A New Criterion for the Riemann Hypothesis or two True Proofs?

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Abstract

There are tens of self-proclaimed proofs for the Riemann Hypothesis and only 2 or 4 disproofs of it in arXiv. I am adding to the Status Quo my very short and clear evidence which uses the peer-reviewed achievement of Dr. Solé and Dr. Zhu, which they published just 4 years ago in a serious mathematical journal INTEGERS.

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I. THE PAPER STRATEGY

It is known that the Riemann Hypothesis is true, if either the Robin inequality [1]

$$\frac{\sigma(n)}{n} \leq \mathrm{e}^{\gamma} \ln\ln n =: u(n) \tag{1}$$

holds, where $\sigma(n)$ is the sum of divisors of n, e.g. $\sigma(6) = 1 + 2 + 3 + 6$, $\gamma \approx 0.577$ is Euler's constant, or the Lagarias inequality [2]

$$\frac{\sigma(n)}{n} < \frac{H_n + \exp(H_n)\,\ln(H_n)}{n} =: U(n)\,, \quad H_n = \gamma + \ln n + O(1/n)\,. \tag{2}$$

holds. Eqs. (1) and (2) are equivalents of the Riemann Hypothesis. If one or even both inequalities are proven to be true, the Riemann Hypothesis is true.

A. "One page" Proof

If one of the equivalent formulations of the Riemann Hypothesis is showing the Riemann Hypothesis false, then all equivalent formulations of Riemann Hypothesis show that the Riemann Hypothesis is false. Because, $u(n) < U(n)$, the Robin formulation allows situation, where Riemann Hypothesis is shown to be false, whereas the Lagarias formulation still shows the Riemann Hypothesis to be true. To avoid this contradiction the Robin formulation must show the Riemann Hypothesis to be true for any n.

Other words: if Robin inequality is violated, then the Riemann Hypothesis is false, which means that mathematics allows the

$$u(n) < \frac{\sigma(n)}{n} < U(n)\,.$$

However, that is not allowed, because if the Riemann Hypothesis is false, then due to Lagarias inequality must be $\frac{\sigma(n)}{n} > U(n)$. We came to contradiction, thus, the $\frac{\sigma(n)}{n} < u(n)$ always holds. And so, the Riemann Hypothesis is true.

B. The evidence using Dr. Solé and Dr. Zhu result

Numerical tests on the Robin inequality have shown that Eqs. (1) and (2) both hold for any needed $5041 < n < N$, because $U(n) > u(n)$. Today the unchecked area of n is given by $n \geq N = \exp(\exp(26)) \gg 1$.

Dr. Solé and Dr. Zhu have proven [3] that for large numbers of n one has

$$u(n) - \frac{\sigma(n)}{n} \geq -\beta(n)\,, \tag{3}$$

where $\beta(n) \geq 0$ is a certain but unknown function which, if non-vanishing, is monotonically decreasing and $\beta(n) = 0$ for $n \to \infty$. The inequality (3) holds in any case, even if the Riemann Hypothesis is false.

From Eqs. (2) and (3) it follows that the Riemann Hypothesis is true, if

$$\beta(n) + u(n) < U(n)\,, \tag{4}$$

which I call "Martila inequality". Following from this inequality, for large n I am showing that the case $\beta(n) = C/n^x$, $x > 0$ and $x \neq 0$, where $C \geq 0$ is an arbitrary constant, satisfies the Martila inequality. This discovery means, that if $\beta(n)$ is an analytical function, or it can be expressed using Taylor series expansion (for small $\epsilon = 1/n^v$, where $v > 0$, e.g. $v = 0.3$), then the Riemann Hypothesis is true. In general, if for large n the $\beta(n) < \beta_0(n)$, where

$$\beta_0(n) + u(n) = U(n)\,, \tag{5}$$

The Riemann Hypothesis is true.

II. PRIOR RESEARCH RESULT

Because the 2018 paper of Dr. Zhu [4] is not published in a peer-review journal (for 4 years) and is very complicated, it could contain a fatal mistake. For this reason, I do not start with the final result called "The probability of Riemann's hypothesis being true is equal to one" but rather with the starting information of the papers [3, 4] (one of the papers is peer-reviewed), where is proven (cf. Theorem 2) that for the "limit inferior" one has

$$\lim_{n\to\infty} \inf d(n) \geq 0\,, \tag{6}$$

where $d(n) = D(n)/n$ and $D(n) = \mathrm{e}^\gamma\, n \ln\ln n - \sigma(n)$. Hereby the Riemann Hypothesis holds true, if $\lim\limits_{n\to\infty} \inf D(n) \geq 0$.

The main problem of the available Riemann Hypothesis proofs is a possible fatal mistake somewhere in the text. If text is complicated enough, the mistake is practically impossible to find. The final result of Ref. [4] comes from too many theorems (theorems 1, 2 and 3 in Ref. [3]), so the risk of having a mistake is very high. However, I will demonstrate that it is enough to hope for the validity of Theorem 2 in Ref. [3], i.e. I can prove the Riemann Hypothesis even without Theorems 1 and 3. Recall that the Riemann Hypothesis has been shown to hold unconditionally for n up to $N = \exp(\exp(26))$, as written in Refs. [3, 5]. Thus, it is enough to check the Riemann Hypothesis for the region $n \gg 1$. Therefore, we do not need Theorem 3, because it is a trivial fact Dr. Zhu is proving that if $D(n) \geq 0$ for $n > N \gg 1$, the Riemann Hypothesis is correct. Also, we do not need Theorem 1, as Theorem 2 already says that Eq. (6) holds.

III. MY PROOF

Today the unchecked area of the Riemann Hypothesis is located at extremely large values $n > \exp(\exp(26))$ (including the unlimitely large n). From Eq. (6) I conclude that for large $n \gg 1$ one has

$$\frac{D(n)}{n} \equiv \mathrm{e}^{\gamma} \ln\ln n - \frac{\sigma(n)}{n} \geq -\beta(n)\,, \tag{7}$$

where the continuous monotonic function $-\beta(n) \leq \inf d(n)$, and $\beta(n) = 0$ for $n \to \infty$. On the other hand, the Riemann Hypothesis is true, if for every $n > 1$ one has [2]

$$\frac{\sigma(n)}{n} < \frac{H_n + \exp(H_n)\,\ln(H_n)}{n}\,, \tag{8}$$

where the harmonic number is

$$H_n = \gamma + \ln(n) + K(n)\,, \tag{9}$$

where $K(n) > 0$, and $K(n) = 0$ for $n \to \infty$. Inserting H_n from Eq. (9) into Eq. (8), one obtains

$$\frac{\sigma(n)}{n} < \mathrm{e}^{\gamma} \ln(\gamma + \ln(n)) + R(n)\,, \tag{10}$$

where $R(n) > 0$. From Eqs. (7) and (10) follows that the Riemann Hypothesis is true, if for large n one has

$$\beta(n) + \mathrm{e}^{\gamma} \ln\ln n < \mathrm{e}^{\gamma} \ln(\gamma + \ln(n)) + R(n)\,. \tag{11}$$

The inequality (11) is satisfied, if

$$0 \leq \beta(n) \leq \beta_0(n)\,, \tag{12}$$

but is violated if $\beta(n) > \beta_0(n)$. Let us find the violation threshold $\beta_0(n)$. From Eq. (11) one has

$$\beta_0(n) = \mathrm{e}^{\gamma} \ln(\gamma + \ln(n)) - \mathrm{e}^{\gamma} \ln\ln n + R(n) = \mathrm{e}^{\gamma} \ln([\gamma/\ln(n)] + 1) + R(n) \,. \tag{13}$$

I am citing from the end of Ref. [3]: "For instance, one cannot rule out the case that $D(n)$ behaves like $-\sqrt{n}$ when $n \to \infty$, which would not contradict the fact that $\liminf_{n\to\infty} d(n) = 0$." This points to my function $\beta(n) = (C\sqrt{n})/n = C/\sqrt{n}$, where $C \geq 0$, e.g. $C = 1$. Because holds $C/\sqrt{n} < \beta_0(n)$, the Riemann Hypothesis is true for such case [6]. And in order to avoid the contradiction with the Robin inequality (which is $D(n) \geq 0$) we have to assign $C = 0$.

Moreover, $\beta(n) = C/n^x \geq 0$ results in $C = 0$ by the same analysis [7] for all fixed powers $x > 0$, $x \neq 0$, e.g. $x = 0.25$. That means that if there exists a Taylor series expansion for $\beta(n)$ for large n (using the small $\epsilon = 1/n^v$ with $v > 0$), the Riemann Hypothesis is proven. But because $\beta(n)$ can be described by monotonic decreasing function, it is well justified that it has a non-zero derivative (when formally the n are taken to be continuous) somewhere in the first Taylor terms. If $\beta(n)$ would be exponential and, therefore, non-analytically approaches zero rapidly, Eq. (12) is still satisfied, $\exp(-n) \ll 1/\sqrt{n}$.

In particular, if exists such x that

$$n^x \, \beta(n) = 0 \,, \quad n \to \infty \,, \tag{14}$$

the Riemann Hypothesis is true.

IV. DISCUSSION

A. Is N large?

A journal referee might say some nonsense like "what if $N = \exp(\exp(26))$ is very small, i.e. maybe $N \sim 1$?" to reject the paper. I disagree! Ref. [3] tells us, that the area where $n > M$ with $M \to \infty$ is decisive. I mean, if the Riemann Hypothesis is wrong, it must be shown wrong at $n \to \infty$. Therefore, you can replace the N with any fixed $M \gg N$ in my analysis.

B. Inequalities are true together

If Robin inequality is violated at some $n = n_0$, then it is certain, that both inequalities (Robin and Lagarias) are violated at some n_h. However, because of this certainty, we must be certain as well to violate them both at the n_0. In the hypothetical situation, where Robin inequality is violated for several finite n_i, but the Lagarias one is violated only for infinite $n_L \to \infty$, the Lagarias inequation has lost the meaning of equivalent formulation of Riemann Hypothesis, which is not possible. In another situation, where Robin inequation is violated only in one single point n_0, the Lagarias one must be violated at this point as well. Thus, if the Riemann Hypothesis is wrong, both ones must be violated together.

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[1] Robin G., Grandes Valeurs de la fonction somme des diviseurs et hypothése de Riemann, J. Math. Pures Appl. **63**, 187–213 (1984); Akbary A.; Friggstad Z., Superabundant numbers and the Riemann hypothesis, Am. Math. Monthly **116** (3), 273–275 (2009).

[2] Lagarias J. C., An elementary problem equivalent to the Riemann hypothesis, The American Mathematical Monthly **109** (6), 534–543 (2002); Sandifer C. E., How Euler Did It, MAA Spectrum, Mathematical Association of America, p. 206 (2007).

[3] Solé P., Zhu Y., An Asymptotic Robin Inequality, INTEGERS, **A81**, 16 (2016), http://math.colgate.edu/∼integers/q81/q81.pdf

[4] Zhu Y., The probability of Riemann's hypothesis being true is equal to 1, arXiv:1609.07555v2 [math.GM] (2016, 2018).

[5] Briggs K., Abundant numbers and the Riemann hypothesis, Experiment. Math. **15** (2), 251–256 (2006).

[6] To demonstrate this, one formally inserts $K(n) \equiv 0$ for all n in Eq. (9), checks the resulting inequality, and restores $K(n) > 0$.

[7] While demonstration I defined $z(x, n) := \beta(n) - \beta_0(n)$ and extracted the critical curve $x = x_c(n)$ from $z(x, n) = 0$; in the limit $n \to \infty$ the $x_c = 0$.