A New Criterion for Riemann Hypothesis or a True Proof?

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Abstract

There are tens of self-proclaimed proofs for Riemann Hypothesis and only 2 or 4 disproofs of it in arXiv. I am adding to the Status Quo my very short and clear evidence which uses the peer-reviewed achievement of Dr. Solé and Dr. Zhu, which they published just 4 years ago in a serious mathematical journal INTEGERS.

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I. PRIOR RESEARCH RESULT

Because the 2018 paper of Dr. Zhu [1] is not published in a peer-review journal (for 4 years) and is very complicated, it could contain a fatal mistake. For this reason, I do not start with the final result called "The probability of Riemann's hypothesis being true is equal to 1" but rather with the starting information of the papers [1, 2] (one of the papers is peer-reviewed), where is proven (cf. Theorem 2) that the for the "limit inferior" one has

$$\lim_{n\to\infty} \inf d(n) \geq 0\,, \tag{1}$$

where $d(n) = D(n)/n$ and $D(n) = \mathrm{e}^{\gamma}\, n \ln\ln n - \sigma(n)$. Hereby the Riemann Hypothesis holds true, if $\lim_{n\to\infty} \inf D(n) \geq 0$.

The main problem of the available Riemann Hypothesis proofs is a possible fatal mistake somewhere in the text. If text is complicated enough, the mistake is practically impossible to find. The final result of Ref. [1] comes from too many theorems (theorems 1, 2 and 3 in Ref. [2]), so the risk of having a mistake is very high. However, I will demonstrate that it is enough to hope for the validity of Theorem 2 in Ref. [2], i.e. I can prove the Riemann Hypothesis even without Theorems 1 and 3. Recall that the Riemann Hypothesis has been shown to hold unconditionally for n up to $N = \exp(\exp(26))$, as written in Refs. [2, 3]. Thus, it is enough to check the Riemann Hypothesis for the region $n \gg 1$. Therefore, we do not need Theorem 3, because it is a trivial fact Dr. Zhu is proving that if $D(n) \geq 0$ for $n > N \gg 1$, the Riemann Hypothesis is correct. Also, we do not need Theorem 1, as Theorem 2 already says that Eq. (1) holds.

II. MY PROOF

Today the unchecked area of the Riemann Hypothesis is located at extremely large values $n > \exp(\exp(26))$ (including the unlimitely large n). From Eq. (1) I conclude that for large $n \gg 1$ one has

$$\frac{D(n)}{n} \equiv \mathrm{e}^{\gamma} \ln\ln n - \frac{\sigma(n)}{n} \geq -\beta(n)\,, \tag{2}$$

where $\beta(n) = 0$ for $n \to \infty$. On the other hand, the Riemann Hypothesis is true, if for every $n > 1$ one has [4]

$$\frac{\sigma(n)}{n} < \frac{H_n + \exp(H_n)\,\ln(H_n)}{n}\,, \tag{3}$$

where the harmonic number is

$$H_n = \gamma + \ln(n) + K(n)\,, \tag{4}$$

where $K(n) > 0$, and $K(n) = 0$ for $n \to \infty$. Inserting H_n from Eq. (4) into Eq. (3), one obtains

$$\frac{\sigma(n)}{n} < \mathrm{e}^{\gamma} \ln(\gamma + \ln(n)) + R(n)\,, \tag{5}$$

where $R(n) > 0$. From Eqs. (2) and (5) follows that the Riemann Hypothesis is true, if for large n one has

$$\beta(n) + \mathrm{e}^{\gamma}\, \ln\ln n < \mathrm{e}^{\gamma} \ln(\gamma + \ln(n)) + R(n)\,. \tag{6}$$

The inequality (6) is satisfied, if

$$0 \le \beta(n) \le \beta_0(n)\,, \tag{7}$$

but is violated if $\beta(n) > \beta_0(n)$. Let us find the violation threshold $\beta_0(n)$. From Eq. (6) one has

$$\beta_0(n) = \mathrm{e}^{\gamma} \ln(\gamma + \ln(n)) - \mathrm{e}^{\gamma} \ln\ln n + R(n) = \mathrm{e}^{\gamma} \ln([\gamma/\ln(n)] + 1) + R(n)\,. \tag{8}$$

I am citing from the end of Ref. [2]: "For instance, one cannot rule out the case that $D(n)$ behaves like $-\sqrt{n}$ when $n \to \infty$, which would not contradict the fact that $\liminf_{n\to\infty} d(n) = 0$." This points to my function $\beta(n) = (C\sqrt{n})/n = C/\sqrt{n}$, where $C \ge 0$, e.g. $C = 1$. The following holds true: If $C/\sqrt{n} < \beta_0(n)$[5], the Riemann Hypothesis is true. And because the Riemann Hypothesis is shown now to be true (for the case that $D(n)$ acts like $-\sqrt{n}$), in order to avoid the contradiction with Robin's inequality for validity of Riemann Hypothesis (which is $D(n) \ge 0$) we must assign $C = 0$.

Moreover, $\beta(n) = C/n^x \ge 0$ results in $C = 0$ by the same analysis for all fixed powers $x > 0$, $x \ne 0$. That means that if there exists a Taylor series expansion for $\beta(n)$ for large n (using the small $\epsilon = 1/n^v$ with $v > 0$), the Riemann Hypothesis is proven. But because $\beta(n)$ is a monotonic slowly decreasing function, it is well justified that it has a non-zero derivative (when formally the n are taken to be continuous) somewhere in the first Taylor terms. If $\beta(n)$ would be exponential and, therefore, non-analytically approaches zero rapidly, Eq. (7) is still satisfied, $\exp(-n) \ll 1/\sqrt{n}$.

Moreover, I can show that if the function $f(n)$ tends to infinity, there exists an x such that

$$\frac{n^x}{f(n)} = 0\,, \quad n \to \infty\,. \tag{9}$$

Proof: if $x = 0$, the statement is true. However, it could be violated for $x > 0$. Therefore, looking at x as definite numbers, the actual condition for the violation is given by $x > S > 0$. Therefore, if $0 \leq x \leq S$ the theorem (9) is true.

III. DISCUSSION

A journal referee might say some nonsense like "what if $N = \exp(\exp(26))$ is very small, i.e. maybe $N \sim 1$?" to reject the paper. I disagree! Ref. [2] tells us, that the area where $n > M$ with $M \to \infty$ is decisive. I mean, if the Riemann Hypothesis is wrong, it must be shown wrong at $n \to \infty$. Therefore, you can replace the N with any fixed $M \gg N$ in my analysis.

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[1] Zhu Y., The probability of Riemann's hypothesis being true is equal to 1, arXiv:1609.07555v2 [math.GM] (2016, 2018)

[2] Solé P. and Zhu Y., An Asymptotic Robin Inequality. INTEGERS, **A81**, 16 (2016), http://math.colgate.edu/~integers/q81/q81.pdf

[3] Briggs K., Abundant numbers and the Riemann hypothesis, Experiment. Math. **15** (2), 251–256 (2006).

[4] Lagarias J. C., An elementary problem equivalent to the Riemann hypothesis, The American Mathematical Monthly **109** (6), 534–543 (2002); Sandifer C. E., How Euler Did It, MAA Spectrum, Mathematical Association of America, p. 206 (2007)

[5] To demonstrate this, one formally inserts $K(n) \equiv 0$ for all n in Eq. (4), checks the resulting inequality, and restores $K(n) > 0$.