An Exact Vacuum Solution of Five-Dimensional Spacetime

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Abstract

This paper is a review of previous paper [1], which is badly written and hard to read. There are three reason of writing this paper. First, Some results were presented immediately without an intermediate process. Second, when i recently read that even i cant find what i want to show. Third, some derivation was wrong.

1 Solution

This section just present the results of the last paper. The form of the solution was as follows (The coordinate are in oder ω, t, r, θ, ϕ, and tilde means five-dimension)

$$\tilde{g}_{ab} = \begin{bmatrix} -N^2 - \kappa^2(1-\frac{r_s r}{\Sigma}) & \kappa(1-\frac{r_s r}{\Sigma}) & 0 & 0 & \kappa\frac{r_s r}{\Sigma}a\sin^2\theta \\ \kappa(1-\frac{r_s r}{\Sigma}) & -(1-\frac{r_s r}{\Sigma}) & 0 & 0 & -\frac{r_s r}{\Sigma}a\sin^2\theta \\ 0 & 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & 0 & \Sigma & 0 \\ \kappa\frac{r_s r}{\Sigma}a\sin^2\theta & -\frac{r_s r}{\Sigma}a\sin^2\theta & 0 & 0 & (r^2+a^2)\sin^2\theta + \frac{r_s r}{\Sigma}a^2\sin^4\theta \end{bmatrix} \tag{1}$$

Where the parameters are:

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{2}$$

$$\Delta = r^2 - r_s r + a^2, \tag{3}$$

$$a = \frac{J}{Mc}, \tag{4}$$

$$r_s = \frac{2GM}{c^2}, \tag{5}$$

$$N = const, \tag{6}$$

$$\kappa = const, \tag{7}$$

where J is angular-momentum. It is interesting to note that N is a constant.

From Eq.(1), the non-zero Christoffel Symbols are (duplicate terms($\Gamma^a_{bc} = \Gamma^a_{cb}$) are omitted)

$\tilde{\Gamma}^t_{\omega r} = -\kappa r_s(a^2+r^2)(r^2-a^2cos^2\theta)/(2\Delta\Sigma^2)$,
$\tilde{\Gamma}^t_{\omega\theta} = \kappa r_s a^2 r\cos\theta\sin\theta/\Sigma^2$,
$\tilde{\Gamma}^t_{tr} = r_s(a^2+r^2)(r^2-a^2cos^2\theta)/(2\Delta\Sigma^2)$,
$\tilde{\Gamma}^t_{t\theta} = -r_s a^2 r\cos\theta\sin\theta/\Sigma^2$,
$\tilde{\Gamma}^t_{r\phi} = -r_s a\sin^2\theta(a^4\sin^2\theta - a^4 + 3r^4 + 2a^2r^2 - a^2r^2\sin^2\theta)/(2\Delta\Sigma^2)$,
$\tilde{\Gamma}^t_{\theta\phi} = r_s a^3 r(2\sin 2\theta - \sin 4\theta)/(8\Sigma^2)$,

$\tilde{\Gamma}^r_{\omega\omega} = -\kappa^2 r_s(a^2\cos^2\theta - r^2)\Delta/(2\Sigma^3)$,
$\tilde{\Gamma}^r_{\omega t} = \kappa r_s(a^2\cos^2\theta - r^2)\Delta/(2\Sigma^3)$,
$\tilde{\Gamma}^r_{\omega\phi} = -\kappa r_s a\sin^2\theta(a^2\cos^2\theta - r^2)\Delta/(2\Sigma^3)$,
$\tilde{\Gamma}^r_{tt} = -r_s(a^2\cos^2\theta - r^2)\Delta/(2\Sigma^3)$,
$\tilde{\Gamma}^r_{t\phi} = r_s a\sin^2\theta(a^2\cos^2\theta - r^2)\Delta/(2\Sigma^3)$,
$\tilde{\Gamma}^r_{rr} = -\Delta(r_s(r^2 - a^2\cos^2\theta) - 2a^2r\sin^2\theta)/(2\Sigma\Delta^2)$,
$\tilde{\Gamma}^r_{r\theta} = -a^2\cos\theta\sin\theta/\Sigma$,
$\tilde{\Gamma}^r_{\theta\theta} = -r\Delta/\Sigma$,
$\tilde{\Gamma}^r_{\phi\phi} = -\sin^2\theta\Delta(2r^5 + 4a^2r^3\cos^2\theta + r_s a^4\cos^2\theta - r_s a^4\cos^4\theta + 2a^4r\cos^4\theta - r_s a^2r^2\sin^2\theta)/(2\Sigma^3)$,

$\tilde{\Gamma}^\theta_{\omega\omega} = -\kappa^2 r_s a^2 r\cos\theta\sin\theta/\Sigma^3$,
$\tilde{\Gamma}^\theta_{\omega t} = \kappa r_s a^2 r\cos\theta\sin\theta/\Sigma^3$,
$\tilde{\Gamma}^\theta_{\omega\phi} = -\kappa r_s ar\cos\theta\sin\theta(a^2+r^2)/\Sigma^3$,
$\tilde{\Gamma}^\theta_{tt} = -r_s a^2 r\cos\theta\sin\theta/\Sigma^3$,
$\tilde{\Gamma}^\theta_{t\phi} = r_s ar\cos\theta\sin\theta(a^2+r^2)/\Sigma^3$,
$\tilde{\Gamma}^\theta_{rr} = a^2\cos\theta\sin\theta/(\Sigma\Delta)$,
$\tilde{\Gamma}^\theta_{r\theta} = r/\Sigma$,
$\tilde{\Gamma}^\theta_{\theta\theta} = -a^2\cos\theta\sin\theta/\Sigma$,
$\tilde{\Gamma}^\theta_{\phi\phi} = -(2\cos\theta\sin\theta(a^2+r^2) + (4r_s a^2 r\cos\theta\sin^3\theta)/\Sigma + (2r_s a^4 r\cos\theta\sin^5\theta)/\Sigma^2)/(2\Sigma)$,

$\tilde{\Gamma}^\phi_{\omega r} = \kappa r_s a(a^2\cos^2\theta - r^2)/(2\Sigma^2\Delta)$,
$\tilde{\Gamma}^\phi_{\omega\theta} = \kappa r_s ar\cos\theta/(\Sigma^2\sin\theta)$,
$\tilde{\Gamma}^\phi_{tr} = -r_s a(a^2\cos^2\theta - r^2)/(2\Sigma^2\Delta)$,
$\tilde{\Gamma}^\phi_{t\theta} = -r_s ar\cos\theta/(\Sigma^2\sin\theta)$,
$\tilde{\Gamma}^\phi_{r\phi} = -(2r_s r^4 - 2r^5 - 4a^2r^3\cos^2\theta - r_s a^4\cos^2\theta + r_s a^4\cos^4\theta - 2a^4r\cos^4\theta + 2r_s a^2r^2\cos^2\theta + r_s a^2r^2\sin^2\theta)/(2\Sigma^2\Delta)$,
$\tilde{\Gamma}^\phi_{\theta\phi} = \cos\theta(\Sigma^2 + r_s a^2 r\sin^2\theta)/(\Sigma^2\sin\ \theta)$.

From above calculations, the Ricci tensor is

$$\tilde{R}_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

This is exactly vacuum solution. This calculation were done by MATLAB [2] and the code can find at APPENDIX A.

In fact, I started with a general form of metric tensor

$$\tilde{g}_{ab} = \begin{bmatrix} -N^2 + \beta^\lambda \beta_\lambda & \beta_\nu \\ \beta_\mu & g_{\mu\nu} \end{bmatrix}. \tag{9}$$

However, with plausible assumptions regarding physical phenomena, the condition of N and β_μ are derived. Then, By restricting $\tilde{R}_{ab} = 0$, the general form of $\tilde{g}_{ab}$ can be derived.

2 Constant N and κ

Among a higher than four-dimension, after take the one which is related with electromagnetism, make up five-dimension. Therefore, the constant N amd κ in Eq.(1) must be related with that. Before deriving them, we need to identify $\tilde{U}^\omega$ of 5-velocity.

The line element is

$$-d\tilde{s}^2 = -(\tilde{c}_* d\tilde{\tau})^2 = \tilde{g}_{ab} dx^a dx^b, \tag{10}$$

where c_* is used related with speed of light. To minimize action, the equation of motion is

$$\frac{D\tilde{U}^a}{D\tilde{\tau}} = \frac{d\tilde{U}^a}{d\tilde{\tau}} + \tilde{\Gamma}^a_{bc} \tilde{U}^b \tilde{U}^c = 0. \tag{11}$$

What we want to get from the above two equations is the Lorentz force

$$\frac{DU^\mu}{D\tau} = \frac{q}{m} F^{\mu\nu} U_\nu. \tag{12}$$

As above equation is written by using four-velocity and τ, we need information about four-velocity from five-velocity which is defined as

$$\tilde{U}^a = \frac{dx^a}{d\tilde{\tau}}. \tag{13}$$

Let define four-velocity as

$$U^\mu \equiv \frac{dx^\mu}{d\tau}. \tag{14}$$

Then we need to find relation with five-veloicty. When five-dimension foliated about the fifth coordinate ω, there are foliated layer, which is four-dimensional hyper-surface. Four-velocity is projection to that.

$$U^\mu = \tilde{U}^\mu. \tag{15}$$

Although this may seems obvious, but this result depends on ansatz form (see Eq.(9)). As four-vector, U^μ, is projected vector onto four-dimensional hyper-surface,

$$U_\mu = g_{\mu\nu} U^\mu. \tag{16}$$

By using above four relation,

$$\tilde{U}_\mu = \tilde{g}_{\mu a} \tilde{U}^a = U_\mu + \beta_\mu \tilde{U}^\omega. \tag{17}$$

To derive Eq.(12), we will use following relation.

$$\frac{D\tilde{U}^a}{D\tau} = \frac{d\tilde{\tau}}{d\tau} \frac{D\tilde{U}^a}{D\tilde{\tau}} = 0 \tag{18}$$

$$\frac{d}{d\tilde{\tau}} = \tilde{U}^a \frac{\partial}{\partial x^a} \tag{19}$$

$$\frac{d}{d\tau} = U^\lambda \frac{\partial}{\partial x^\lambda} \tag{20}$$

The Eq.(18) is

$$\frac{D\tilde{U}^\mu}{D\tau} = \frac{d\tilde{\mathbf{U}}}{d\tau} * \tilde{e}^\mu = \frac{d\tilde{U}^a}{d\tau} \delta^\mu_a + \tilde{U}^a \left(\frac{\partial \tilde{e}_a}{\partial x^\nu} * \tilde{e}^\mu \right) U^\nu = \frac{d\tilde{U}^\mu}{d\tau} + \tilde{\Gamma}^\mu_{a\nu} \tilde{U}^a U^\nu = 0 \tag{21}$$

And this can be rewritten as

$$\frac{dU^\mu}{d\tau} + \tilde{\Gamma}^\mu_{\lambda\nu} U^\lambda U^\nu = -\tilde{\Gamma}^\mu_{\omega\nu} \tilde{U}^\omega U^\nu \tag{22}$$

At this point, with comparing with Eq.(12), some parameter should have test particle's imformation (charge and mass). The only possible case is

$$\tilde{U}^\omega \propto \frac{q}{m}. \tag{23}$$

And $\tilde{U}^\omega$ is assumed equal to q/m (Magnification is optional), which is constant of motion.
With this assumption and Eq.(9) (see APPENDIX.B-(a)),

$$\tilde{\Gamma}^\omega_{ab} = 0, \tag{24}$$

$$\tilde{\Gamma}^\mu_{\lambda\nu} = \Gamma^\mu_{\lambda\nu}, \tag{25}$$

$$\tilde{\Gamma}^\mu_{\omega\nu} = g^{\mu\rho} \left(\partial_\nu \left(\frac{\beta_\rho}{2} \right) - \partial_\rho \left(\frac{\beta_\nu}{2} \right) \right) \tag{26}$$

Then the Eq.(22) can be rewritten as

$$\frac{dU^\mu}{d\tau} + \Gamma^\mu_{\lambda\nu} U^\lambda U^\nu = \frac{q}{m} g^{\mu\rho} \left(\partial_\rho \left(\frac{\beta_\nu}{2} \right) - \partial_\nu \left(\frac{\beta_\rho}{2} \right) \right) U^\nu \tag{27}$$

By comparing Eq.(12), we can identify β_μ as vector potential $2A_\mu$ then Eq.(27) is same with Eq.(12). Finally, we can get κ,

$$\kappa = \frac{Qc}{4\pi\epsilon_0 GM} \tag{28}$$

The remaining parts is to determining N. In the previous paper, N was determined as

$$\frac{1}{N^2} = \frac{8\pi G/c^4}{2\mu_0} = \frac{4\pi\epsilon_0 G}{c^2} \tag{29}$$

If we just list the previous results (The derivation can find at APPENDIX.C),

$$\tilde{R}_{\omega\omega} = -\Box\phi + F^{\lambda\rho}F_{\lambda\rho} = 0, \tag{30}$$

$$\tilde{R}_{\omega\mu} = \nabla^{\nu}F_{\mu\nu} = \mu_0 J_\mu = 0, \tag{31}$$

which is field equation as expected, and the five-dimension Ricci-scalar is

$$\tilde{R} = R - \frac{1}{N^2}\left(F_{\lambda\rho}F^{\lambda\rho} - 2\mu_0 J^\lambda A_\lambda\right) + \frac{1}{N^2}\Box\phi. \tag{32}$$

As can found from coefficient of second term, it is the reason why we determined N as Eq.(29). And $(-)$ sign of $F_{\lambda\rho}F^{\lambda\rho}$ is only derived by considering time-like fifth coordinate. We have already noticed that the above result, which is $(-)$ sign, is given also when we consider the fifth coordinate as space-like with following ansatz [3],

$$\tilde{g}_{ab} = \begin{bmatrix} \phi^2 & \phi^2 A_\nu \\ \phi^2 A_\mu & g_{\mu\nu} + \phi^2 A_\mu A_\nu \end{bmatrix}. \tag{33}$$

But it becomes reversed by considering ansatz form, Eq.(9). But when we convert $\tilde{g}_{\omega\omega}$ to space-like form, $N^2 - \kappa^2\left(1 - \frac{r_s r}{\Sigma}\right)$, which gives vacuum solution, was mathematically also possible.

In the previous paper, several conceptual error have been discovered. It is not recommended to read it for the purpose of seeing them.

3 Chage Q, Mass M and Angular-Momentum J

The Eq.(1), have three-symmetry and there are conserved quantities corresponding to each symmetry:

$$\frac{\partial \tilde{g}_{ab}}{\partial \omega} = 0,\ \partial_\omega \to Q, \tag{34}$$

$$\frac{\partial \tilde{g}_{ab}}{\partial t} = 0,\ \partial_t \to M, \tag{35}$$

$$\frac{\partial \tilde{g}_{ab}}{\partial \phi} = 0,\ \partial_\phi \to J. \tag{36}$$

In this section, we will derive explicitly these quantities. The concept is as follows:

$$\int \rho_{charge} dV = \int j^0_{charge} dV = Q. \tag{37}$$

To express above equation in general relativity, the following relation are used:

$$\tilde{\nabla}_a \tilde{T}^{ab} = 0, \tag{38}$$

$$\tilde{\nabla}_a \eta_b + \tilde{\nabla}_b \eta_a = 0, \tag{39}$$

where η is killing vector. As $T^{\mu\nu}$ is symmetric tensor

$$\tilde{\nabla}_a \left(\tilde{T}^{ab} \eta_b \right) = 0. \tag{40}$$

Then, we consider j^a as $\tilde{T}^{ab}\eta_b$. In other words,

$$\eta = \partial_\omega \rightarrow \int \tilde{T}^{0b} \eta_b \; dV \propto Q, \tag{41}$$

$$\eta = \partial_t \rightarrow \int \tilde{T}^{0b} \eta_b \; dV \propto M, \tag{42}$$

$$\eta = \partial_\phi \rightarrow \int \tilde{T}^{0b} \eta_b \; dV \propto J, \tag{43}$$

where upper index 0 means time component. The invariant form is as follows:

$$\int \tilde{T}^{0b} \eta_b \sqrt{\tilde{g}} \; d^5 x. \tag{44}$$

One can think that this integration will get zero as $\tilde{R}_{ab}$ equal to zero. In fact, $\tilde{R}_{ab}$ equal to zero at 'outside' of Black-Hole. The integration should be divided by two parts, inside and outside.

Although this paper does not treated the inside of Black-Hole, we can calculate this integral. There is analogous to this problem: when we calculate total charge of the charged sphere, if we know outside electric-field, we do not need to know the interior charge distribution. In other words, if we can convert Eq.(44) to surface integral we can calculate that. We will using following relation:

$$\tilde{T}_{ab} = \frac{1}{\kappa} \left(\tilde{R}_{ab} - \frac{1}{2} \tilde{R} \tilde{g}_{ab} \right), \tag{45}$$

$$\tilde{R}_{ab} \eta^b = \tilde{R}_{ba} \eta^b = \tilde{R}^c_{bca} \eta^b = \left(\tilde{\nabla}_c \tilde{\nabla}_a - \tilde{\nabla}_a \tilde{\nabla}_c \right) \eta^c. \tag{46}$$

Because η follows killing equation, $\tilde{\nabla}_a \eta^a = 0$, second term of right side of Eq.(46) vanishes. Then, Eq.(44) can be written as:

$$\frac{1}{\kappa} \int \left(\tilde{\nabla} a \left(\tilde{\nabla}^0 \eta^a \right) - \tilde{R} \tilde{g}^{0b} \eta_b \right) \sqrt{\tilde{g}} d^5 x \tag{47}$$

For now, ignore second term. From first integrand term, five-dimensional volume integral can be converted to four-dimensional surface integral. To calculate easily, by using killing equation, we can rewrite this as follows:

$$\frac{1}{\kappa} \int \tilde{\nabla} a \left(\tilde{\nabla}^0 \left(\frac{\eta^a}{2} \right) - \tilde{\nabla}^a \left(\frac{\eta^0}{2} \right) \right) \sqrt{\tilde{g}} d^5 x \tag{48}$$

In section 2, if $\eta = \partial_\omega$, η^μ is identified as $2A^\mu$. As noted, Then integrand term is,

$$\tilde{\nabla}_a F^{0b}, \tag{49}$$

where F is field strength tensor and $F^{0i} \propto E^i$. Now to calculate Eq.(48), we consider $r = const$

surface and we can ignore dx^4 and dx^5. Because of five-divergence, although $\tilde{g}_{ab}$ is not function of t and ω, there are delta function for hidden t and ω also. Note that this result is independent of surface shape and Eq.(44) guarantees this. It was too long to calculate this.

$$\frac{1}{\kappa}\int\left(\tilde{\nabla}^0\left(\frac{\eta^1}{2}\right)-\tilde{\nabla}^1\left(\frac{\eta^0}{2}\right)\right)\sqrt{\tilde{g}}d\theta d\phi \tag{50}$$

From Eq.(1),

Case1. $\eta=\partial_\omega$

$$-\frac{c^4}{8\pi G}\int\kappa\left[\frac{r_s\left(r^2+a^2\right)\left(-r^2+a^2\cos^2\theta\right)}{\left(r^2+a^2\cos^2\theta\right)^3}\right]N\left(r^2+a^2\cos^2\theta\right)\sin\theta d\theta d\phi$$

$$=-\frac{c^4}{8\pi G}N\kappa\int_0^{2\pi}\left[\frac{r_s\left(r^2+a^2\right)\cos\theta}{r^2+a^2\cos^2\theta}\right]_0^\pi d\phi=-N\kappa Mc^2=-Q\times N\frac{c^3}{4\pi\epsilon_0 G} \tag{51}$$

Case2. $\eta=\partial_{\mathbf{t}}$

$$\frac{c^4}{8\pi G}\int\left[\frac{r_s\left(r^2+a^2\right)\left(-r^2+a^2\cos^2\theta\right)}{\left(r^2+a^2\cos^2\theta\right)^3}\right]N\left(r^2+a^2\cos^2\theta\right)\sin\theta d\theta d\phi$$

$$=\frac{c^4}{8\pi G}N\int_0^{2\pi}\left[\frac{r_s\left(r^2+a^2\right)\cos\theta}{r^2+a^2\cos^2\theta}\right]_0^\pi d\phi=M\times Nc^2 \tag{52}$$

Case3. $\eta=\partial_\phi$

$$\int\left[\frac{ar_s\sin^3\theta\left(-a^2r^2\sin^2\theta+a^4\sin^2\theta+3r^4+2a^2r^2-a^4\right)}{\left(a^2\cos^2(\theta)+r^2\right)^3}\right]N(r^2+a^2\cos\theta)\sin\theta d\theta d\phi$$

$$=\frac{c^4}{8\pi G}N\int_0^{2\pi}\left[\frac{ar_s\cos\theta\left(\left(r^2-a^2\right)\cos^2\theta-3r^2-a^2\right)}{a^2\cos^2\theta+r^2}\right]_0^\pi d\phi=\frac{Nar_sc^4}{G}=J\times 2Nc \tag{53}$$

4 Over the Five-Dimension

As you may have noticed, mathematically, we can get vacuum solution of 'Any N-Dimension', which based on N-1 dimension. I checked this up to six-dimension, doing more was meaningless. Although this method does not guarantees the general solution, method is simple. Let we know a vacuum solution of N-1-dimension. Let the form of N-dimension metric,

$${}^N g_{\mathbb{AB}}=\begin{bmatrix}\mp N_c^2+\beta^{\mathbf{w}}\beta_{\mathbf{w}} & \beta_{\mathbf{v}}\\ \beta_{\mathbf{u}} & {}^{N-1}g_{\mathbf{uv}}\end{bmatrix}. \tag{54}$$

And assume that ${}^{N-1}g_{\mathbf{uv}}$ have symmetry,

$$\frac{\partial^{N-1}g_{\mathbf{uv}}}{\partial x^\alpha}=0. \tag{55}$$

Then if we simply put

$$\beta_u = k \times^{N-1} g_{\alpha\mathbf{u}}, \tag{56}$$

$$N_c = const, \tag{57}$$

where k is constant. Then we can get N-dimension vacuum solution.

5 Summary

1. It is 5D vacuum solution,

$$\tilde{g}_{ab} = \begin{bmatrix} \mp N^2 - \kappa^2(1-\frac{r_s r}{\Sigma}) & \kappa(1-\frac{r_s r}{\Sigma}) & 0 & 0 & \kappa\frac{r_s r}{\Sigma} a\sin^2\theta \\ \kappa(1-\frac{r_s r}{\Sigma}) & -(1-\frac{r_s r}{\Sigma}) & 0 & 0 & -\frac{r_s r}{\Sigma} a\sin^2\theta \\ 0 & 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & 0 & \Sigma & 0 \\ \kappa\frac{r_s r}{\Sigma} a\sin^2\theta & -\frac{r_s r}{\Sigma} a\sin^2\theta & 0 & 0 & (r^2+a^2)\sin^2\theta + \frac{r_s r}{\Sigma} a^2\sin^4\theta \end{bmatrix}$$

2. If we set $\tilde{U}^5 = \frac{q}{m}$, with general form of metric tensor,

$$\tilde{g}_{ab} = \begin{bmatrix} \mp N^2 + \beta^\lambda\beta_\lambda & \beta\nu \\ \beta_\mu & g_{\mu\nu} \end{bmatrix},$$

Lorentz equation can be derived,

$$\frac{dU^\mu}{d\tau} + \Gamma^\mu_{\lambda\nu}U^\lambda U^\nu = \frac{q}{m} g^{\mu\rho}\left(\partial_\rho\left(\frac{\beta_\nu}{2}\right) - \partial_\nu\left(\frac{\beta_\rho}{2}\right)\right)U^\nu.$$

Then we can identify β_μ as $2A_\mu$. So, from Eq.(1),

$$\kappa = \frac{Qc}{4\pi\epsilon_0 GM}$$

3. The Eq.(1) have three symmetry. And there are conserved quantities. For following equation,

$$\frac{1}{\kappa}\int \tilde{\nabla}a\left(\tilde{\nabla}^0\left(\frac{\eta^a}{2}\right) - \tilde{\nabla}^a\left(\frac{\eta^0}{2}\right)\right)\sqrt{|\tilde{g}|}d^5x,$$

the calculated result is

$$\eta = \partial_\omega \to Q \times N\frac{c^3}{4\pi\epsilon_0 G}$$

$$\eta = \partial_t \to M \times Nc^2$$

$$\eta = \partial_\phi \to J \times 2Nc$$

6 APPENDIX.A

```
syms w t r u v       %coordinate
syms Sig Del         %variable
syms Rs N K a real   %constant
D=5;
X=[w t r u v ];
g=[ +N^2-K^2*(1-Rs*r/Sig) K*(1-Rs*r/Sig)        0         0      K*Rs*r/Sig*a*sin(u)^2                   ;
    K*(1-Rs*r/Sig)         -(1-Rs*r/Sig)         0         0      -Rs*r/Sig*a*sin(u)^2                    ;
    0                      0                     Sig/Del   0      0                                       ;
    0                      0                     0         Sig    0                                       ;
    K*Rs*r/Sig*a*sin(u)^2 -Rs*r/Sig*a*sin(u)^2  0         0      (r^2+a^2)*sin(u)^2+Rs*r/Sig*a^2*sin(u)^4 ];
Sig=r^2+a^2*cos(u)^2; Del=r^2-Rs*r+a^2;
g=subs(g); g=simplify(g);
G=simplify(inv(g));
'==========================1.Christoffel Simbol==============================Chr'
for i=1:D
 for j=1:D
  for k=1:D
   chr=0;
   for l=1:D
    chr=chr+G(i,l)*(1/2)*(diff(g(l,j),X(k))+diff(g(l,k),X(j))-diff(g(j,k),X(l)));
   end
   Chr(i,j,k)=chr;
  end
 end
end
Chr=simplify(Chr);
'==============================2.Ricci Tensor===============================Ricci'

for i=1:D
 for j=1:D
  ricci1=0;
  ricci2=0;
  for l=1:D
   for m=1:D
    ricci2=ricci2+Chr(l,m,l)*Chr(m,j,i)-Chr(l,m,j)*Chr(m,l,i);
   end
   ricci1=ricci1+diff(Chr(l,i,j),X(l))-diff(Chr(l,l,i),X(j));
  end
  Ricci(i,j)=ricci1+ricci2;
 end
end
Ricci=simplify(Ricci)
```

Figure 1: Code for Ricci Tensor

Note that (+) sign of N^2 were used for check. Originally i used (-) sign. And both gives $\tilde{R}_{ab} = 0$. To calculate Ricci tensor, $\tilde{\Gamma}^i_{jk}$ is used as Chr(i,j,k). With this simple code you can calculate Ricci tensors of all dimensions according to your computer performance. This was done in a few seconds on my personal computer.

7 APPENDIX.B

The metric tensor form is

$$\tilde{g}_{ab} = \begin{bmatrix} -N^2 + \beta^\lambda \beta_\lambda & \beta_\nu \\ \beta_\mu & g_{\mu\nu} \end{bmatrix}, \quad \tilde{g}^{ab} = \begin{bmatrix} -\frac{1}{N^2} & \frac{\beta^\nu}{N^2} \\ \frac{\beta^\mu}{N^2} & g^{\mu\nu} - \frac{\beta^\mu \beta^\nu}{N^2} \end{bmatrix}. \tag{58}$$

We impose two symmetry to metric tensor,

$$\partial_\omega \tilde{g}_{ab} = \partial_t \tilde{g}_{ab} = 0. \tag{59}$$

The Christoffel Symbols, Γ^ω_{ab}, are,

$$\begin{aligned} \tilde{\Gamma}^\omega_{\mu\nu} &= \tilde{g}^{\omega\omega}\tilde{\Gamma}_{\omega\mu\nu} + \tilde{g}^{\omega\lambda}\tilde{\Gamma}_{\lambda\mu\nu} \\ &= -\frac{1}{2N^2}\left(\partial_\mu \beta_\nu + \partial_\mu \beta_\nu - \partial_\omega g_{\mu\nu}\right) + \frac{\beta^\lambda}{2N^2}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right) \\ &= -\frac{1}{2N^2}\left(\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu - \partial_\omega g_{\mu\nu}\right) \\ &= -\frac{1}{2N^2}\left(\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu\right) \end{aligned} \tag{60}$$

$$\begin{aligned} \tilde{\Gamma}^\omega_{\omega\mu} &= \tilde{g}^{\omega\omega}\tilde{\Gamma}_{\omega\omega\mu} + \tilde{g}^{\omega\lambda}\tilde{\Gamma}_{\lambda\omega\mu} \\ &= -\frac{1}{2N^2}\partial_\mu\left(-N^2 + \beta^\sigma \beta_\sigma\right) + \frac{\beta^\lambda}{2N^2}\left(\partial_\mu \beta_\lambda - \partial_\lambda \beta_\mu + \partial_\omega g_{\lambda\mu}\right) \\ &= \frac{1}{N}\partial_\mu N - \frac{1}{N^2}\beta^\sigma \nabla_\mu \beta_\sigma + \frac{\beta^\lambda}{2N^2}\left(\nabla_\mu \beta_\lambda - \nabla_\lambda \beta_\mu + \partial_\omega g_{\lambda\mu}\right) \\ &= \frac{1}{N}\partial_\mu N - \frac{1}{N^2}\beta^\sigma \nabla_\mu \beta_\sigma + \frac{\beta^\lambda}{2N^2}\left(\nabla_\mu \beta_\lambda - \nabla_\lambda \beta_\mu\right) \end{aligned} \tag{61}$$

$$\begin{aligned} \tilde{\Gamma}^\omega_{\omega\omega} &= \tilde{g}^{\omega\omega}\tilde{\Gamma}_{\omega\omega\omega} + \tilde{g}^{\omega\lambda}\tilde{\Gamma}_{\lambda\omega\omega} \\ &= -\frac{1}{2N^2}\partial_\omega\left(-N^2 + \beta^\sigma \beta_\sigma\right) + \frac{\beta^\lambda}{2N^2}\left(2\partial_\omega \beta_\lambda - \partial_\lambda\left(-N^2 + \beta^\sigma \beta_\sigma\right)\right) \\ &= -\frac{\beta^\lambda}{2N^2}\partial_\lambda\left(-N^2 + \beta^\sigma \beta_\sigma\right) \end{aligned} \tag{62}$$

If we set $\tilde{U}^5 = \frac{q}{m}$ (constant of motion), from equation of motion,

$$\tilde{\Gamma}^\omega_{ab} = 0, \tag{63}$$

then parameter β and N is constrained.

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0. \tag{64}$$

Using above equation, by substitute $\nabla_\lambda \beta_\mu$ of Eq.(61) to $-\nabla_\mu \beta_\lambda$, only remaining term is

$$N \partial_\mu N = 0 \tag{65}$$

As can notice from Eq.64, which is killing equation, the only possible case is

$$\beta_\mu = \alpha g_{0\mu}. \tag{66}$$

Then, β^μ is equal to $(\alpha, 0, 0, 0)$. So Eq.(62) gives nothing. Now we have only two condition

$$\beta_\mu = \alpha g_{0\mu}, \quad N = const, \tag{67}$$

where α is constant. The other parts based on this condition are

$$\begin{aligned}
\tilde{\Gamma}^{\mu}_{\lambda\nu} &= \tilde{g}^{\mu\omega}\tilde{\Gamma}_{\omega\lambda\nu} + \tilde{g}^{\mu\rho}\tilde{\Gamma}_{\rho\lambda\nu} \\
&= \frac{\beta^{\mu}}{N^2}\tilde{\Gamma}_{\omega\lambda\nu} + \left(g^{\mu\rho} - \frac{\beta^{\mu}\beta^{\rho}}{N^2}\right)\tilde{\Gamma}_{\rho\lambda\nu} \\
&= g^{\mu\rho}\tilde{\Gamma}_{\rho\lambda\nu} + \frac{\beta^{\mu}}{N^2}\left(\tilde{\Gamma}_{\omega\lambda\nu} - \beta^{\rho}\tilde{\Gamma}_{\rho\lambda\nu}\right) \\
&= \Gamma^{\mu}_{\lambda\nu}
\end{aligned} \tag{68}$$

(Parenthese of third line equal to zero by condition, and $\tilde{\Gamma}^{\mu}_{\lambda\nu} = \Gamma^{\mu}_{\lambda\nu}$ is used.)

$$\begin{aligned}
\tilde{\Gamma}^{\mu}_{\omega\nu} &= \tilde{g}^{\mu\omega}\tilde{\Gamma}_{\omega\omega\nu} + \tilde{g}^{\mu\rho}\tilde{\Gamma}_{\rho\omega\nu} \\
&= \frac{\beta^{\mu}}{N^2}\tilde{\Gamma}_{\omega\omega\nu} + \left(g^{\mu\rho} - \frac{\beta^{\mu}\beta^{\rho}}{N^2}\right)\tilde{\Gamma}_{\rho\omega\nu} \\
&= g^{\mu\rho}\tilde{\Gamma}_{\rho\omega\nu} + \frac{\beta^{\mu}}{N^2}\left(\tilde{\Gamma}_{\omega\omega\nu} - \beta^{\rho}\tilde{\Gamma}_{\rho\omega\nu}\right) \\
&= g^{\mu\rho}\left(\partial_{\nu}\left(\frac{\beta_{\rho}}{2}\right) - \partial_{\rho}\left(\frac{\beta_{\nu}}{2}\right)\right)
\end{aligned} \tag{69}$$

(Parenthese of third line equal to zero by condition)

$$\begin{aligned}
\tilde{\Gamma}^{\mu}_{\omega\omega} &= \tilde{g}^{\mu\omega}\tilde{\Gamma}_{\omega\omega\omega} + \tilde{g}^{\mu\rho}\tilde{\Gamma}_{\rho\omega\omega} \\
&= 0 + \left(g^{\mu\rho} - \frac{\beta^{\mu}\beta^{\rho}}{N^2}\right)\tilde{\Gamma}_{\rho\omega\omega} \\
&= g^{\mu\rho}\tilde{\Gamma}_{\rho\omega\omega} \\
&= -\nabla^{\mu}\phi
\end{aligned} \tag{70}$$

(where $\phi \equiv \frac{\tilde{g}_{\omega\omega}}{2}$)

8 APPENDIX.C

In this section use results of APPENDIX.B.

$$\begin{aligned}
\tilde{R}_{\mu\nu} &= \tilde{R}^{a}_{\mu a\nu} \\
&= \partial_{[a}\tilde{\Gamma}^{a}_{\nu]\mu} - \tilde{\Gamma}^{b}_{\mu[a}\tilde{\Gamma}^{a}_{\nu]b} \\
&= \partial_{[\lambda}\tilde{\Gamma}^{\lambda}_{\nu]\mu} - \tilde{\Gamma}^{\rho}_{\mu[\lambda}\tilde{\Gamma}^{\lambda}_{\nu]\rho} \\
&= \partial_{[\lambda}\Gamma^{\lambda}_{\nu]\mu} - \Gamma^{\rho}_{\mu[\lambda}\Gamma^{\lambda}_{\nu]\rho} \\
&= R^{\lambda}_{\mu\lambda\nu} \\
&= R_{\mu\nu}
\end{aligned} \tag{71}$$

(The third line used $\tilde{\Gamma}^{\omega}_{ab} = 0$, fourth line used $\tilde{\Gamma}^{\mu}_{\lambda\rho} = \Gamma^{\mu}_{\lambda\rho}$)

$$\begin{aligned}
\tilde{R}_{\mu\omega} &= \tilde{R}^{a}_{\mu a\omega} \\
&= \partial_{[a}\tilde{\Gamma}^{a}_{\omega]\mu} - \tilde{\Gamma}^{b}_{\mu[a}\tilde{\Gamma}^{a}_{\omega]b} \\
&= \partial_{[\lambda}\tilde{\Gamma}^{\lambda}_{\omega]\mu} - \tilde{\Gamma}^{\rho}_{\mu[\lambda}\tilde{\Gamma}^{\lambda}_{\omega]\rho} \\
&= \partial_{\lambda}F_{\mu}{}^{\lambda} - \Gamma^{\rho}_{\mu\lambda}F_{\rho}{}^{\lambda} + \Gamma^{\lambda}_{\lambda\rho}F_{\mu}{}^{\rho} \\
&= \nabla_{\lambda}F_{\mu}{}^{\lambda}
\end{aligned} \tag{72}$$

$$
\begin{aligned}
\tilde{R}_{\omega\omega} &= \tilde{R}^{a}_{\omega a\omega} \\
&= \partial_{[a}\tilde{\Gamma}^{a}_{\omega]\omega} - \tilde{\Gamma}^{b}_{\omega[a}\tilde{\Gamma}^{a}_{\omega]b} \\
&= \partial_{[\lambda}\tilde{\Gamma}^{\lambda}_{\omega]\omega} - \tilde{\Gamma}^{\rho}_{\omega[\lambda}\tilde{\Gamma}^{\lambda}_{\omega]\rho} \\
&= \partial_{\lambda}\left(-\nabla^{\lambda}\phi\right) - F_{\lambda}{}^{\rho}F_{\rho}{}^{\lambda} + \Gamma^{\lambda}_{\lambda\rho}\left(-\nabla^{\rho}\phi\right) \\
&= -\Box\phi + F_{\lambda\rho}F^{\lambda\rho}
\end{aligned} \tag{73}
$$

(By substitutue $\beta_\mu = 2A_\mu$, $\tilde{\Gamma}^{\lambda}_{\omega\rho}$ is used as $F_{\rho}{}^{\lambda}$)

The five-dimension Ricci scalar is

$$
\begin{aligned}
\tilde{R} &= \tilde{R}_{ab}\tilde{g}^{ab} \\
&= -\frac{1}{N^2}\left(-\Box\phi + F_{\lambda\rho}F^{\lambda\rho}\right) + 2\beta^{\mu}\nabla_{\lambda}F_{\mu}{}^{\lambda} + \left(g^{\mu\nu} - \frac{\beta^{\mu}\beta^{\nu}}{N^2}\right)R_{\mu\nu} \\
&= R - \frac{1}{N^2}F_{\lambda\rho}F^{\lambda\rho} + \frac{4\mu_0}{N^2}J_{\mu}A^{\mu} - \frac{1}{N^2}R_{\mu\nu}\beta^{\mu}\beta^{\nu} + \frac{1}{N^2}\Box\phi
\end{aligned} \tag{74}
$$

References

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