Proof of the Beal conjecture and Fermat- Catalan conjecture (summary)

Nguyen Van Quang
Hue - Vietnam, 05-2020

Abstract

This article includes the theorems [7], [8] and the lemmas, using them to prove the Beal conjecture and the Fermat - Catalan conjecture, through which we learn more about rational and irrational numbers. I think the method of proof will be useful for solving other Math - problems, and they need more research

1 The theorems

Theorem 1. *For positive integers* x, y, z, k_i *and A,B coprime integers, and* $a_k, a_{k-1}, ..., a_1, a_0$ *are fixed numbers.*
$\sqrt[z]{A^x \pm B^y} = a_k s^k + a_{k-1} s^{k-1} t + a_{k-2} s^{k-2} t^2 + ... + a_1 s t^{k-1} + a_0 s^k$ *for any s,t coprime integers.*

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{3}{LCM(x, y, z)} > 1$$

In other worlds, the $C = \sqrt[z]{A^x \pm B^y}$ $(A, B, C \neq 1,$ *coprime)can not be expressed as* $a_k s^k + a_{k-1} s^{k-1} t + a_{k-2} s^{k-2} t^2 + ... + a_1 s t^{k-1} + a_0 s^k$ *with all fixed coefficients , s,t coprime integer if*

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{3}{LCM(x, y, z)} \leqslant 1$$

Notes:
LCM (x,y,z): least common multiples of x,y and z.
Except $A = 3, B = 2, \sqrt[z]{3^2 - 2^3} = 1$

Theorem 2. *For positive integers* $x, y, z,$ *and* $A, B, C \neq \pm 1,$ *coprime integers:*

The equation $A^x + B^y = C^z \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{3}{LCM(x, y, z)} > 1$

In other worlds, the equation $A^x + B^y = C^z$ *has no solution* $(A, B, C \neq \pm 1,$ *coprime)in integer if*

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{3}{LCM(x, y, z)} \leqslant 1$$

Notes:
LCM (x,y,z): least common multiples of x,y and z.

As above, if A, B and C have a common factor, we always find a solution as below:
Let $a_0^x + b_0^x = c_0$ then $c_0^x a_0^x + c_0^x b_0^x = c_0^{x+1}$
so $A^x + B^x = C^{x+1}$ ($A = c_0 a_0, B = c_0 b_0, C = c_0$)
Or
Let $a_0^x + b_0^y = c_0$ then $c_0^{xy} a_0^x + c_0^{xy} b_0^y = c_0^{xy+1}$
so $A^x + B^y = C^{xy+1}$, $(A = c_0^y a_0, B = c_0^x b_0, C = c_0)$

2 The lemmas

Lemma 1. *For any integer N ,s,t (s,t = 1), there exist integers* $a_k, a_{k-1}, a_{k-2}, ..., a_0$ *such that :*
$N = a_k s^k + a_{k-1}s^{k-1}t + a_{k-2}s^{k-2}t^2 + ... + a_1 st^{k-1} + a_0 t^k$
In others wolds, any integer N can be expressed as:
$N = a_1 s + a_0 t$
$N = a_2 s^2 + a_1 st + a_0 t^2$
...
$N = a_k s^k + a_{k-1}s^{k-1}t + a_{k-2}s^{k-2}t^2 + ... + a_1 st^{k-1} + a_0 t^k$

Proof. It is known, given any integer s, t coprime, then exit the integers N_1, N_0, such that any integer N can be express as
$N = N_1 s + N_0 t$
then, we express for N_1 and N_0:
$N_1 = N_{11}s + N_{10}t$
$N_0 = N_{01}s + N_{00}t$
it gives:
$N = (N_{11}s + N_{10}t)s + (N_{01}s + N_{00}t)t = N_{11}s^2 + (N_{10} + N_{01})st + N_{00}t^2$
Then we express for $N_{11}, (N_{10} + N_{01}), N_{00}$
and so on we obtain the express $N = a_k s^k + a_{k-1}s^{k-1}t + a_{k-2}s^{k-2}t^2 + ... + a_1 st^{k-1} + a_0 t^k$
□

* Clearly, expression for N above is also true when N is not a integer, then all a_i are also not simultaneously integers.

Lemma 2. *The equation* $u_k s^k + u_{k-1}s^{k-1}t + u_{k-2}s^{k-2}t^2 + ... + u_1 st^{k-1} + u_0 t^k = 0$ *for any s, t coprime if the coefficients* u_i*satify :*

1. Each coefficients satify :
$u_k = h_k t$
$u_{k-1} = -h_k s + h_{k-1}t$
$u_{k-2} = -h_{k-1}s + h_{k-2}t$
....
$u_1 = -h_2 s + h_1 t$
$u_0 = -h_1 s$

2.All coefficients (u_i) *equal to zero(all* $h_i = 0$*)*

Note that the number h_k, h_1 appear once, the remaining h_i appear twice in two successive $i^{th}, (i-1)^{th}$ equations with opposite signs. Change each other's sign, the result will remain the same.

Proof. From the equation $u_k s^k + u_{k-1}s^{k-1}t + u_{k-2}s^{k-2}t^2 + ... + u_1 st^{k-1} + u_0 t^k = 0 \Rightarrow u_k = h_k t, u_0 = -h_1 s$, substitute into this equation, it gives:
$h_k s^k t + u_{k-1}s^{k-1}t + u_{k-2}s^{k-2}t^2 + ... + u_1 st^{k-1} - h_1 st^k = 0$
$st(h_k s^{k-1} + u_{k-1}s^{k-2} + u_{k-2}s^{k-1}t + ... + u_1 t^{k-2} - h_1 t^{k-1}) = 0$
$(h_k s + u_{k-1})s^{k-2} + u_{k-2}t^{k-3}s + ... + (u_1 - h_1 t)t^{k-2} = 0$
$\Rightarrow h_k s + u_{k-1} = -h_{k-1}t \Rightarrow u_{k-1} = -h_k s + h_{k-1}t$
and $\Rightarrow u_1 - h_1 t = -h_2 s \Rightarrow u_1 = -h_2 s + h_1 t$
and so on. □

3 Proof of theorem 1 and 2

$$A^x + B^y = C^z$$

1. Case A: $x = y = z$ (Fermat Last's theorem)
Express A,B, and C as polynomial of the same degree by using lemma 1:

$A = a_k s^k + a_{k-1}s^{k-1}t + a_{k-2}s^{k-2}t^2 + ... + a_1 st^{k-1} + a_0 t^k$
$B = b_k s^k + b_{k-1}s^{k-1}t + b_{k-2}s^{k-2}t^2 + ... + b_1 st^{k-1} + b_0 t^k$
$C = c_k s^k + c_{k-1}s^{k-1}t + c_{k-2}s^{k-2}t^2 + ... + c_1 st^{k-1} + c_0 t^k$
And we obtain the equation below:

$(a_k s^k + a_{k-1}s^{k-1}t + a_{k-2}s^{k-2}t^2 + ... + a_1 st^{k-1} + a_0 t^k)^x$
$+(b_k s^k + b_{k-1}s^{k-1}t + b_{k-2}s^{k-2}t^2 + ... + b_1 st^{k-1} + b_0 t^k)^x$
$= (c_k s^k + c_{k-1}s^{k-1}t + c_{k-2}s^{k-2}t^2 + ... + c_1 st^{k-1} + c_0 t^k)^x$
Combine coefficients with common $s^i t^j$ we obtain the equation:

$(a_k^x + b_k^x - c_k^x)s^{kx} + x(a_k^{x-1}a_{k-1} + b_k^{x-1}b_{k-1} - c_k^{x-1}c_{k-1})s^{kx-1}t + ... + (a_0^x + b_0^x - c_0^x)t^{kx} = 0$
Using the lemma 2, we get kx + 1 equations below:
$1^{th} : a_k^x + b_k^x - c_k^x = h_k t$

$2^{th} : a_k^{x-1}a_{k-1} + b_k^{x-1}b_{k-1} - c_k^{x-1}c_{k-1} = -h_k s + h_{k-1}t$
...
$(kx+1)^{th} : a_0^x + b_0^x - c_0^x = -h_1 s$
If all coefficients a_i, b_i, c_i of A, B and C polynomials are fixed numbers, then:
$1^{th} : a_k^x + b_k^x - c_k^x = 0$

$2^{th} : a_k^{x-1}a_{k-1} + b_k^{x-1}b_{k-1} - c_k^{x-1}c_{k-1} = 0$
...
$(kx+1)^{th} : a_0^x + b_0^x - c_0^x = 0$
And the number of variables, for $a_i : k+1$, for $b_i : k+1$,for $c_i : k+1$
The total of variables is 3k + 3.
The total of equations is kx + 1.

2. Case B: x, y and z are not the same, we need to homogenize the degree:
- Homogenization for degree:
-We denote l = LCM(x,y,z) as the least common multiple of x,y and z, we convert the equation to the equation with the same degree,(degree= kl) by selecting the polynomials below:

$A = a_{kl/x}s^{kl/x} + a_{(kl/x)-1}s^{(kl/x)-1}t + a_{(kl/x)-2}s^{(kl/x)-2}t^2 + ... + a_1 st^{(kl/x)-1} + a_0 t^{kl/x}$
$B = b_{kl/y}s^{kl/y} + b_{(kl/y)-1}s^{(kl/y)-1}t + b_{(kl/y)-2}s^{(kl/y)-2}t^2 + ... + b_1 st^{(kl/y)-1} + b_0 t^{kl/y}$
$C = c_{kl/z}s^{kl/x} + c_{(kl/z)-1}s^{(kl/y)-1}t + c_{(kl/z)-2}s^{(kl/z)-2}t^2 + ... + c_1 st^{(kl/z)-1} + c_0 t^{kl/z}$

Since l = LCM(x,y,z),hence $x \mid l, y \mid l, z \mid l$, that means kl/x, kl/y, kl/z are integers.
And we obtain equation below:

$(a_{kl/x}s^{kl/x} + a_{(kl/x)-1}s^{(kl/x)-1}t + a_{(kl/x)-2}s^{(kl/x)-2}t^2 + ... + a_1 st^{(kl/x)-1} + a_0 t^{kl/x})^x +$
$(b_{kl/y}s^{kl/y} + b_{(kl/y)-1}s^{(kl/y)-1}t + b_{(kl/y)-2}s^{(kl/y)-2}t^2 + ... + b_1 st^{(kl/y)-1} + b_0 t^{kl/y})^y =$
$(c_{kl/z}s^{kl/z} + c_{(kl/z)-1}s^{(kl/z)-1}t + c_{(kl/z)-2}s^{(kl/z)-2}t^2 + ... + c_1 st^{(kl/z)-1} + c_0 t^{kl/z})^z$

with degree = kl.
Combine coefficients with common $s^i t^j$ we obtain the equation:

$(a^x_{kl/x}+b^y_{kl/y}-c^z_{kl/z})s^{kl}+(xa^{x-1}_{kl/x}a_{(kl/x)-1}+yb^{y-1}_{kl/y}b_{(kl/y)-1}+zc^{z-1}_{kl/z}c_{(kl/z)-1})s^{kl-1}t+...+(a^x_0+b^y_0-c^z_0)t^{kl}=0$
Using the lemma 2, we get kl + 1 equations below:

$1^{th}: a^x_{kl/x}+b^y_{kl/y}-c^z_{kl/z}=h_k t$

$2^{th}: xa^{x-1}_{kl/x}a_{(kl/x)-1}+yb^{y-1}_{kl/y}b_{(kl/y)-1}-zc^{z-1}_{kl/z}c_{(kl/z)-1}=-h_k s+h_{k-1}t$
...
$(kl+1)^{th}: a^x_0+b^y_0-c^z_0=-h_1 s$
If all coefficients a_i, b_i, c_i of A, B and C polynomials are fixed numbers, then
$1^{th}: a^x_{kl/x}+b^y_{kl/y}-c^z_{kl/z}=0$

$2^{th}: xa^{x-1}_{kl/x}a_{(kl/x)-1}+yb^{y-1}_{kl/y}b_{(kl/y)-1}-zc^{z-1}_{kl/z}c_{(kl/z)-1}=0$
...
$(kl+1)^{th}: a^x_0+b^y_0-c^z_0=0$
And the number of variables, for $a_i: (kl/x)+1$, for $b_i: (kl/y)+1$,for $c_i: (kl/z)+1$
The total of variables is $kl/x+kl/y+kl/z+3$.
The total of equations is kl + 1

The system of equations is overdetermined if the number of equations is higher than the number of variables $\Rightarrow kl/x+kl/y+kl/z+3<kl+1 \Longleftrightarrow kl/x+kl/y+kl/z+3\leqslant kl$

Divide both sides by kl we obtain:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{3}{kl}\leqslant 1$$

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{3}{kl}$ is max if $k=1$, we get

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{3}{l}\leqslant 1$$

* Special case $x=y=z=l$

This formula is mentioned in the theorem 1 and 2.

If all coefficients a_i, b_i, c_i of A, B and C polynomials are not simultaneously fixed number for any k, then at least one of A, B and C is not determined.Thus, for any degree of A, B and C, there must be exist k so that all coefficients are fixed, meaning that for this value k, then all right sides are equal to 0.

If all right sides are equal to 0,then for the case:$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{3}{kl}\leqslant\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{3}{l}\leqslant 1$ the system of equations is overdetermined, and all equations in the system above are independent, hence the system is inconsistent , and have no non- trivial solutions.

Theorem 1 is proven, the theorem 2 is also true [8], the algorithm of the proof for general theorems [7], [8] is similar.

And adding argument in [7]the Beal conjecture and the Fermat - Catalan conjecture are proven.

References

[1] Beal conjecture - Wikipedia

[2] Fermat Catalan conjecture - Wikipedia

[3] Catalan's conjecture - Wikipedia

[4] Euler's Sum of powers conjecture - Wikipedia

[5] Quang N V, Theorem for W^n and Fermat's Last theorem Vixra: 1811.0072 v2(NT)

[6] Quang N V, A proof of the four color theorem by induction Vixra: 1601.0247 (CO)

[7] Quang N V, $A^x + B^y = C^z$ - Part 1: My theorem Vixra: 1910.0563 (NT)

[8] Quang N V, $A^x + B^y = C^z$ - Part 2: Another version of my theorem and infinite ascent Vixra: 2004.0690 (NT)

Email:
nguyenvquang67@gmail.com
quangnhu67@yahoo.com.vn