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The *abc* Conjecture is False

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To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

Abstract

In this note, I give the proof that the *abc* conjecture is false because, in the case $c > rad(abc)$, for $0 < \epsilon < 1$ we can not find the constant $K(\epsilon)$ so that $c < K(\epsilon).rad^{1+\epsilon}(abc)$ for c very large. A counter-example is given.

1. *Introduction*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**1**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1. Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad^{1+\epsilon}(abc), \quad K(\epsilon) \text{ depending only of } \epsilon. \tag{1.3}$$

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in November 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [**2**]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**1**]. A conjecture was proposed that $c < rad^2(abc)$ [**3**]:

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Conjecture 2. Let a, b, c positive integers relatively prime with $c = a + b$, then:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{1.4}$$

After studying the abc conjecture using different choices of the constant $K(\epsilon)$ and having attacked the problem from diverse angles, I have arrived to conclude that, assuming that $c < rad^2(abc)$ or $c < rad^{1.63}$ is true, the abc conjecture does not hold when $0 < \epsilon < 1$. Then the abc conjecture as it was defined is false. In this note, I give a counter-example that the abc conjecture is not true, in the case $rad(abc) < c$ taking $\epsilon \in]0,1[$ without assuming one of the two open questions : $c < rad^2(abc)$ and $c < rad^{1.63}(abc)$ that was proposed in 1996 by A. Nitaj [**4**].

The paper is organized as follows: in the second section, we give a counter-example that abc conjecture is false in the case $rad(abc) < c$, choosing $\epsilon \in]0,1[$.

2. *Proof the abc Conjecture is False*

We note $R = rad(ac)$ in the case $c = a + 1$ (respectively $R = rad(abc)$ if $c = a + b$).

2.1. *Case $c < R$:*

As $c < R \Longrightarrow c < R \Longrightarrow c < K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0$ since we choose $K(\epsilon) \geq 1$ and the conjecture (1) is verified.

2.2. *Case $c = R$*

Case to reject as a, b, c (respectively a, c) are relatively prime.

2.3. *Case $R < c$*

I will consider the case $c = a + 1$. I give the following counter example:

$$8^n = 2^{3n} = (7+1)^n = 7^n + 7^{n-1}n + \ldots + 7n + 1 = 7(7^{n-1} + n7^{n-2} + \ldots + n) + 1 \Longrightarrow$$
$$2^{3n} = 7(7^{n-1} + n7^{n-2} + \ldots + n) + 1 \tag{2.1}$$

We suppose that for n odd and large, the abc conjecture holds taking $\epsilon = \epsilon_0 \in]0,1[$. Then $\exists\ K(\epsilon_0) > 0$ and:

$$2^{3n} < K(\epsilon_0)R^{1+\epsilon_0} \tag{2.2}$$

We obtain:

$$rad(c) = rad(2^{3n}) = 2$$
$$rad(a) = rad(7(7^{n-1} + n7^{n-2} + \ldots + n)) = 7.rad(7^{n-1} + n7^{n-2} + \ldots + n) \Longrightarrow$$
$$rad(a) \leq 7.(7^{n-1} + n7^{n-2} + \ldots + n) \leq n.7^n \Longrightarrow rad(a) \leq 7^n n \tag{2.3}$$

We re-write the equation (2.2) in detail:

$$2^{3n} < K(\epsilon_0)2^{1+\epsilon_0}n^{1+\epsilon_0}7^{n(1+\epsilon_0)} \tag{2.4}$$

That we can write as:

$$e^{3nLog2\left(1 - (1+\epsilon_0)\dfrac{Log7}{3Log2} - \dfrac{1+\epsilon_0}{3Log2}\dfrac{Logn}{n}\right)} < K(\epsilon_0)2^{1+\epsilon_0} \tag{2.5}$$

We choose $\epsilon_0 = 0.06$ and we consider that n is very large ($n \longrightarrow +\infty$), then we obtain:

$$e^{3nLog2(1-0.99193)} \leq K(0.06)2^{1.06} \Longrightarrow +\infty \leq K(0.06)2^{1.06} \tag{2.6}$$

Hence the contradiction, and the *abc* conjecture is false for the value $\epsilon_0 = 0.06$.

We can announce the following theorems that are very easy to verify:

THEOREM 2.1. ***(The truncated abc conjecture:)*** *Let a, b, c positive integers relatively prime with $c = a + b$, and assuming $c < rad^2(abc)$ is true, then for each $\epsilon \geq 1$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{2.7}$$

where $K(\epsilon)$ is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad , \epsilon \geq 1$$

and:

THEOREM 2.2. ***(The truncated abc conjecture:)*** *Let a, b, c positive integers relatively prime with $c = a + b$, and assuming $c < rad^{1.63}(abc)$ is true, then for each $\epsilon \geq 0.63$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{2.8}$$

where $K(\epsilon)$ is a constant depending of ϵ proposed as :

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad , \epsilon \geq 0.63$$

Ouf! The end of the mystery!

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References

1. M. WALDSCHMIDT, On the abc Conjecture and some of its consequences presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, (2013).

2. K. KREMMERZ for Quanta Magazine, Titans of Mathematics Clash Over Epic Proof of ABC Conjecture. The Quanta Newsletter, 20 September 2018. www.quantamagazine.org. (2018).

3. P. MIHĂILESCU, Around ABC. European Mathematical Society Newsletter **N° 93**, September 2014. pp 29-34. (2014).

4. A. NITAJ, Aspects expérimentaux de la conjecture *abc*. Séminaire de Théorie des Nombres de Paris (1993-1994), London Math. Soc. Lecture Note Ser., Vol **n°235**. Cambridge Univ. Press. pp. 145-156. (1996)

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