8\ln 1)+\frac{1}{2^{0.9}}\cdot\sin(-1.8\ln 2)+\frac{1}{3^{0.9}}\cdot\sin(-1.8\ln 3)+...\right]=0.5555...-0.4189...i$$

when I first started to read about the zeta function I didn't know what are those "Assigned Values" or "Analytic Continuation" and how and why people are trying to give a value for divergent series And why that specific value and not something else? I wanted an explanation other then its analytic continuation

Those "origin points" did the trick!

the simplest origin point to understend is $\eta(-1)=1-2+3-4+5-6+...$



the value 1/4 is not the summation of $\eta(-1)$ it's the analytic continuation value for the summation
it's simply represents the intersection points of the two lines
or as i like to describe it as the origin point of the spiral on the complex plane

I also submitted to Vixra on Dirichlet Eta Function Negative Integer Formula
https://vixra.org/pdf/2005.0048v3.pdf

If you are assigning a value for a series that decreases to a specific value (case #1)
Then you can assigning a value for a series that increases **from** a specific value (case #2) ← origin point

Other then those two cases there is one more

This is when the spiral at some point start to spin around a specific value with a “fixed radius” creating a circle those cases appears at the zeta function $\zeta(a+ib)$ when $a=1$ and the radius will be $1/b$
meaning that this is a divergent series with a “fixed radius” and the center of the circle is the analytic continuation

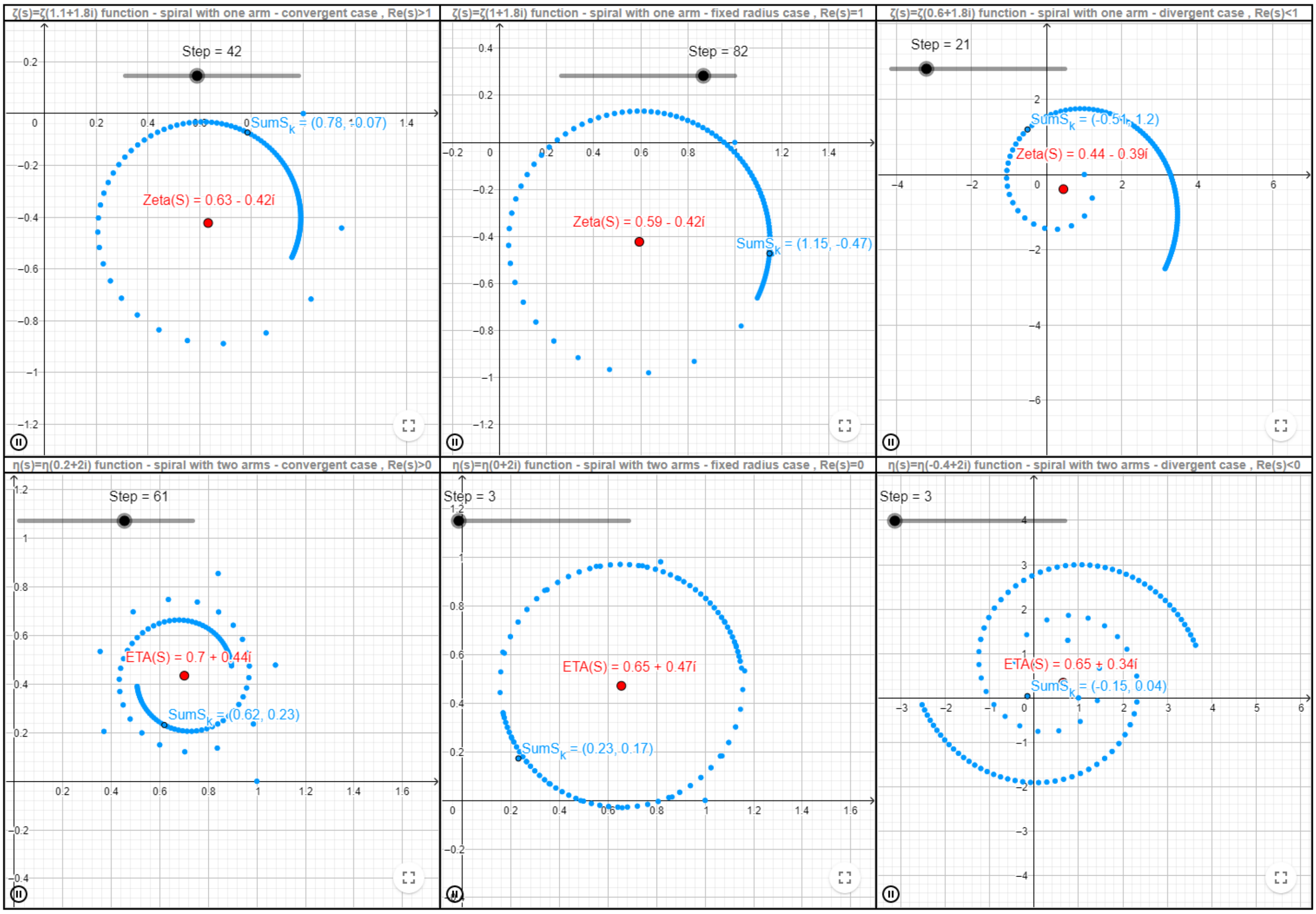
now lets show the eta function spirals

Its true that the zeta function spirals have 3 cases but they are all spirals with **one arm**
Now at the eta function the spirals have **two arms** (that is because of the +/- swapping) with the same 3 cases

$$\eta(a+ib)=\left[\vec{V}_1\cdot\cos(\theta_1)-\vec{V}_2\cdot\cos(\theta_2)+\vec{V}_3\cdot\cos(\theta_3)-\ldots\right]+i\cdot\left[\vec{V}_1\cdot\sin(\theta_1)-\vec{V}_2\cdot\sin(\theta_2)+\vec{V}_3\cdot\sin(\theta_3)-\ldots\right]$$

By the way the fixed radius circles appears at the eta function $\eta(a+ib)$ when $a=0$

If you like to know more I am providing further details at http://myzeta.125mb.com



Removing the Riemann Hypothesis from the Complex Plane

$$\eta(a+ib) = \left[\vec{V_1}\cdot\cos(\theta_1) - \vec{V_2}\cdot\cos(\theta_2) + \vec{V_3}\cdot\cos(\theta_3) - ...\right] + i\cdot\left[\vec{V_1}\cdot\sin(\theta_1) - \vec{V_2}\cdot\sin(\theta_2) + \vec{V_3}\cdot\sin(\theta_3) - ...\right] = x + iy$$

moving on the xAxis: $x = \left[\vec{V_1}\cdot\cos(\theta_1) - \vec{V_2}\cdot\cos(\theta_2) + \vec{V_3}\cdot\cos(\theta_3) - \vec{V_4}\cdot\cos(\theta_4) + ...\right]$

moving on the yAxis: $y = \left[\vec{V_1}\cdot\sin(\theta_1) - \vec{V_2}\cdot\sin(\theta_2) + \vec{V_3}\cdot\sin(\theta_3) - \vec{V_4}\cdot\sin(\theta_4) + ...\right]$

when x=0 and y=0 then $\eta(s) = 0$ meaning that also $\xi(s) = \dfrac{\eta(s)}{(1-2^{1-s})} = 0$

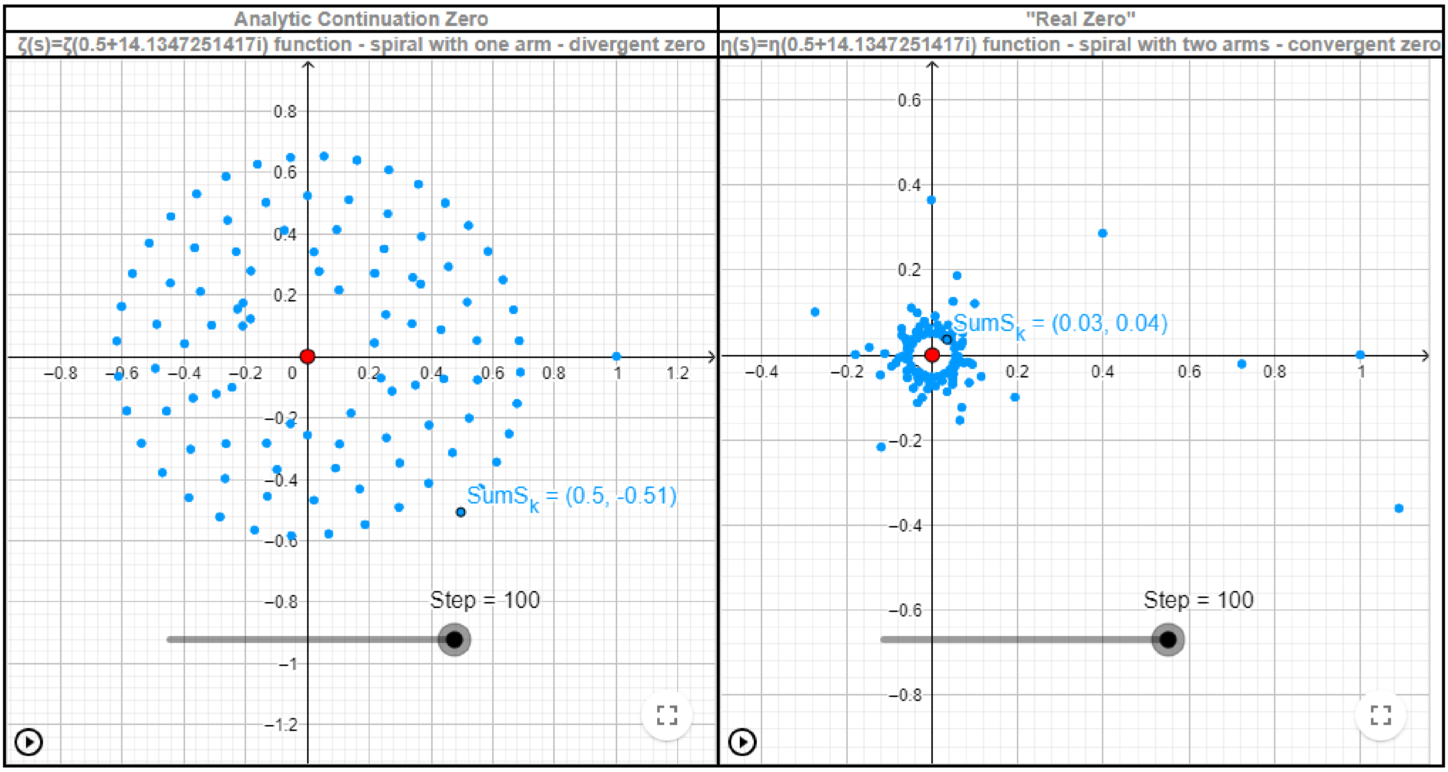
this helps extend the zeta function from Re(s) > 1 to the larger domain

The Riemann hypothesis equivalent to:

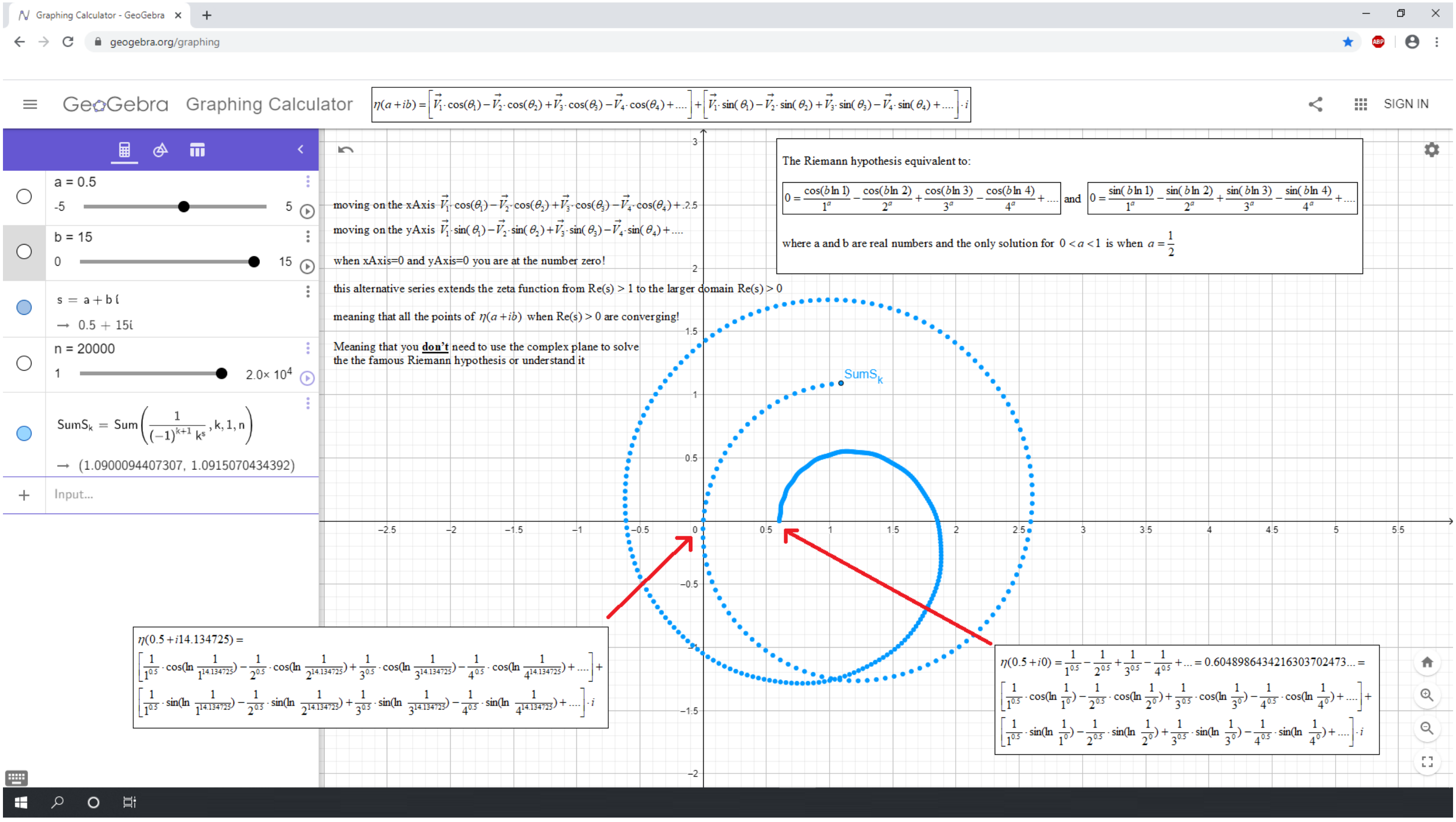
$$0 = \frac{\cos(b\ln 1)}{1^a} - \frac{\cos(b\ln 2)}{2^a} + \frac{\cos(b\ln 3)}{3^a} - .... \quad \text{and} \quad 0 = \frac{\sin(b\ln 1)}{1^a} - \frac{\sin(b\ln 2)}{2^a} + \frac{\sin(b\ln 3)}{3^a} - ....$$

where a and b are real numbers and the only solution for $0 < a < 1$ is when $a = \dfrac{1}{2}$

lets look at the two points $\xi(s) = \xi(0.5 + i\cdot 14.1347251417...) = 0$ and $\eta(s) = \xi(0.5 + i\cdot 14.1347251417...) = 0$



zeta function will give us analytic continuation zero where as eta function will give us a real zero convergents this will help us to see the behavior of convergent points on the critical line with real numbers later this next image is the movement along the line of a=0.5 using the eta function

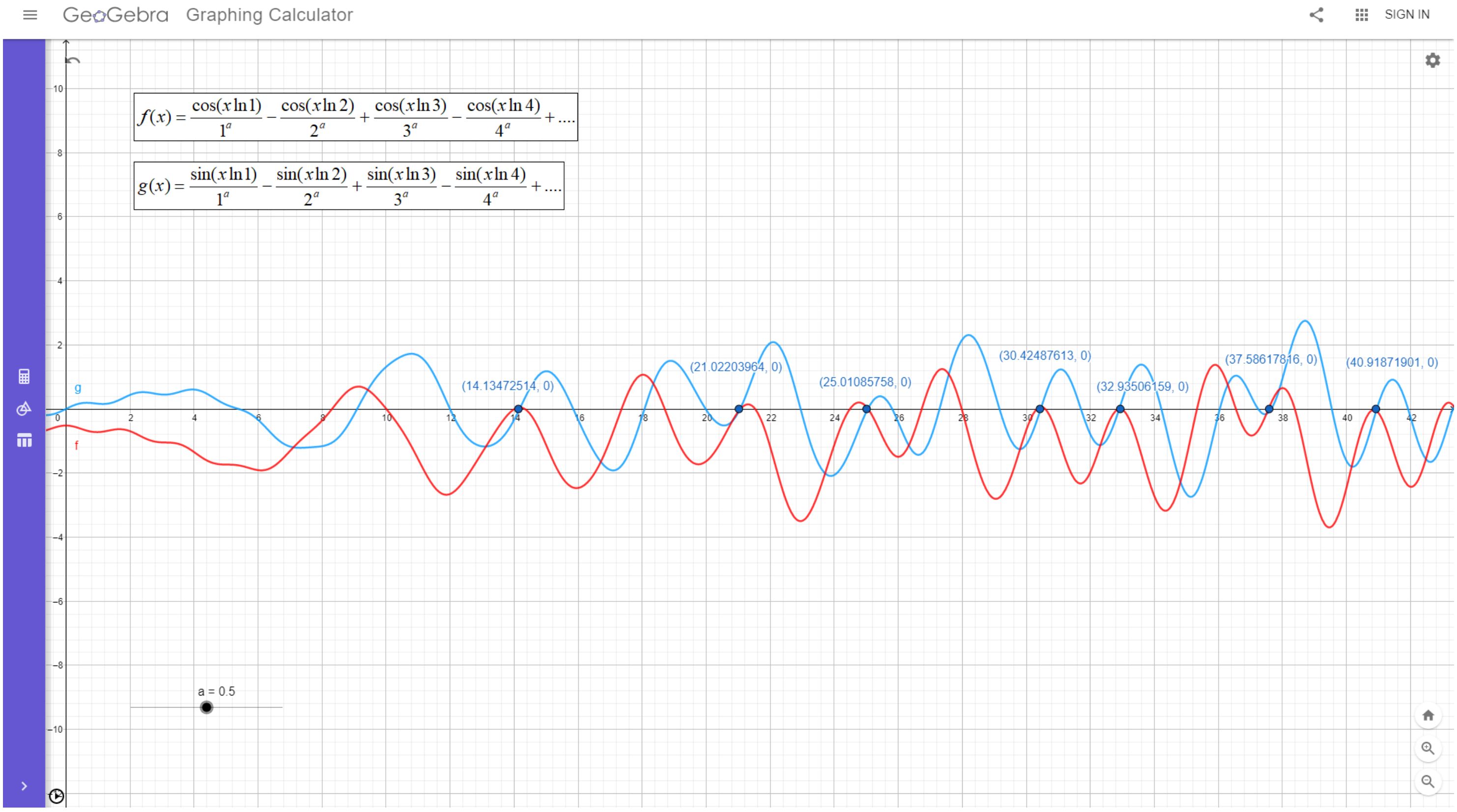


$$\eta(0.5 + i\cdot 14.134...) = \left[\frac{\cos(14.134...\cdot\ln 1)}{1^a} - \frac{\cos(14.134...\cdot\ln 2)}{2^a} + \frac{\cos(14.134...\cdot\ln 3)}{3^a} - ...\right]$$

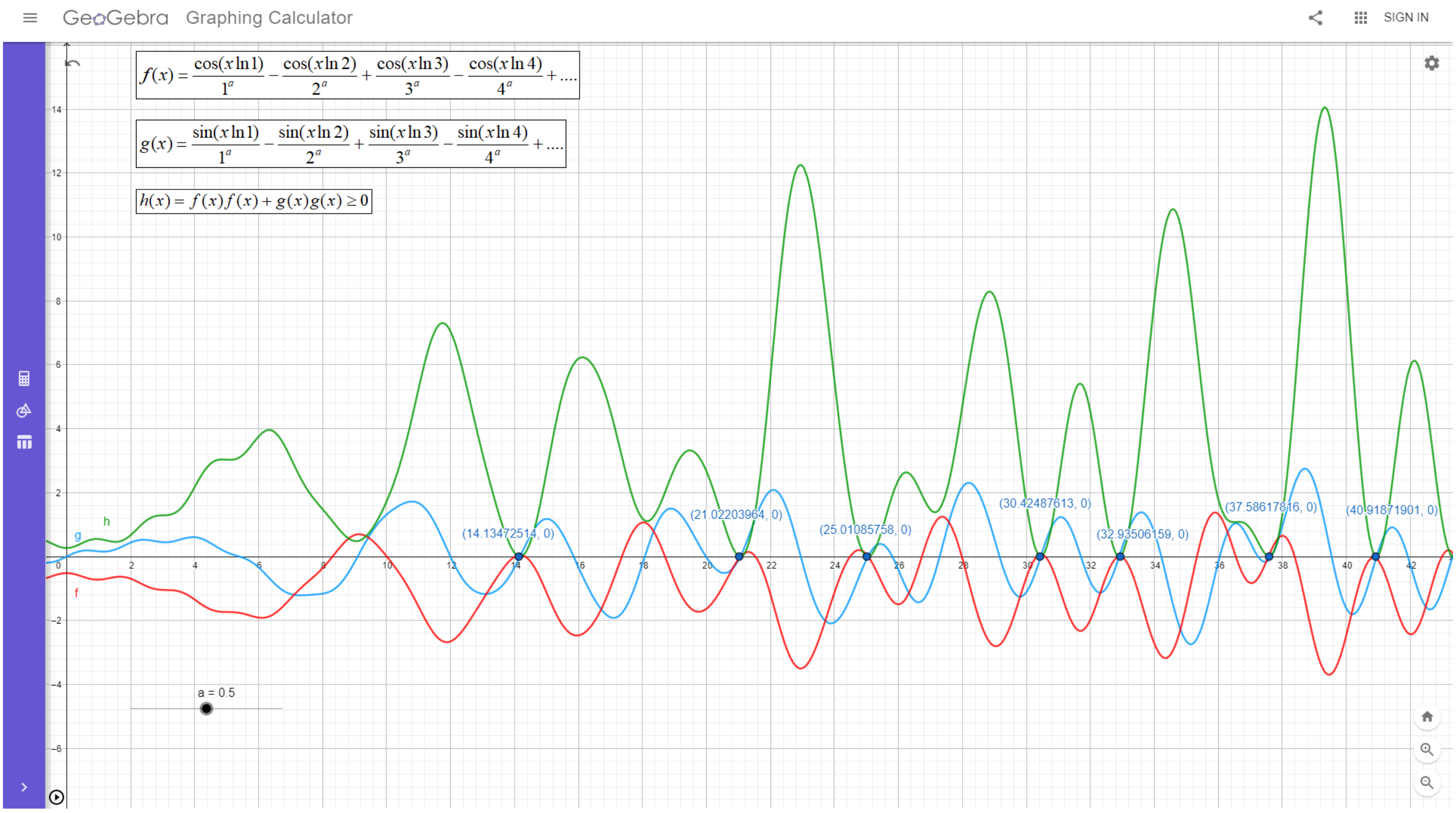
$$+ i\cdot\left[\frac{\sin(14.134...\cdot\ln 1)}{1^a} - \frac{\sin(14.134...\cdot\ln 2)}{2^a} + \frac{\sin(14.134...\cdot\ln 3)}{3^a} - ...\right] = 0$$

Where are all the non trivial zeroes?

$$f(x)=\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\ldots\qquad g(x)=\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\ldots$$



I am going to make a new function $h(x)$ that will include both cases and will be 0 only when both functions $f(x)$ and $g(x)$ are 0 as well. The simplest way is to have $h(x)=f(x)f(x)+g(x)g(x)$ where $h(x)\geq 0$ This way $h(x)=0$ only when you have non-trival zeros



$$f(x)=\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+....$$

$$f(x)\cdot f(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(x\ln n)}{n^a}\cdot\left(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos(x\ln k)}{k^a}\right)=$$

$$+\frac{\cos(x\ln 1)}{1^a}\cdot\left(\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+...\right)$$
$$-\frac{\cos(x\ln 2)}{2^a}\cdot\left(\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+...\right)$$
$$+\frac{\cos(x\ln 3)}{3^a}\cdot\left(\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+...\right)$$
$$-\frac{\cos(x\ln 4)}{4^a}\cdot\left(\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 4)}{4^a}+...\right)$$

$$f(x)\cdot f(x)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+n}\frac{\cos(x\ln n)}{n^a}\cdot\frac{\cos(x\ln k)}{k^a}=$$

$$+\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 4)}{4^a}+...$$
$$-\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 1)}{1^a}+\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 2)}{2^a}-\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 3)}{3^a}+\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 4)}{4^a}-...$$
$$+\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 1)}{1^a}-\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 3)}{3^a}-\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 4)}{4^a}+...$$
$$-\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 1)}{1^a}+\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 2)}{2^a}-\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 3)}{3^a}+\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 4)}{4^a}-...$$

$$f(x)\cdot f(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln n)}{n^a}\cdot\frac{\cos(x\ln k)}{k^a}+\sum_{k=1}^{\infty}\frac{\cos(x\ln k)}{k^a}\cdot\frac{\cos(x\ln k)}{k^a}$$

$$+\frac{\cos(x\ln 1)}{1^a}\cdot\frac{\cos(x\ln 1)}{1^a}$$
$$-2\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 1)}{1^a}+\frac{\cos(x\ln 2)}{2^a}\cdot\frac{\cos(x\ln 2)}{2^a}$$
$$+2\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 1)}{1^a}-2\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 2)}{2^a}+\frac{\cos(x\ln 3)}{3^a}\cdot\frac{\cos(x\ln 3)}{3^a}$$
$$-2\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 1)}{1^a}+2\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 2)}{2^a}-2\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 3)}{3^a}+\frac{\cos(x\ln 4)}{4^a}\cdot\frac{\cos(x\ln 4)}{4^a}-...$$

$$g(x)=\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+....$$

$$g(x)\cdot g(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(x\ln n)}{n^a}\cdot\left(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin(x\ln k)}{k^a}\right)=$$

$$+\frac{\sin(x\ln 1)}{1^a}\cdot\left(\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+...\right)$$
$$-\frac{\sin(x\ln 2)}{2^a}\cdot\left(\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+...\right)$$
$$+\frac{\sin(x\ln 3)}{3^a}\cdot\left(\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+...\right)$$
$$-\frac{\sin(x\ln 4)}{4^a}\cdot\left(\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 4)}{4^a}+...\right)$$

$$g(x)\cdot g(x)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+n}\frac{\sin(x\ln n)}{n^a}\cdot\frac{\sin(x\ln k)}{k^a}=$$

$$+\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 4)}{4^a}+...$$
$$-\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 1)}{1^a}+\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 2)}{2^a}-\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 3)}{3^a}+\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 4)}{4^a}-...$$
$$+\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 1)}{1^a}-\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 3)}{3^a}-\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 4)}{4^a}+...$$
$$-\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 1)}{1^a}+\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 2)}{2^a}-\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 3)}{3^a}+\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 4)}{4^a}-...$$

$$g(x)\cdot g(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\sin(x\ln n)}{n^a}\cdot\frac{\sin(x\ln k)}{k^a}+\sum_{k=1}^{\infty}\frac{\sin(x\ln k)}{k^a}\cdot\frac{\sin(x\ln k)}{k^a}$$

$$+\frac{\sin(x\ln 1)}{1^a}\cdot\frac{\sin(x\ln 1)}{1^a}$$
$$-2\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 1)}{1^a}+\frac{\sin(x\ln 2)}{2^a}\cdot\frac{\sin(x\ln 2)}{2^a}$$
$$+2\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 1)}{1^a}-2\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 2)}{2^a}+\frac{\sin(x\ln 3)}{3^a}\cdot\frac{\sin(x\ln 3)}{3^a}$$
$$-2\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 1)}{1^a}+2\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 2)}{2^a}-2\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 3)}{3^a}+\frac{\sin(x\ln 4)}{4^a}\cdot\frac{\sin(x\ln 4)}{4^a}-...$$

$$f(x)\cdot f(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln n)}{n^a}\cdot\frac{\cos(x\ln k)}{k^a}+\sum_{k=1}^{\infty}\frac{\cos(x\ln k)}{k^a}\cdot\frac{\cos(x\ln k)}{k^a}$$

$$g(x)\cdot g(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\sin(x\ln n)}{n^a}\cdot\frac{\sin(x\ln k)}{k^a}+\sum_{k=1}^{\infty}\frac{\sin(x\ln k)}{k^a}\cdot\frac{\sin(x\ln k)}{k^a}$$

now lets combine the two functions

$$h(x)=f(x)f(x)+g(x)g(x)$$

$$\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\left(\frac{2\cos(x\ln n)}{n^a}\cdot\frac{\cos(x\ln k)}{k^a}+\frac{2\sin(x\ln n)}{n^a}\cdot\frac{\sin(x\ln k)}{k^a}\right)+\sum_{k=1}^{\infty}\left(\frac{\cos(x\ln k)}{k^a}\cdot\frac{\cos(x\ln k)}{k^a}+\frac{\sin(x\ln k)}{k^a}\cdot\frac{\sin(x\ln k)}{k^a}\right)$$

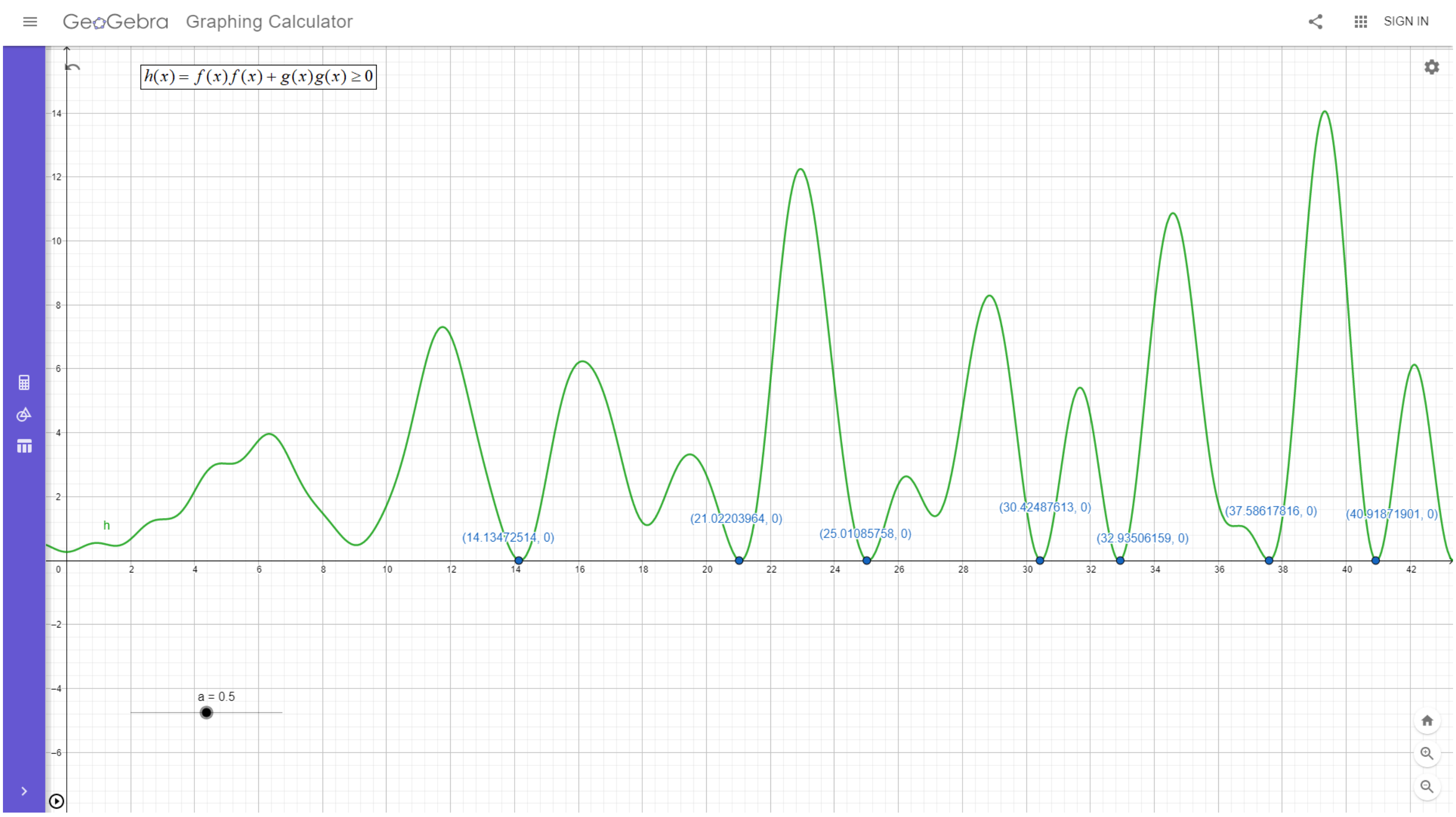
$$\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{(-1)^{k+n}}{n^ak^a}\left(2\cos(x\ln n)\cdot\cos(x\ln k)+2\sin(x\ln n)\cdot\sin(x\ln k)\right)+\sum_{k=1}^{\infty}\frac{1}{k^ak^a}\left(\cos(x\ln k)\cdot\cos(x\ln k)+\sin(x\ln k)\cdot\sin(x\ln k)\right)$$

$$\boxed{\cos(a)\cos(b)+\sin(a)\sin(b)=\cos(a-b)}$$

$$h(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\frac{(-1)^{k+n}}{n^ak^a}\left(2\cos(x\ln n-x\ln k)\right)+\sum_{k=1}^{\infty}\frac{1}{k^ak^a}\left(\cos(x\ln k-x\ln k)\right)$$

$$h(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln(n/k))}{(nk)^a}+\sum_{k=1}^{\infty}\frac{1}{k^{2a}}$$

$$h(x)=\zeta(2a)+\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln(n/k))}{(nk)^a}$$



$$0 \le h(x) = \zeta(2a) + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln(n/k))}{(nk)^{a}}$$

$$-\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n}\frac{2\cos(x\ln(n/k))}{(nk)^{a}} \le \zeta(2a)$$

$$q(x) = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^{a}} \le \zeta(2a)$$

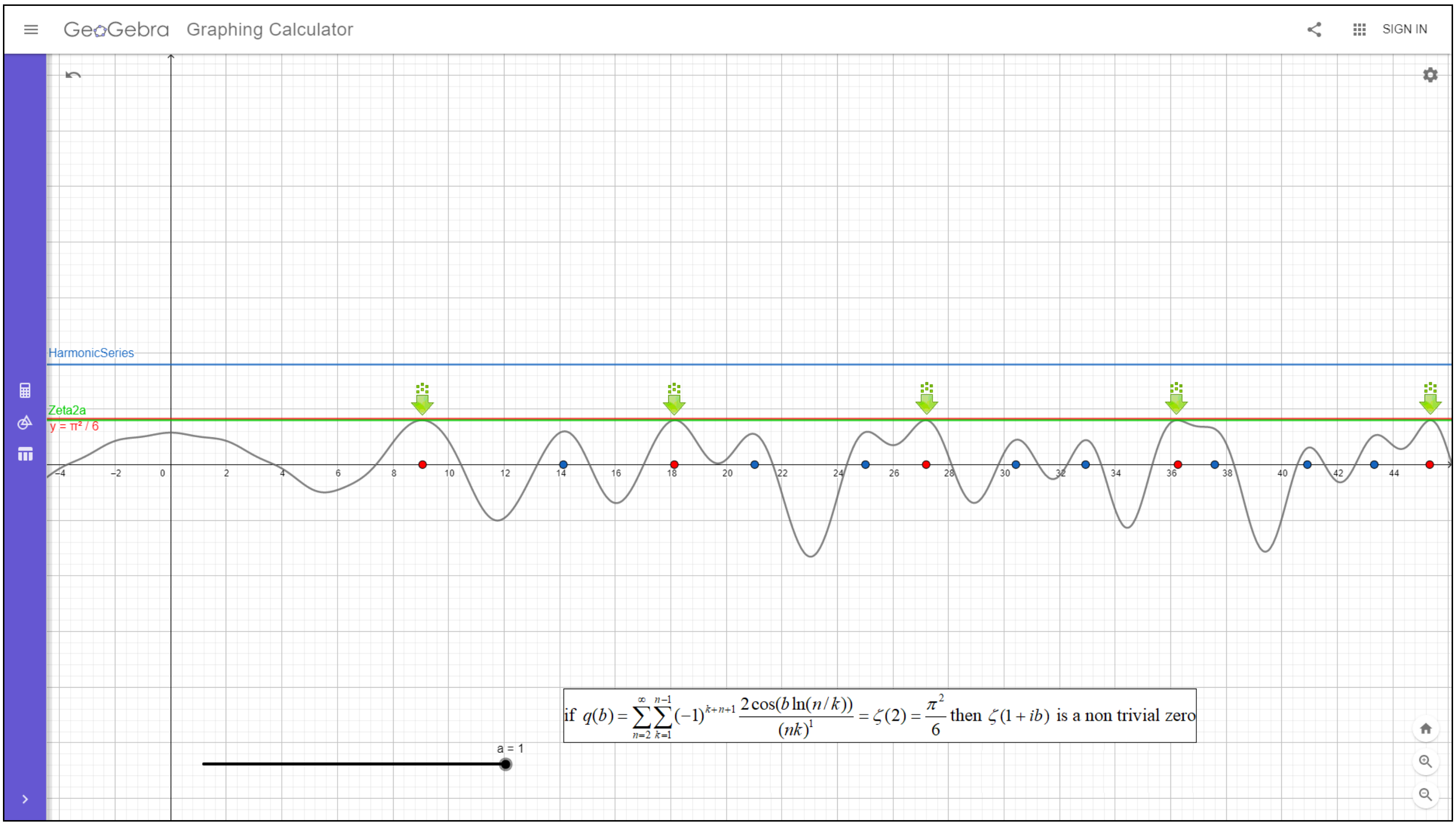
When $q(b) = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(b\ln(n/k))}{(nk)^{a}} = \zeta(2a)$ then $\zeta(a+ib)$ is a non trivial zero (because $h(b)=0$)

(For $1 < a$ there are no non trivial zeros this is a known fact so I wont even going to use the formula for that)

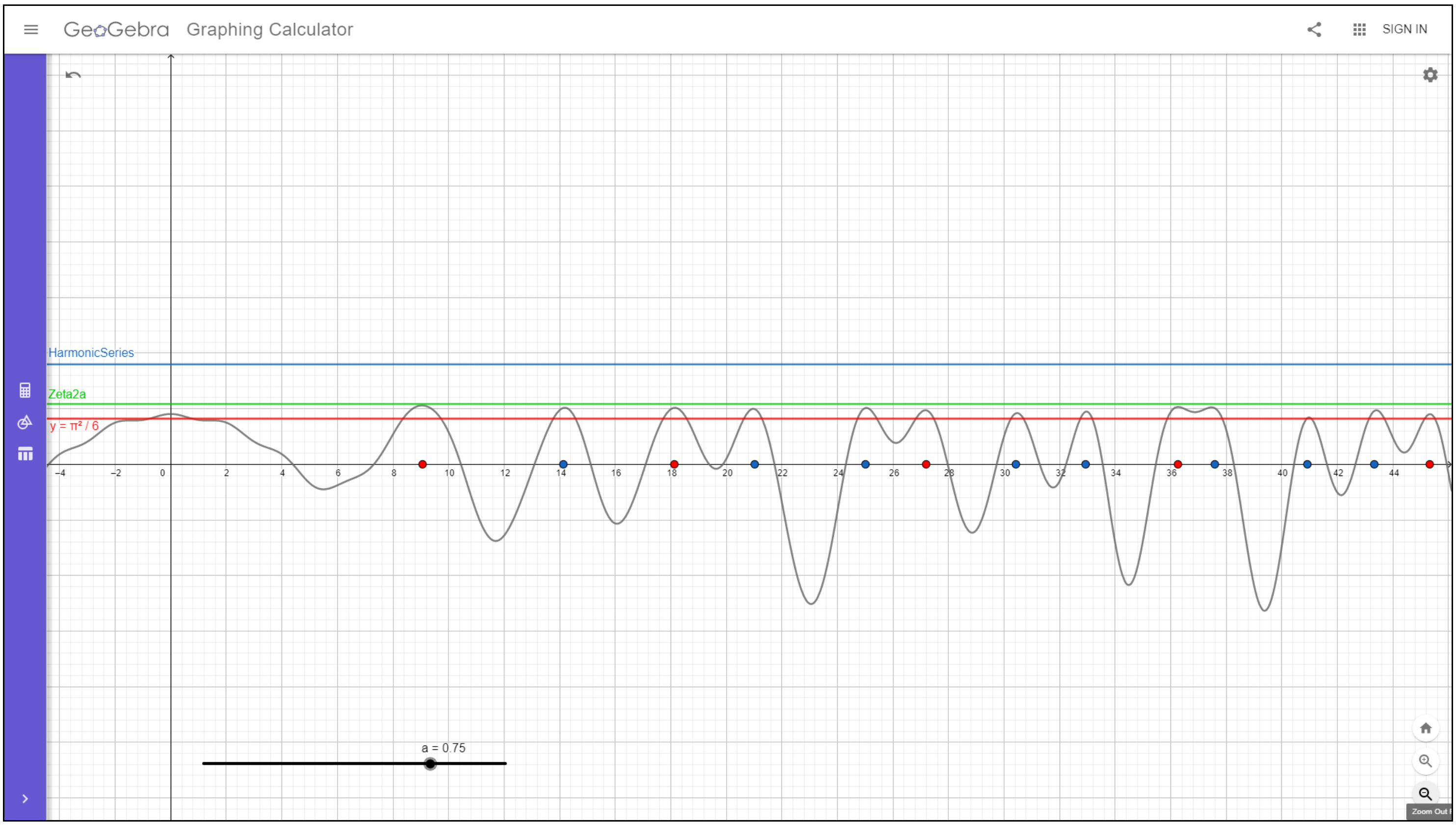
i used eta function summation to get $h(x)$

because $\left(1-\frac{2}{2^{s}}\right)\zeta(s) = \eta(s)$ then $s = 1 + i\frac{2\pi}{\ln 2}n$ where n is any nonzero integer

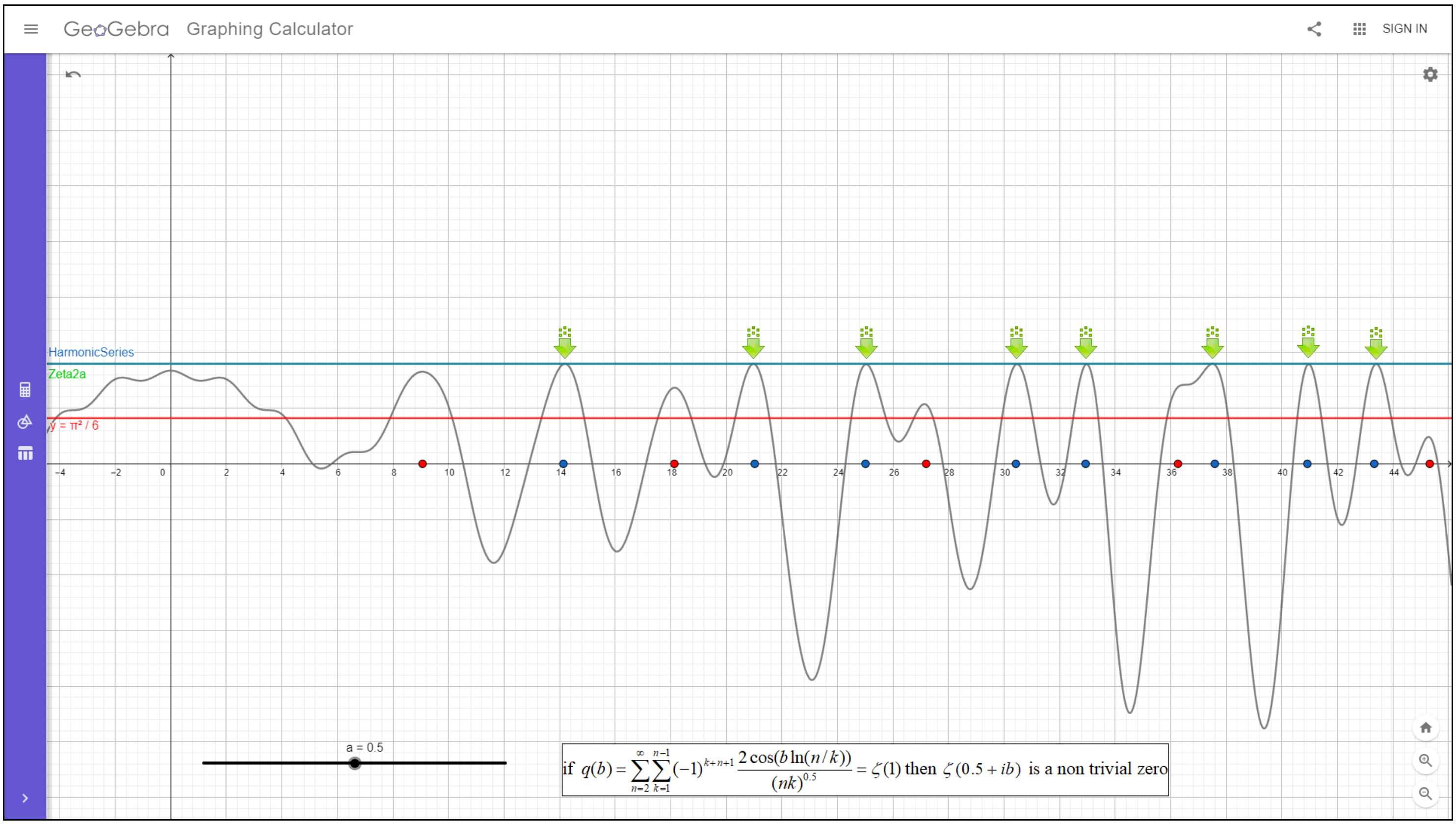
for $b = \frac{2\pi t}{\ln 2}$ when $a = 1$ we get $q(b) = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(b\ln(n/k))}{(nk)^{1}} = \zeta(2) = \frac{\pi^{2}}{6}$



When looking for non trivial zeros on the line a=0.75 we get: $\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(b\ln(n/k))}{(nk)^{0.75}}=\zeta(1.5)=2.61237...$



When looking non trivial zeros on the line a=0.5 we get: $\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(b\ln(n/k))}{(nk)^{0.5}}=\zeta(1)$



Convergence tests

p-Series test:

$$\sum_{k=1}^{\infty}\frac{1}{k^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots$$

$\sum_{k=1}^{\infty}\frac{1}{k^p}$ converges if $p>1$, and diverges if $p\le 1$.

Alternating series test:

$$a_k=\frac{1}{k^{2n}}$$

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2n}}=\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\dots$$

When $0<n$

$|a_k|=\left|\frac{1}{k^{2n}}\right|$ decreases monotonically

$$\lim_{k\to\infty}a_k=\lim_{k\to\infty}\frac{1}{k^{2n}}=0$$

the alternating series converges

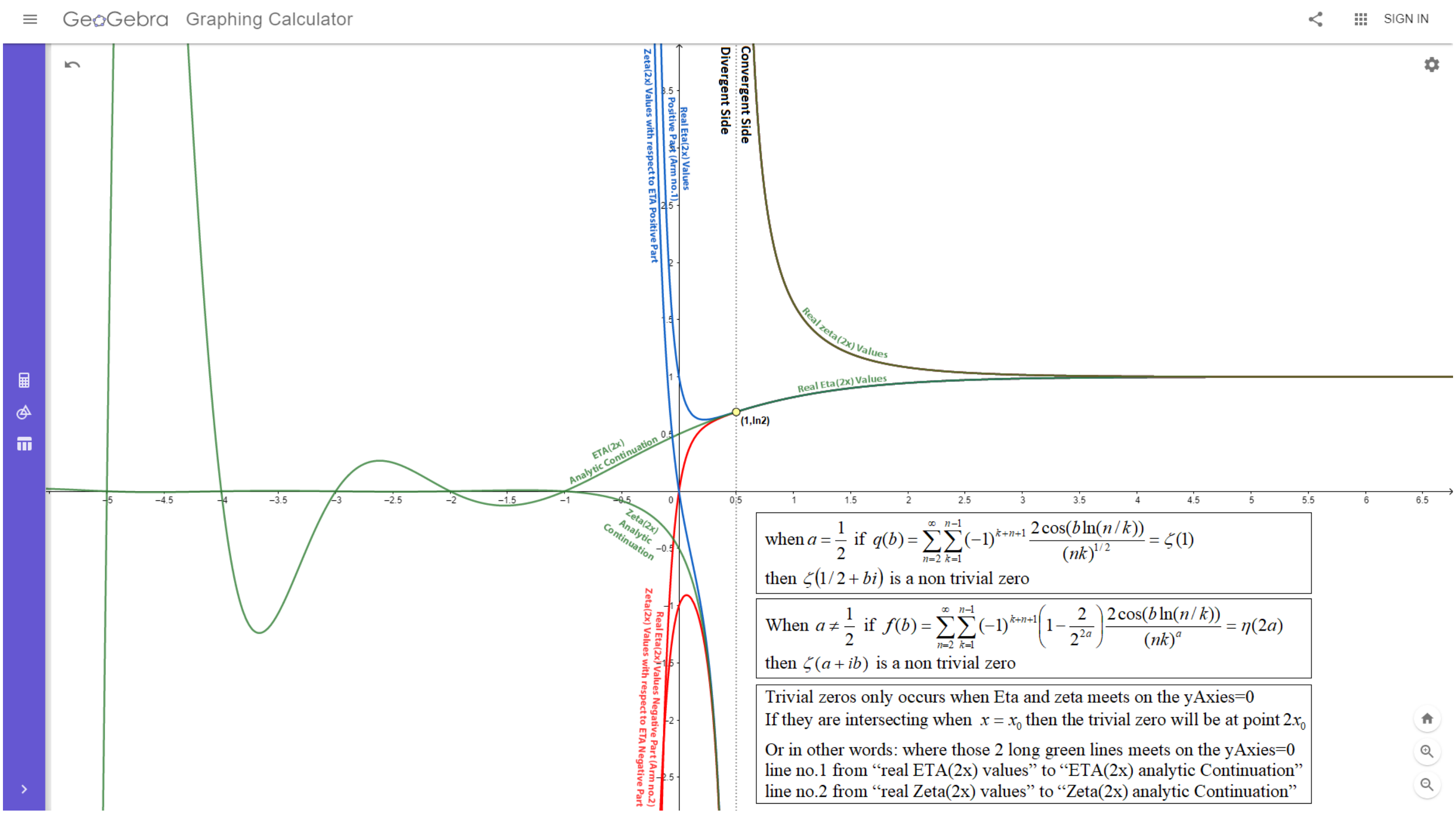
Absolute convergence test:

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}a_k\right|=\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{k^{2n}}\right|=\sum_{k=1}^{\infty}\left|\frac{1}{k^{2n}}\right|=\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\dots$$

when $0<2n\le 1$ (by p-Series test) the series diverges

$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2n}}=\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\dots$ when $0<n\le 0.5$ the series converges Conditionally!

Critical Strip



When $q(b)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(b\ln(n/k))}{(nk)^{a}}=\zeta(2a)$ then $\zeta(a+ib)$ is a non trivial zero

Case #1

for the range $0.5<a<1$ we can multiply by $\left(1-\frac{2}{2^{2a}}\right)\neq 0$

$$\left(1-\frac{2}{2^{2a}}\right)\zeta(2a)=\eta(2a) \qquad \Rightarrow \qquad j(x)=\left(1-\frac{2}{2^{2a}}\right)\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^{a}}=\left(1-\frac{2}{2^{2a}}\right)\zeta(2a)=\eta(2a)$$

on the right side of the equation the the series $\eta(2a)$ is **<u>convergence absolutely</u>** in the range $0.5<a<1$

meaning the function $j(x)=\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\left(1-\frac{2}{2^{2a}}\right)\frac{2\cos(x\ln(n/k))}{(nk)^{a}}$ has a sup value

for every a in the range $0.5<a<1$ and because of that

the function $q(x)$ (theoretically) can have values of x that will result $q(x)=0$

Case #2

for the range $0 < a < 0.5$ we can multiply by $\left(1 - \frac{2}{2^{2a}}\right) \neq 0$

$$\left(1 - \frac{2}{2^{2a}}\right)\zeta(2a) = \eta(2a) \qquad \Rightarrow \qquad j(x) = \left(1 - \frac{2}{2^{2a}}\right)\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\frac{2\cos(x\ln(n/k))}{(nk)^a} = \left(1 - \frac{2}{2^{2a}}\right)\zeta(2a) = \eta(2a)$$

the right side $\eta(2a)$ **converges conditionally** in the range $0 < a < 0.5$

meaning the function $j(x) = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\left(1 - \frac{2}{2^{2a}}\right)\frac{2\cos(x\ln(n/k))}{(nk)^a}$ has no ("fixed") sup value!

The sup value should have been $\eta(2a)$ but this is not a fixed value in the range $0 < a < 0.5$ and because of that the values of x cant get a fixed value on the cos function summation.

Proof by Contradiction

Assumption: there are zero points on the line $a = a_0$ where $0 < a_0 < 0.5$

If there are zero points on the line $a = a_0$ then:

$$j(b) = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(-1)^{k+n+1}\left(1 - \frac{2}{2^{2a_0}}\right)\frac{2\cos(b\ln(n/k))}{(nk)^{a_0}} = \eta(2a_0)$$

if $0 < a_0 \leq 0.5$ then $\eta(2a_0)$ converges conditionally

so if there are any b value $b_1, b_2, b_3, b_4, ...$ that do satisfies the equation for a given value of $\eta(2a_0)$
then they don't have a fixed value because the series $\eta(2a_0)$ converges conditionally
and can be rearranged to converge to any value!

but the points $\eta(a_0 + ib_1), \eta(a_0 + ib_2), \eta(a_0 + ib_3), ...$ are all fixed points on the complex plane

This is a contradiction, and therefore our assumption that there are zero points on the line $a = a_0$ where $0 < a_0 < 0.5$ is wrong

Thus there are no zero points on the line $a = a_0$ where $0 < a_0 < 0.5$

Functional equation gives us:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

Because in the range $0 < a < 0.5$ the function has no zeros
that means that in the range $0.5 < a < 1$ there are no zeros as well!

Case #3

when $a = 0.5$ the function $q(x) = \zeta(1)$ is divergent to infinity

$$\lim_{M \to \infty} \sum_{n=2}^{M} \sum_{k=1}^{n-1} (-1)^{k+n+1} \frac{2\cos(x_0 \ln(n/k))}{(nk)^{1/2}} = \lim_{M \to \infty} \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{M} \right] = \lim_{M \to \infty} \sum_{n=1}^{M} \frac{1}{k} = \zeta(1)$$

and we already know there are infintly many zeroes on the critical line (Hardy 1914)