Proof of Fermat Last Theorem based on successive presentations of pairs of odd numbers

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A simpler proof of Fermat Last Theorem (FLT), based mostly on new concepts, is suggested. FLT was formulated by Fermat in 1637, and proved by A. Wiles in 1995. The initial equation *x^n + y^n = z^n* is considered not in natural, but in integer numbers. It is subdivided into four equations based on parity of terms and their powers. Cases 1, 3 and 4 can be converted to case 2, which is studied using presentations of pairs of odd numbers with a successively increasing presentation factor of 2^*r*. The proposed methods and ideas can be used for studying other problems in number theory.

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1. Introduction

One of the reasons that FLT still attracts people is that the known solution [1], in their view, is too complicated for the problem. Here, we introduce a simpler approach based on earlier work [2].

2. FLT sub-equations

Let us consider an equation.

$$x^a + y^a = z^a \tag{1}$$

The power *a* is a natural number $a \geq 3$. Unlike in the original FLT equation, here, *x*, *y*, *z* belong to the set of integer numbers **Z**. Combinations with zero values are not considered as solutions. We assume that variables *x*, *y*, *z* have no common divisor. Indeed, if they have such a divisor *d*, both parts of equation can be divided by d^a, so that the new variables $x_1 = x/d$, $y_1 = y/d$, $z_1 = z/d$ will have no common divisor. We will call such a solution, without a common divisor, a *primitive solution*. From the formulas above, it is clear that any non-primitive solution can be reduced to a primitive solution by dividing by the greatest common divisor. The reverse is also true, that is any non-primitive solution can

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be obtained from a primitive solution by multiplying the primitive solution by a certain number. So, it is suffice to consider primitive solutions only.

Values x, y, z in (1) cannot be all even. Indeed, if this is so, this means that the solution is not primitive. By dividing it by the greatest common divisor, it can be reduced to a primitive solution. Obviously, x, y, z cannot be all odd. So, the only possible combinations left are when x and y are both odd, then z is even, or when one of the variables, x or y, is even, and the other is odd. In this case, z is odd. Thus, equation (1) can be subdivided into the following cases, which cover all permissible permutations of equation's parameters.

1. $a = 2n$; $x = 2k+1$; $y = 2p+1$. Then, z is even, $z = 2m$. (2)
2. $a = 2n+1$; $x = 2p+1$; $y = 2m$. Then, z is odd, $z = 2k+1$. (3)
3. $a = 2n+1$; $x = 2k+1$; $y = 2p+1$. Then, z is even, $z = 2m$. (4)
4. $a = 2n$; $x = 2p+1$; $y = 2m$. Then, z is odd, $z = 2k+1$. (5)

3. Conversion of cases 1, 3 and 4 to case 2

It will be shown in section 4.10 that case 3 is equivalent to case 2. For cases 1 and 4, as Dr. M. J. Leamer noted in his comment, there is a well known way to show that considering equation (1) is equivalent (in terms of existence of solution) to the case when exponent a is represented as a product of number four and (or) odd prime numbers. Indeed, we can assume that $a = fp$, where p is a product of one or more prime factors, so that p is odd; $f \geq 1$ is an even natural number (since a is even for cases 1 and 4). (Certainly, prime factors of a can be distributed between f and p.) We can rewrite (1) as

$$(x^f)^p + (y^f)^p = (z^f)^p$$

Then, if there is no integer solution for the odd exponent p>1, then (1) has no integer solution too. Indeed, if one assumes that {x,y,z} is a solution of (1), then { x^f , y^f , z^f } would be integers representing a solution for the above equation. However, by assumption, it has no solution. The obtained contradiction means that {x,y,z} is not a solution of (1).

When a has no prime factors, $p = 1$. Since $a \geq 3$, $f \geq 4$. When f is divisible by four, we can use Corollary 2 on p. 53 in [3] that (1) has no solution in natural numbers for $a = 4$, representing the terms of (1) - say, the first one, as $(x^{f/4})^4$. Note that because of the even power, the Corollary extends on integer numbers too. However, then {x,y,z} also cannot be a solution of (1), since in this case $x^{f/4}$, $y^{f/4}$, $z^{f/4}$ will be integers satisfying to (1), contrary to the aforementioned Corollary 2.

When f is divisible by two, but not four, that would mean that $p \geq 3$ (since $a \geq 3$), and we again can convert (1) to an equation with an odd power.

Thus, cases 1 and 4 with even powers can be converted to cases 3 and 2 accordingly. Case 3 is equivalent to case 2. So, it is suffice to only prove that there is no integer solution for case 2. Independent solutions for cases 1 and 4 are presented in [2], v. 9.

4. Cases 2 and 3

We will need several Lemmas.

4.1. Presentation of numbers in a binary form

Lemma 1: *Each non-negative integer number n can be presented in a form*

$$n = \sum_{i=0}^{r} 2^i K_i \qquad (6)$$

where $K_i = \{0,1\}$.

Proof: Effectively, this Lemma states the fact that any non-negative number can be presented in a binary form.

From Lemma 1, the following Corollary follows.

Corollary 1: Any negative integer number *n* can be presented as $n = \sum_{i=0}^{r} 2^i B_i$, where $B_i = \{-1{,}0\}$.

4.2. Presentation of equation (1) for cases 2 and 3

For the case 3, we have $a = 2n+1$; $x = 2k_1+1$; $y = 2p_1+1$; $z = 2m$. Then, (1) transforms to

$$(2k_1+1)^{2n+1} + (2p_1+1)^{2n+1} = (2m)^{2n+1} \qquad (7)$$

For the case 2, the power $a = 2n+1$; $x = 2p+1$; $y = 2m$. Then, *z* is odd, $z = 2k+1$.

$$(2p+1)^{2n+1} + (2m)^{2n+1} = (2k+1)^{2n+1}$$

It can be rewritten in a form

$$(2k+1)^{2n+1} - (2p+1)^{2n+1} = (2m)^{2n+1} \qquad (8)$$

We can present *m* in (8) as $m = 2^{\mu} m_1$, where $\mu \geq 0$, and m_1 is an odd number. Then, (8) transforms to

$$(2k+1)^N - (2p+1)^N = 2^{N(\mu+1)} m_1^N \qquad (9)$$

where $N = 2n+1$. Note that the value $r_t = N(\mu+1)$ is a threshold one. If we divide both parts of the equation by 2^r, then for $r < r_t$ the right part is even, for $r = r_t$ it is odd, and for $r > r_t$ it is rational.

In the following, we will use a presentation of pairs of odd numbers with a factor of 2^r, where $r \geq 1$, whose properties are considered below.

4.3. Presentation of pairs of odd numbers with a factor of 2^r

4.3.1. Introducing pairs of presentation terms

Let us consider an infinite set of *pairs* of odd integers produced by a pair of terms $[(2k+1),(2p+1)]$, *k* and *p* are *integer* variables without duplicate values, $(-\infty < k, p < \infty)$. The set, produced by a term $(2k+1)$, can be presented by two subterms $(4t+1)$ and $(4t+3)$, $(-\infty < t < \infty)$, with a factor of four (for the even and odd *k* (*k*=2*t*, *k*=2*t*+1)). Numbers 1 and 3 are *free coefficients* (abbreviated as FC and FCs in this paper). Each such subterm represents a subset of odd numbers. Similarly, the set of odd numbers $\{(2p+1)\}$, produced by a term $(2p+1)$, can be presented by two subterms $(4s+1)$ and $(4s+3)$, $(-\infty < s < \infty)$. (Here and in the entire paper such parameters as *k*, *p*, *t* and *s*, defining sets of numbers, are *integer* variables without duplicate values, defined on the range $(-\infty, \infty)$). Thus, the original set, produced by a pair of terms $[(2k+1),(2p+1)]$, can be presented by four possible pairs from the above subterms with a presentation factor of four (2^2). In the following, such a pair of terms (for an arbitrary presentation level) will be called a *pair of presentation terms* (PPT). Note that in this paper we will consider only the terms presenting odd numbers. PPT *defines* a set of pairs of odd numbers. Such, each *one* PPT, presented above, defines an *infinite* set of pairs of odd numbers.

In essence, PPT *is* a set of numbers it produces. The distinction between PPTs and sets of pairs of odd numbers they produce is subtle; it emerges only when one begins splitting presentation terms (and consequently splitting the corresponding sets). In this case, instead of operating on infinite sets one could operate on presentation terms, producing these sets. Of course, the presentation terms have to have certain properties, which make operations on them equivalent to desirable operations on infinite sets.

Table 1 shows four possible PPTs, expressed with a factor of four. Such a presentation produces a *complete set* of pairs of odd integer numbers, since we considered all possible combinations of parities of *k* and *p*. (The completeness of such a presentation will be proved later for a general case of presentation with a factor 2^r).

We can continue presenting sets of pairs of odd numbers by PPTs using a successively increasing factor of 2^r. Initial PPTs for the next presentation level with a factor of 2^3 are in cells (2,1)-(2,4). Table 2 shows the presentation with a factor of 2^3 for two initial PPTs (cells (2,3), (2,4) in Table 1). Note that index '3' for variables *t*, *s* corresponds to power *r*=3 in a presentation factor 2^r. Such correspondence of the index to the power *r* of two in a presentation factor will be used throughout the paper.

Table 1. All possible PPTs, defining sets of pairs of odd numbers, expressed with a factor of four.

	0	1	2	3	4
1	k p	$2t_2$ $2s_2+1$	$2t_2+1$ $2s_2$	$2t_2$ $2s_2$	$2t_2+1$ $2s_2+1$
2	$2k+1$ $2p+1$	$\mathbf{4t_2+1}$ $\mathbf{4s_2+3}$	$\mathbf{4t_2+3}$ $\mathbf{4s_2+1}$	$4t_2+1$ $4s_2+1$	$4t_2+3$ $4s_2+3$

Table 2. PPTs, expressed with a factor of eight (2^3), corresponding to initial PPTs $[4t_2+1, 4s_2+1]$, $[4t_2+3, 4s_2+3]$ from Table 1.

	0	1	2	3	4
1	t_2 s_2	$2t_3$ $2s_3+1$	$2t_3+1$ $2s_3$	$2t_3$ $2s_3$	$2t_3+1$ $2s_3+1$
2	$4t_2+1$ $4s_2+1$	$\mathbf{8t_3+1}$ $\mathbf{8s_3+5}$	$\mathbf{8t_3+5}$ $\mathbf{8s_3+1}$	$8t_3+1$ $8s_3+1$	$8t_3+5$ $8s_3+5$
3	$4t_2+3$ $4s_2+3$	$\mathbf{8t_3+3}$ $\mathbf{8s_3+7}$	$\mathbf{8t_3+7}$ $\mathbf{8s_3+3}$	$8t_3+3$ $8s_3+3$	$8t_3+7$ $8s_3+7$

Note: Of course, odd values *k*=2*t*+1 and *p*=2*s*+1 could be presented as *k*=2*t*-1 and *p*=2*s*-1, in which case the subterms will be accordingly $(4t-1)$ and $(4s-1)$. However, since $(-\infty < t < \infty)$, the subterms $(4t-1)$ and $(4t+3)$ produce the same subsets of odd numbers.

4.3.2. Presentation terms and infinite subsets of integer numbers

So, we have two tightly related entities: PPTs and the corresponding infinite sets of pairs of odd numbers they produce. Eventually, we need to prove that (1) has no solution for all possible *pairs of odd numbers*. However, the proof is based on consideration of PPTs, which are producers of infinite sets and subsets of pairs of odd numbers. On the surface, one observes that the number of PPTs at each presentation level is finite, while they produce subsets composed of infinite number of pairs of odd numbers, which could be perceived as an issue.

In fact, the same issue is implicitly present in all problems, dealing with infinite sets of numbers. However, in those problems, the sets and the appropriate terms, producers of these sets, relate to one presentation level, so that one even doesn't think about such an issue, taking for granted that the expression-producer, indeed, represents the infinite set; in fact, *is* this set. For instance, if one considers an infinite set of odd integer numbers, then a term, producing this set, is $(2k+1)$, where *k* is an integer. Once one proves that a certain problem has no solution for $(2k+1)$, that is for the *term*, this implicitly assumes that the problem has no solution for the *set* of all odd integers. In this case, there is no question about the legitimacy of the used approach.

The issue emerges when we represent the term $(2k+1)$ as a union of two terms $(4t+1)$ and $(4t+3)$. Since *k* can only be odd or even, these two terms describe all possible odd integers, the same as the initial term $(2k+1)$. Each of these terms is a producer of the associated infinite subset of odd integers. Then, can one say that the union of these two infinite subsets represents the *whole* set of odd integers, same as the term $(2k+1)$ does? Intuitively, this is obviously so. We have two non-intersecting subsets, which together comprise *all possible* odd integers. We just have to prove that such *subsets*, indeed, are non-intersecting, and the operations of splitting them and assembling back - by appropriate splitting and assembling their presentation terms - are unique and reversible. The uniqueness is understood as follows. Suppose subset *S* is produced by a term *T*, which then is split into several subterms $t_1, t_2, ...t_m$ for another presentation level. Each of these subterms accordingly produces subsets $s_1, s_2, ...s_m$. Then, such a splitting of the term *T* is unique (and the appropriate operation of splitting subset *S* into subsets $s_1, s_2, ...s_m$ is unique too), *if for any element in S there is one and only one such element in one of the subsets* $s_1, s_2, ...s_m$. The uniqueness of the reverse operation, assembling subterms $t_1, t_2, ...t_m$ into the term *T* (with appropriate assembling of subsets $s_1, s_2, ...s_m$ into set *S*) is defined in the same way.

Infinite subsets of numbers are produced by the subterms. There is no infinite subset of numbers without its producer, the term, which *entirely* defines the properties of the subset. Operating on the terms, we operate on the produced by them subsets of numbers. If subterms uniquely add to a term, generating a certain set of numbers, this is the same as adding corresponding subsets of numbers to produce such a set. We just have to make sure that the operations of adding subterms are unique.

What embarrasses people in the above procedures though, is splitting and assembling *infinite* sets, which they mentally detach from the presentation terms, quickly forgetting that these terms in essence *are* the sets in question. Such confusion might come from the notion of asymptotic density [4,5], which is used for characterization of infinite sets with relation to the set of integer numbers. Once people come across the aforementioned splitting of infinite subsets, they begin to think of their characterization in terms of *asymptotic densities*, which is *absolutely not the case* in our situation. By definition, asymptotic density is rather a *stochastic* notion, while we consider *only deterministic* values. The fact that for a certain infinite set the density converges to value d_s does not mean that *all* elements of the set satisfy to a certain criterion, say, not being a solution of some equation. Any finite number of such elements-exceptions, for which the equation does have a solution, won't change the value of asymptotic density d_s, since the number of elements in the set is infinite. In our case, we need *all* elements of the considered sets and subsets to satisfy the same criterion, that is to not be a solution of equation (1), with absolutely no possibility of any exception. The notion of density is certainly unsuitable for such a purpose, and so *it is not used* in this proof, as some commenters wrongly assumed.

4.4. The concept of the proof

Each PPT in Tables 1 and 2, and in subsequent presentations, defines an *infinite* set of pairs of odd numbers. However, the number of PPTs at each presentation level is *finite*. All PPTs, belonging to the same presentation level, together produce the ***whole** set of pairs of odd numbers*.

A subset of PPTs from one level can be uniquely transformed to a subset of PPTs at another presentation level. The subsets of pairs of odd numbers, corresponding to initial PPTs and the transformed ones, are *the same*. Using such transformations of PPTs, we can distribute the set of PPTs from the initial level across different presentation levels, and vice versa (that is to combine PPTs from different upper presentation levels back to initial or other lower level). Accordingly, the pairs of odd numbers, associated with PPTs at different presentation levels, will be associated with one presentation level again. Such transformations are one-to-one transformations in both directions, meaning that for each pair of odd numbers at one presentation level there is one and only one such pair at any other presentation level. The properties of a subset of pairs of odd numbers (and of associated PPTs), acquired at other presentation levels (say, that (1) has no solution on this subset), remain with this subset at another presentation level (because these are just the same combinations of numbers).

It will be proved in Lemma 3 that the infinite sets, defined by PPTs at the same level, are unique and do not intersect. At each presentation level, equation (8) has no solution for a certain fraction of PPTs. Such "no solution" fractions accumulate through subsequent presentation levels, producing a greater and greater total fraction of PPTs, for which (8) has no solution. In the limit, this total "no solution" fraction becomes equal to one, which can be considered as if all "no solution" PPTs sums up to the initial PPT $[(2k+1), (2p+1)]$, which defines a set of all possible pairs of odd numbers. From this, one may conclude that (8) has no solution for all possible pairs of odd numbers - the result we aim for. However, connecting the limit of one with the whole set of pairs of odd numbers was considered by some reviewers as a too subtle proposition. Therefore, in this paper, we present an entirely different *conventional* mathematical approach, based on the proof that with the increase of presentation level any pair of odd numbers will eventually belong to a "no solution" PPT (section 4.9).

It is important to note that we deal with purely *deterministic* (but not stochastic!) values and relationships. The "no solution" fractions, associated with finite number of PPTs at different presentation levels, are such deterministic values.

The illustration of the concept of the proof below *is not* an actual proof, which is different. It only illustrates *one* of the possible considerations, related to parity, while the actual proof uses also rationality and zero values of equations' terms (without which the proof would be unlikely possible).

We begin the illustration with the *whole set* of all possible pairs of odd integer numbers, defined by a PPT $[(2k+1),(2p+1)]$. At the next presentation level *r*=2, we have four PPTs listed in Table 1. Equations, corresponding to a *half* of these PPTs, have no solution. This half of PPTs (that is two of them) is set aside (they constitute the "no solution" fraction f_{2ns}). The remaining half of PPTs compose an "uncertain" fraction of PPTs, for which solution is uncertain. The "uncertain" fraction is equal to $f_{2u}=1-f_{2ns}=1/2$.

Equation (8) can be transformed as a difference of two numbers in odd powers.

$$2(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^{i}=(2m)^{2n+1}$$

Dividing both parts by two, one obtains

$$(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^{i}=m(2m)^{2n}$$

Here, the sum is odd as an odd quantity of odd numbers. If the factor $(k-p)$ is odd, then the left part is odd, while the right part is even (since $n>0$). This means that there is no solution in this case. The value of $(k-p)$ is odd when one of the terms is odd and the other is even, which are the values of *k* and *p* in cells (1,1), (1,2) in Table 1, corresponding to PPTs $[4t+1,4s+3]$ and $[4t+3,4s+1]$ with a presentation factor of 2^2. The change of algebraic signs of *k* and *p* does not change the parity of the left part. So, the result is valid for *integer* numbers *k* and *p*. When $(k-p)=0$, the left part is zero, while the right part is an integer. So, there is no solution in this case.

When $(k-p)$ is even, both parts of equation are even, and solution is uncertain. This corresponds to values of *k* and *p* in cells (1,3), (1,4) in Table 1, with corresponding PPTs $[4t+1,4s+1]$ and $[4t+3,4s+3]$. These "uncertain" PPTs should be used as initial ones for the next presentation level with a factor of 2^3 in Table 2.

At the presentation level with *r*=3, we again find that a half of new PPTs (four PPTs in bold in Table 2) correspond to a "no solution" fraction, which is found as $f_{3ns}=f_{2u}\times 1/2=1/4$. The fraction of remaining uncertain PPTs is accordingly $f_{3u}=f_{2u}-f_{3ns}=1/2-1/4=1/4$. Therefore, two presentation levels produce the following total fraction of PPTs, for which (8) has no solution, $F_{3NS}=f_{2ns}+f_{3ns}=1/2+1/4=3/4$. The "uncertain" fraction $f_{3u}=1-3/4=1/4$, gives initial PPTs for the next presentation level (*r*=4), and so forth, until in infinity the "no solution" fraction accumulates to one (that is to the reference value, corresponding to initial term $[(2k+1),(2p+1)]$, which produces the whole set of pairs of odd numbers). (The real situation with the "no solution" fractions is more complicated, since some PPTs may have no solution for the entire PPT, and such a branch is closed. However, the total "no solution" fraction is still equal to one in the limit.) Fig. 1 illustrates the concept.

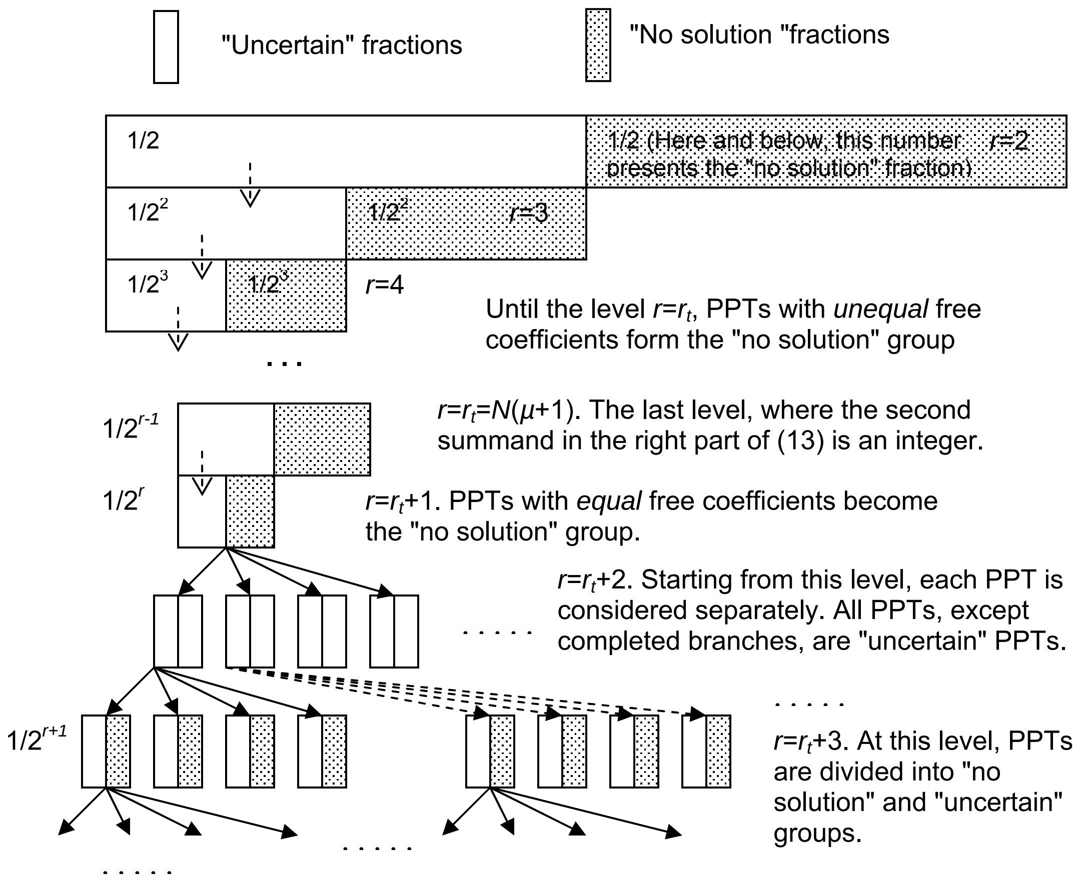

Fig. 1. Graphical presentation of how the "No solution" fraction accumulates through presentation levels, and the appropriate decrease of "Uncertain" fraction. The value of $r=r_t=N(\mu+1)$ is a threshold value, after which the groups of PPTs with unequal FCs seize to be "No solution" PPTs, and become "Uncertain" ones.

4.4.1. Properties of PPTs with a factor of 2^r

Lemma 2: *Successive presentations of odd numbers with a factor of* 2^r *cannot contain a FC, whose module is greater or equal to* 2^r.

Proof: Presentations of odd numbers with factors 2^2 and 2^3 satisfy this requirement. Let us assume that this is true for a presentation level *r*, that is the FC *v* in a term $(2^r t_r + v)$ satisfies the condition $|v| < 2^r$. At a presentation level (*r* + 1), this term is presented as $(2^{r+1}t_{r+1} + 2^r + v)$ or $(2^{r+1}t_{r+1} + v)$. In the latter term, the condition is already fulfilled. In the first term, $0 < 2^r + v < 2^r + 2^r = 2^{r+1}$, since $|v| < 2^r$ is true for level *r* by assumption (zero value corresponds to negative *v*). So, assuming that the condition is fulfilled at the level *r*, we obtained that it is also fulfilled at the level (*r* + 1). According to principle of mathematical induction, this means the validity of the assumption. This proves the Lemma.

Note: PPT with a negative FC can be always transformed to a PPT with a positive one. Let $v > 0$. Then one can present an odd number with negative FC as $(2^r t_r - v) = 2^r(t_r - 1) + (2^r - v) = 2^r t_{1r} + v_1$, where $t_{1r} = t_r - 1$, $v_1 = 2^r - v$. According to Lemma 2, the FC $v_1 = (2^r - v)$ is positive. Since $(-\infty < t_{1r}, t_r < \infty)$, both PPTs $(2^r t_r - v)$ and $(2^r t_{1r} + v_1)$ define the same set of odd integers. So, without restricting generality, we can consider only PPTs with positive FCs.

The substitution $t_{1r} = t_r - 1$ affects the parity of certain equation terms, which will be considered later. Thus, if some PPT has negative FCs, it *first* has to be converted to a PPT with positive FCs, and *only then* the parity and other considerations should be applied to the corresponding equation.

The number of PPTs grows for successive *complete* presentations in a geometrical progression with a common ratio of *four*, since each initial PPT produces four new PPTs at the next presentation level. (Each new PPT corresponds to one of the four possible parity combinations of input parameters, like t_2, s_2 in Table 2, whose parity is expressed through t_3, s_3.)

For the following, we need to prove that (a) such a presentation produces the whole set of pairs of odd numbers at each level; (b) the presentation is unique, that is for each pair of odd integers at one level there is one and only one such pair at any other presentation level.

Lemma 3: *Successive PPTs with a factor of* 2^r, $r \geq 2$*, produce the same whole set of pairs of odd numbers at each presentation level. Such presentations are unique, that is for any pair of odd numbers there is one and only one the same pair at any other presentation level. Each PPT at the same presentation level produces a unique set of pairs of odd numbers.*

Proof: First let us note that each next presentation level (r+1) is obtained prom the previous one through branching of each initial PPT from level r into *all* four possible combinations of parities of parameters t_r and s_r, so that there are no any other possible combinations of parities, and so no element of the original set, defined by the initial PPT $[(2^r t_r + v), (2^r s_r + w)]$, could be missing at the next presentation level. This means that any PPT from an arbitrary level r is fully represented at level (r+1), although in the form of four new PPTs. In fact, this is just a different form of presentation of the same PPT. However, this also means that the initial set from level r is also fully represented at level (r+1).

Let us show that each new PPT at level (r+1) produces a unique set. The term $(2^r t_r + v)$ of the initial PPT can be presented at level (r+1) only in two forms (for even and odd values of t_r), that is as $2^r(2t_{r+1}) + v = 2^{r+1}t_{r+1} + v$, or $2^r(2t_{r+1}+1) + v = 2^{r+1}t_{r+1} + 2^r + v$. Similarly, the term $(2^r s_r + w)$ can be represented in the same two forms only. So, only four combinations of these terms, containing both t and s parameters, are possible. These combinations are unique, because the combinations of FCs are unique. Indeed, the FCs' combinations are as follows: [v,w], $[2^r + v, w]$, [v, $2^r + w$], $[2^r + v, 2^r + w]$, and neither one can be obtained from another on the interval (0, 2^{r+1}). Thus, each of the four new PPTs produces unique set of pairs of odd integers, and these sets do not intersect. At the same time, together they constitute the whole set of pairs of odd numbers, produced by the initial PPT.

Now, we should show that there are no duplicate elements in each set defined by new PPTs. Suppose there are such duplicate pairs of odd numbers in the set defined by PPT [$2^{r+1}t_{r+1} + v$, $2^{r+1}s_{r+1} + w$], corresponding to a pair of FCs [v,w]. Transformation from level r to level (r+1) is a one-to-one transformation, that is one value of $t_{r,1}$ can correspond to only one value of t_{r+1} at level (r+1). (And the same is true for $s_{r,1}$.) Indeed, according to transformation rule (see an example in the above paragraph) $2^{r+1}t_{r+1} = 2^r(2t_{r,1})$, which is a one-to-one transformation. So, if we have duplicate pairs, then they should be produced by different values of t_r and s_r, say by $t_{r,2}$ and $s_{r,2}$. So, one can write the following: $2^r(t_{r,1}) + v = 2^r(t_{r,2}) + v = 2^{r+1}t_{r+1} + v$, or $2^r(t_{r,1}) + v = 2^r(t_{r,2}) + v$, from which $t_{r,1} = t_{r,2}$ follows, which is contrary to our assumption that $t_{r,1} \neq t_{r,2}$. So, it is invalid. (Similarly, we can prove that $s_{r,1} = s_{r,2}$.) Thus, the set defined by PPT [$2^{r+1}t_{r+1} + v$, $2^{r+1}s_{r+1} + w$], contrary to our assumption, has no duplicate elements.

The above considerations are similarly applicable to other PPTs, corresponding to remaining three pairs of FCs. The sets, corresponding to these PPTs, also have no duplicate pairs. Since all four such sets have no duplicates, and these sets do not intersect, this means that the whole presentation level has no duplicate elements (provided the initial level has no duplicates, which was earlier stated).

So, we found that (a) the subsets defined by new PPTs at level (*r*+1) are unique in that regard that each of them contains unique elements; (b) each of these subsets has no duplicate elements; (c) four new PPTs at level (*r*+1) together define the same set of pairs of odd numbers as the initial PPT at level *r*.

The reverse is also true, that is four PPTs at presentation level (*r*+1) uniquely converge to one initial PPT at lower level *r*. Indeed, two terms with parameter *t* converge to the same term $(2^r t_r + v)$.

$$2^{r+1} t_{r+1} + v = 2^r (2t_{r+1}) + v = 2^r t_r + v \qquad (10)$$

$$2^{r+1} t_{r+1} + 2^{r+1} + v = 2^r (2t_{r+1} + 1) + v = 2^r t_r + v \qquad (11)$$

where $t_r = \{2t_{r+1}, 2t_{r+1} + 1\}$.

Since (10) and (11) represent one-to-one transformations, the same value of t_{r+1} cannot produce two different elements at level *r*. Also, two different values of $t_{r+1,1}$ an $t_{r+1,2}$ cannot produce the same value of t_r at level *r*. The proof is similar to what we did to prove the uniqueness of transformation from level *r* to level (*r*+1), that is we assume that two different values of $t_{r+1,1}$ and $t_{r+1,2}$ produce the same value of $2^r t_r + v$. Equating, for instance, presentations $2^r (2t_{r+1,1}) + v = 2^r (2t_{r+1,2}) + v$, we obtain that $t_{r+1,1} = t_{r+1,2}$, contrary to the assumption that $t_{r+1,1} \neq t_{r+1,2}$.

The same convergence to a single term can be obtained for a general case of presenting two terms at level (*r*+1) using Lemma 1, and then transforming them to level *r*.

$$2^{r+1} t_{r+1} + \sum_{i=1}^{r} 2^i K_i + 1 = 2^r (2t_{r+1} + K_r) + \sum_{i=1}^{r-1} 2^i K_i + 1 = 2^r t_r + \sum_{i=1}^{r-1} 2^i K_i + 1 \qquad (12)$$

For a positive number, $K_r = \{0,1\}$, so $t_r = \{2t_{r+1}, 2t_{r+1} + 1\}$, that is the same set of integer numbers, on which t_r was defined originally. The same is true for negative numbers.

Similarly, one can convert two terms with parameter s_{r+1} at level (*r*+1) to a single term with parameter s_r at level *r*. So, four PPTs at level (*r*+1), indeed, converge to one PPT [$2^r t_r + v$, $2^r s_r + w$] at level *r*, which, accordingly, defines the same initial set of pairs of odd integers.

Therefore, such transformations of presentation forms from level *r* to level (*r*+1) and backward are unique; they neither remove nor add new pairs of odd numbers compared to the initial set. The initial set does not have duplicate entries, and so four sets at the next presentation level do not have duplicate entries too. So, for each pair of odd integers at level *r* there is one and only one pair of odd numbers at level (*r*+1), and vice versa. This proves the Lemma.

It follows from Lemma that all PPTs of each presentation level together define the same set of pairs of odd integers, as the initial PPT $[(2k+1), (2p+1)]$ does, that is the *whole* set of pairs of odd integers. This result can be formulated as a Corollary.

Corollary 2: All PPTs of each presentation layer together define the whole set of pairs of odd integers. This set has no duplicate entries.

4.5. Properties of equations, corresponding to pairs of odd numbers with a factor of 2^r

This section introduces an equation, to which all equations, corresponding to pairs of odd numbers, can be transformed, and explores its properties.

Lemma 4: *Let us consider an equation*

$$(2^r t_r + v)^N - (2^r s_r + w)^N = 2^{N(\mu+1)} m_1^N \qquad (13)$$

where t_r and s_r are integers; N=2n+1; m_1 is odd; v, w are positive odd (possibly equal) numbers, obtained through successive PPTs (and consequently through successive presentations of pairs of odd numbers). Then, for any $r \ge 3$, such equations can be transformed to the following form

$$(t_r - s_r)A_r = A_r c / 2^r + m_1^N / 2^{r-r_t} \quad (14)$$

where $A_r = \sum_{i=0}^{N-1}(2^r t_r + v)^{N-1-i}(2^r s_r + w)^i$ *is an odd integer; c is an even integer;* $r_t = N(\mu+1)$.

Proof: Equation (13) is equation (8), rewritten for a presentation with a factor of 2^r.

$$[2^r(t_r - s_r) + (v-w)]\sum_{i=0}^{N-1}(2^r t_r + v)^{N-1-i}(2^r s_r + w)^i = 2^{N(\mu+1)} m_1^N \quad (15)$$

The sum in (15) is odd, because it presents the sum of odd quantity of odd values. Let us denote it

$$A_r = \sum_{i=0}^{N-1}(2^r t_r + v)^{N-1-i}(2^r s_r + w)^i$$

Since *v* and *w* are odd, (*v* - *w*) is even. Also, in successive presentation of odd numbers, according to Lemma 2, $v < 2^r$, $w < 2^r$. Since both values are positive (we can assume this without losing generality, according to a note after Lemma 2), their absolute difference is also less than 2^r. According to Lemma 1 and Corollary 1, (*v* - *w*) can be presented as a sum of powers of two, where non-zero coefficients (equal to one by module) have the same algebraic sign. Since $|v - w| < 2^r$, such a sum cannot contain a summand with a power greater than 2^{r-1}. It also cannot contain two in a zero power, since in this case the sum could be odd, while (*v* - *w*) is even.

$$[2^r(t_r - s_r) + \sum_{i=1}^{r-1} 2^i K_i]A_r = 2^{N(\mu+1)} m_1^N \quad (16)$$

Then, (16) can be rewritten as follows.

$$2^r(t_r - s_r)A_r = -\left(\sum_{i=1}^{r-1} 2^i K_i\right)A_r + 2^{N(\mu+1)} m_1^N \quad (17)$$

Let us denote $c = -\sum_{i=1}^{r-1} 2^i K_i$. Since $c = w - v$, when $w = v$ (that is FCs are equal), $c = 0$. When $w \ne v$, the value of $c \ne 0$ and it is even, as the difference of two odd FCs. Dividing both parts of (17) by 2^r, and taking into account that $r_t = N(\mu+1)$, we obtain

$$(t_r - s_r)A_r = A_r c / 2^r + m_1^N / 2^{r-r_t} \quad (18)$$

This proves the Lemma.

Lemma 5: *If $c \ne 0$ in (18), then $A_r c / 2^r$ is a rational number.*

Proof: It was indicated in Lemma 4 that when FCs *w* and *v* are unequal, $c \ne 0$. According to Lemmas 1 and 4, we can always use a presentation $c = -\sum_{i=1}^{r-1} 2^i K_i$ with values $K_i = \{0,1\}$, $1 \le i \le r-1$, when $c < 0$, and $K_i = \{-1,0\}$ when $c > 0$. Then

$$|c| = \left|\sum_{i=1}^{r-1} 2^i K_i\right| \le \sum_{i=1}^{r-1} 2^i = 2(2^{r-1} - 1)/(2-1) = 2^r - 2 \quad (19)$$

(Here, we substituted the sum of a geometrical progression with a common ratio of two and the first term of two.) Accordingly

$$\left|A_r c / 2^r\right| \le |A_r|(1 - 1/2^{r-1}) \quad (20)$$

Dividing inequality (20) by a positive number $|A_r|$, one obtains

$$\left|c/2^r\right| \le (1-1/2^{r-1}) \tag{21}$$

Thus, $c/2^r$ is a rational number. The term A_r is an odd number, which, consequently, contains no dividers of two. In turn, this means that $A_r c/2^r$ is a rational number. This proves the Lemma.

Lemma 6: *Equation (14) has no solution for PPTs with unequal FCs when* $r \le N(\mu+1)$*, while solution is uncertain for PPTs with equal FCs.*

Proof: For $r \le N(\mu+1) = r_t$, the term $2^{r_t-r} m_1^N$ in (14) is an integer. According to Lemma 5, the summand $A_r c/2^r$ is rational for PPTs with unequal FCs. So, the right part of (14) is rational. On the other hand, the left part is an integer when $(t_r - s_r) \ne 0$. This means that (14) has no solution in this case. When $(t_r - s_r) = 0$, (14) presents equality of zero (in the left part), and of a rational number, which is impossible too. So, (14) has no solution for PPTs with unequal FCs.

When FCs are equal, c = 0, and (14) transforms to

$$(t_r - s_r)A_r = 2^{r_t-r} m_1^N \tag{22}$$

For $r < r_t$, the right part is even, for $r = r_t$ it is odd. The left part can be odd, or even, or zero. So, the solution of this equation is uncertain. Consequently, the PPTs, whose terms have equal FCs, should be used as initial PPTs for the next presentation level. This proves the Lemma.

Now, we should establish relationships between the sizes of groups, corresponding to PPTs with equal and unequal FCs, and the parity of the term $(t_r - s_r)$ in (14).

Lemma 7: *When initial PPTs, obtained from r-level of presentation for level (r+1), have equal FCs, the number of PPTs with equal and unequal FCs at level (r+1) is the same and is equal to 1/2 of the total number of PPTs. The group of PPTs with equal FCs correspond to even values of* $(t_r - s_r)$*, while PPTs with unequal FCs correspond to odd* $(t_r - s_r)$*, so that it is equivalent subdividing the PPTs based on parity of* $(t_r - s_r)$*, or on the basis of equal and unequal FCs.*

Proof: It follows from Table 1 that for $r_2 = 2$ the quantities of PPTs with equal and unequal FCs are equal. Consequently, each group constitutes a half of all PPTs. Odd values of $(t_{r_2} - s_{r_2})$ correspond to PPTs with unequal FCs at level r = 3. Accordingly, even values of $(t_{r_2} - s_{r_2})$ correspond to PPTs with equal FCs. Let us assume that the same is true for an initial PPT with equal FCs at a greater level r, $r \ge 3$. The presentation for all possible parity combinations of t_r and s_r at level (r+1) is shown in Table 3 for one generic PPT with equal FCs.

Table 3. Presentation with a factor 2^r for a PPT with equal FCs.

	0	1	2	3	4
1	t_r s_r	$2t_{r+1}$ $2s_{r+1}+1$	$2t_{r+1}+1$ $2s_{r+1}$	$2t_{r+1}$ $2s_{r+1}$	$2t_{r+1}+1$ $2s_{r+1}+1$
2	$2^r t_r+v_i$ $2^r s_r+v_i$	$2^{r+1}t_{r+1}+v_i$ $2^{r+1}s_{r+1}+2^r+v_i$	$2^{r+1}t_{r+1}+2^r+v_i$ $2^{r+1}s_{r+1}+v_i$	$2^{r+1}t_{r+1}+v_i$ $2^{r+1}s_{r+1}+v_i$	$2^{r+1}t_{r+1}+2^r+v_i$ $2^{r+1}s_{r+1}+2^r+v_i$

It follows from Table 3 that the number of PPTs with equal and unequal FCs is the same, and is equal to 1/2 of quantity of all PPTs. Unequal FCs correspond to odd values of $(t_r - s_r)$, while even values $(t_r - s_r)$ correspond to PPTs with equal FCs. So, we obtained the same results as for $r = 2$. Since the rest of initial PPTs have the same form (in all of them FCs are equal), depending on the parity of $(t_r - s_r)$, they also produce a half of PPTs with equal FCs, and a half with unequal ones. According to principle of mathematical induction, this means that the found properties are valid for any presentation level $r \geq 2$. This proves the Lemma.

Corollary 3: *Consider successive presentations of pairs of odd integers by PPTs having a factor of* 2^r*, which use initial PPTs with equal FCs from the previous level, beginning with one PPT. Then, the number of initial PPTs at level r is equal to* 2^{r-1}.

Proof: For a factor of two, we have one PPT; for a factor of 2^2 there are two PPTs with equal FCs (Table 1); for a factor of 2^3 there are 2^2 such PPTs (Table 2), and so forth. The total number of PPTs increases by four times for the next presentation level (since each initial PPT produces four new PPTs, one per parity combination of t_r, s_r). From this amount, a half of PPTs correspond to PPTs with equal FCs, according to Lemma 7. The value of 2^{r-1} reflects on the fact that at each presentation level the number of PPTs with equal FCs doubles. This proves the Corollary.

Corollary 4: *For* $r \leq r_t = N(\mu+1)$*, the fraction of PPTs, for which equation (8) has no solution at a presentation level r, is equal to*

$$f_r = (1/2)^{r-1} \qquad (23)$$

Proof: It was shown in Lemma 6 that in this case (13) has no solution for PPTs with unequal FCs, while, according to Lemma 7, these PPTs constitute half of all PPTs at a given presentation level. Thus, (23) is true for $r = 2$. Let us assume that Lemma is valid for the value of $r > 2$. According to Lemma 6, for $r \leq r_t$, the corresponding equations have no solution for PPTs with unequal FCs, so that initial PPTs for the next level are always PPTs with equal FCs. Then, the fraction f_{ru} of PPTs, for which solution is uncertain, is the same, as the fraction of "no solution" PPTs, that is $f_{ru} = (1/2)^{r-1}$. This fraction contains initial PPTs for the presentation level (*r*+1). At this level, all PPTs are again divided into two equal groups of "no solution" and "uncertain" PPTs (Lemma 7), so that the "no solution" fraction is

$$f_{r+1} = f_{ru} \times (1/2) = (1/2)^{r-1}/2 = (1/2)^r,$$

which is formula (23) for the level (*r*+1). According to principle of mathematical induction, this means validity of (23). This proves the Corollary.

Lemma 8: *At each next presentation level (r+1), the number of PPTs, corresponding to odd and even values of* $(t_r - s_r)$*, are equal.*

Proof: Suppose we have p_{r+1} initial PPTs at a presentation level (*r*+1). Each initial PPT produces four PPTs at level (*r*+1), one PPT per each possible parity combination of terms t_r, s_r, listed in the first row of Table 3. These parity combinations do not depend, whether the initial PPTs have equal or unequal FCs, and also do not depend on the value of *r* compared to r_t. Two of these parity combinations (in cells (1,1), (1,2) in Table 3) produce odd values of $(t_r - s_r)$, namely when t_r, s_r are equal to [$2t_{r+1}, 2s_{r+1}+1$], [$2t_{r+1}+1, 2s_{r+1}$]. Two other combinations, in cells (1,3), (1,4), produce even values of $(t_r - s_r)$ for PPTs [$2t_{r+1}, 2s_{r+1}$], [$2t_{r+1}+1, 2s_{r+1}+1$]. So, the number of PPTs, for which $(t_r - s_r)$ is odd is equal to $2p_{r+1}$. The number of PPTs, for which $(t_r - s_r)$ is even, is also $2p_{r+1}$. So, quantities of PPTs, corresponding to odd and even values of $(t_r - s_r)$, are equal. This proves the Lemma.

Note: At the presentation level (*r*+1), odd values $(t_r - s_r)$ cannot be zero, given the presentation of t_r and s_r through t_{r+1} and s_{r+1} in Table 3. Even values of $(t_r - s_r)$ can be zero. However, from the perspective of existence of a solution, such a zero term can be transformed to a non-zero even term (such a transition is addressed by Lemma 9).

4.6. Finding fraction of "no solution" PPTs for presentation levels with $r \le r_t = N(\mu+1)$

We found so far that for $r \le r_t = N(\mu+1)$ the following is true:

(a) Initial PPTs with equal FCs, taken from level *r*, produce equal number of PPTs with equal and unequal FCs at a presentation level (*r*+1), Lemma 7;
(b) Corresponding to PPTs equations have no solution for PPTs with unequal FCs, while solution is uncertain for PPTs with equal FCs, Lemma 6;
(c) Each presentation level adds a "no solution" fraction of PPTs equal to $f_r = (1/2)^{r-1}$;
(d) Sets of pairs of odd integers, defined by "no solution" PPTs, do not intersect. This follows from Lemma 3, since each PPT of the next level defines unique set of pairs of odd integers compared to sets defined by other PPTs of the same level.

So, each previous level supplies to the next presentation level "uncertain" PPTs, which constitute half of all PPTs of the previous level. These initial PPTs have equal FCs. This allows finding a "no solution" fraction of PPTs from successive presentations with a factor of 2^r. Since each level adds 1/2 of PPTs to a "no solution" fraction, the total such fraction F_r is equal to a sum of geometrical progression with a common ratio $q = 1/2$, and the first term $f_2 = 1/2$ (the "no solution" fraction at level *r*=2). Fig. 1 illustrates this consideration.

So, we can write

$$F_r = \sum_{i=2}^{r} f_i = f_2 \sum_{i=2}^{r} q^{i-2} = f_2(1-q^{r-1})/(1-q) \tag{24}$$

Note that if such a progression is valid to infinity, the total fraction in the limit would be

$$\lim_{r\to\infty} F_r = f_2/(1-q) = (1/2)/(1/2) = 1 \tag{25}$$

(Here, the limit is understood as an ordinary Cauchy's limit.) The obtained limit of one would mean that the union of all "no solution" PPTs converges to initial PPT $[(2k+1),(2p+1)]$, from which the presentation of pairs of odd integer numbers with a factor of 2^r began, and whose fraction was taken as a reference value of one. However, in order to realize such considerations, one needs to confirm that such a progression is true for $r > r_t = N(\mu+1)$.

4.7. Transcending presentation levels above the threshold value r_t

Presentation level (r_t+1)

Table 4 shows PPTs for level (r_t+1). The number of initial PPTs is defined by Corollary 3, and is equal to 2^{r_t} for this level. For PPTs with *equal* FCs (columns 3 and 4 in Table 4), (14) transform to

$$(t_{r_t+1} - s_{r_t+1})A_{r_t+1,ij} = m_1^N/2 \tag{26}$$

The right part of (26) is rational (m_1 is an odd number). The left part is an integer. So, (26) has no solution for PPTs with equal FCs (and, consequently, for even $(t_{r_t} - s_{r_t})$, according to Lemmas 7 and 8). When $(t_{r_t} - s_{r_t}) = 0$, the left part is zero, while the right part is rational. So, (26) has no solution too. This group of PPTs constitutes 1/2 of all PPTs (Lemma 8), so that the common ratio remains equal to 1/2, and formula (24) stays valid.

For PPTs with unequal FCs (and consequently odd $(t_{r_t} - s_{r_t})$, Lemma 8), (14) transforms to

$$(t_{r_t+1} - s_{r_t+1})A_{r_t+1,j} = A_{r_t+1,j}c/2^{r_t+1} + m_1^N/2 \tag{27}$$

The right part can be rational, an integer or zero. Since the sums $A_{r+1,j}$ are all odd, parity of the left part in (27) is defined by the term $(t_{r_t+1} - s_{r_t+1})$, which can be odd, even or zero. So, solution of (27) for odd

$(t_{r_t} - s_{r_t})$ is uncertain, and such PPTs should be used as initial PPTs for the next presentation level $(r_t + 2)$. As it was mentioned (a note after Lemma 8), for odd $(t_{r_t} - s_{r_t})$, the term $(t_{r_t+1} - s_{r_t+1}) \neq 0$.

Table 4. PPTs with a factor of 2^{r_t+1}. It is assumed that $r = r_t$.

	0	1	2	3	4
	t_r s_r	$2t_{r+1}$ $2s_{r+1}+1$	$2t_{r+1}+1$ $2s_{r+1}$	$2t_{r+1}$ $2s_{r+1}$	$2t_{r+1}+1$ $2s_{r+1}+1$
1	$2^r t_r + v_{r1}$ $2^r s_r + v_{r1}$	$2^{r+1} t_{r+1} + v_{r1}$ $2^{r+1} s_{r+1} + 2^r + v_{r1}$	$2^{r+1} t_{r+1} + 2^r + v_{r1}$ $2^{r+1} s_{r+1} + v_{r1}$	$2^{r+1} t_{r+1} + v_{r1}$ $2^{r+1} s_{r+1} + v_{r1}$	$2^{r+1} t_{r+1} + 2^r + v_{r1}$ $2^{r+1} s_{r+1} + 2^r + v_{r1}$
2	$2^r t_r + v_{r2}$ $2^r s_r + v_{r2}$	$2^{r+1} t_{r+1} + v_{r2}$ $2^{r+1} s_{r+1} + 2^r + v_{r2}$	$2^{r+1} t_{r+1} + 2^r + v_{r2}$ $2^{r+1} s_{r+1} + v_{r2}$	$2^{r+1} t_{r+1} + v_{r2}$ $2^{r+1} s_{r+1} + v_{r2}$	$2^{r+1} t_{r+1} + 2^r + v_{r2}$ $2^{r+1} s_{r+1} + 2^r + v_{r2}$
…					
2^r	$2^r t_r + v_{rR}$ $2^r s_r + v_{rR}$	$2^{r+1} t_{r+1} + v_{rR}$ $2^{r+1} s_{r+1} + 2^r + v_{rR}$	$2^{r+1} t_{r+1} + 2^r + v_{rR}$ $2^{r+1} s_{r+1} + v_{rR}$	$2^{r+1} t_{r+1} + v_{rR}$ $2^{r+1} s_{r+1} + v_{rR}$	$2^{r+1} t_{r+1} + 2^r + v_{rR}$ $2^{r+1} s_{r+1} + 2^r + v_{rR}$

Recall that before the level $(r_t + 1)$ PPTs with *unequal* FCs had no solution, while (26) has no solution for *even* $(t_{r_t+1} - s_{r_t+1})$, corresponding to PPTs with *equal* FCs. In this regard, the level $(r_t + 1)$ reverses the groups of PPTs. The "uncertain" group of PPTs is now composed of PPTs with *unequal* FCs (and accordingly with *odd* $(t_{r_t} - s_{r_t})$). These PPTs (in columns 1 and 2 in Table 4) should be used as initial PPTs at the next presentation level $(r_t + 2)$.

Transition in the presentation level $(r_t + 2)$

Level $(r_t + 1)$ supplied initial PPTs with unequal FCs. This means that we do not have anymore distinct groups with equal and unequal FCs at level $(r_t + 2)$, as before, since the initial PPTs with unequal FCs produce mostly PPTs with unequal FCs, with occasional inclusion of PPTs with equal ones. Previously, we have seen that the parity of parameter $(t_r - s_r)$ was defining the absence or uncertainty of solution. However, beginning from level $(r_t + 2)$, this parameter lost association with groups of PPTs with equal and unequal FCs. This is due to the fact that the right part of equation (27) can be an integer, a rational number, or zero *per PPT basis*, and so we should consider the use of parameter $(t_r - s_r)$ this way. We will still have a half of "no solution" and a half of "uncertain" PPTs, but only for a block of four PPTs, corresponding to each initial PPT. This is the assembly of such "uncertain" PPTs from each block, which goes to the next level (Fig. 1). Table 5 shows PPTs for level $(r_t + 2)$.

When r=(r_t+2), (14) transforms to

$$(t_{r_t+2} - s_{r_t+2}) A_{r_t+2,ij} = A_{r_t+2,ij} c / 2^{r_t+2} + m_1^N / 4 \qquad (28)$$

where index '*ij*' denotes the cell number. The right part of (28) can be rational, an integer, or zero. When the left part is an integer (the case, when it's zero, will be considered later), (28) has no solution for any $(t_{r_t+2} - s_{r_t+2})$ for the rational or zero right part, and, consequently, this branch is completed. (Compared to continuing branches, the completed branch delivers *double* fraction of PPTs, for which (8) has no solution, since in this case two equal fractions of PPTs compose one "no solution" fraction.) If the right part is an integer, (28) has no solution when $(t_{r_t+2} - s_{r_t+2})$ has the opposite parity, and the solution is uncertain for another parity of $(t_{r_t+2} - s_{r_t+2})$. The number of combinations of parameters t_{r_t+2} and s_{r_t+2}, corresponding to each parity, is equal to two from four in this case, and so we still have equal division between the "no solution" and "uncertain" PPTs. However, at this level, we have no distinction between the odd and even values of t_{r_t+2} and s_{r_t+2} in the same way, as before, when there was association with equal and unequal FCs. Such distinction can be done *only* at the next presentation level (r_t+3). All PPTs at level r_t+2 correspond to "uncertain" equations, except for the cases when the PPT's branch is completed.

Table 5. PPTs with a factor of 2^{r_t+2}, obtained from initial PPTs in Table 4, for which $(t_r - s_r)$ is odd. First two rows correspond to cells (1,1), (1,2) in Table 4. It is assumed that $r = r_t$.

	0	1	2
	t_{r+1} s_{r+1}	$2t_{r+2}$ $2s_{r+2}+1$	$2t_{r+2}+1$ $2s_{r+2}$
1	$2^{r+1}t_{r+1}+v_{r1}$ $2^{r+1}s_{r+1}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$
2	$2^{r+1}t_{r+1}+2^r+v_{r1}$ $2^{r+1}s_{r+1}+v_{r1}$	$2^{r+2}t_{r+2}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+v_{r1}$
. . .	. . .	. . .	. . .
2^{r+1}	. . .	. . .	. . .

Table 5 continued

3	4
$2t_{r+2}$ $2s_{r+2}$	$2t_{r+2}+1$ $2s_{r+2}+1$
$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$
$2^{r+2}t_{r+2}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+v_{r1}$
. . .	. . .
. . .	. . .

The case of $(t_{r_t+2} - s_{r_t+2}) = 0$ is also an "uncertain" one, since there is a possibility that two terms in the right part are equal in absolute values and have the opposite algebraic signs.

Note that values $A_{r_t+2,ij}$ are different, so that the right parts of corresponding equations, transformed to a form (14), may have dissimilar parities (as well as may be rational or zeros) for different PPTs. (The right part can be an integer, provided $c \neq 0$ in (14), otherwise the right part is equal to $m_1^N / 2^{r-r_t}$, which is always rational for $r > r_t$, so that such a branch is completed.) This is why one should consider each PPT *separately* (Fig. 1). (In fact, it is possible to show that at level (r_t+2), when $c \neq 0$, integer right parts of these equations have the same parity. However, this is not necessarily true for the next levels, so we use the same generic approach for this and higher levels of presentation.)

With regard to accumulation of a total "no solution" fraction, we have the same common ratio of 1/2, although it is obtained differently - not per group, as previously, but per PPT, and then such "per PPT" fractions are summed up, in order to obtain the total "no solution" fraction. We will consider this assembling process in detail later.

So, we found that the corresponding equations for PPTs in both groups (meaning groups of PPTs, having either even or odd values of $(t_{r_t+2} - s_{r_t+2})$) converge to equations, which have no solution for one parity of $(t_{r_t+2} - s_{r_t+2})$, and accordingly for one half of PPTs (according to Lemma 8), while solution is uncertain for the other parity, corresponding to the second half of PPTs. So, the common ratio for a geometric progression, defining fractions of "no solution" PPTs, will remain equal to 1/2. However, because we can specify particular PPTs, corresponding to odd or even $(t_{r_t+2}-s_{r_t+2})$, at the next level only, this common ratio accordingly should be assigned to a presentation level, where such a specification actually happens; in this case, this is the next level (r_t+3). At level r_t+2, all equations, corresponding to initial PPTs, have the same form (14), and consequently, the same "uncertain" status. All PPTs (except for completed ones) are "uncertain" PPTs.

Presentation level (r_t+3)

We will need the following Lemma to address zero values of $(t_r-s_r)=0$ in equation (14). Note that $(t_r-s_r)=0$ only when both parameters are equal, including when both are equal to zero. When (t_r-s_r) is odd (parameters have different parity), $(t_r-s_r)\neq 0$.

Lemma 9: *Equation (14), that is* $(t_r-s_r)A_r=A_rc/2^r+m_1^N/2^{r-r_t}$, *is equivalent to equation* $(t_{1r}-s_{1r})A_r=2(a-b)A_r+A_rc/2^r+m_1^N/2^{r-r_t}$ *in terms of parities of both parts, with the substitutions* $t_r=t_{1r}-2a$ *and* $s_r=s_{1r}-2b$, *where a and b are integers. If the second equation has no solution based on parity or rationality considerations, then the first equation also has no solution, and vice versa.*

Proof: According to the notion of presentation of odd numbers with a factor of 2^r, the terms t_r and s_r are integers, having ranges of definition $(-\infty<t_r<\infty)$ and $(-\infty<s_r<\infty)$. The only property, which is of importance with regard to such a presentation, is that these parameters should be defined on the whole set of integer numbers, in order to include *all* possible numbers, corresponding to a particular presentation; for instance, the term $(2^rt_r+v_r)$ should produce the whole set of the appropriate "stroboscopic" numbers in the range $(-\infty,\infty)$, located at the distance 2^r from each other. As long as this condition is fulfilled, that is such a set can be reproduced, we can make an equivalent substitution for parameters t_r, s_r. Let us consider the substitution $t_r=t_{1r}-2a$ and show that it is an equivalent one. Indeed, it preserves the range of definition $(-\infty<t_{1r}<\infty)$, and accordingly produces all numbers, which parameter t_r produces (only with a shift of $(-2a\times 2^r)$ for the same values of t_r and t_{1r}). However, this shift makes no difference with regard to the range of produced numbers, since our range $(-\infty,\infty)$ is infinite in both directions. On the other hand, when $(t_r-s_r)=0$, we have $(t_{1r}-s_r)\neq 0$, and when $(t_{1r}-s_r)=0$, $(t_r-s_r)\neq 0$. So, for $(t_r-s_r)=0$, such a substitution produces an equation with a non-zero left part. Value of A_r remains the same, since, due to the substitution $t_r=t_{1r}-2a$, $s_r=s_{1r}-2b$, $A_r(t_r,s_r)=A_r(t_{1r}-2a,s_{1r}-2b)$.

Substituting $t_r=t_{1r}-2a$ into (14), one obtains the equation

$$(t_{1r}-s_r)A_r=2aA_r+A_rc/2^r+m_1^N/2^{r-r_t} \quad (29)$$

When $(t_r-s_r)=0$, we have $(t_{1r}-s_r)=2a\neq 0$. Also, the appearance of the even term $2aA_r$ does not change the parity of the right part, nor this substitution changes the parity of the left part (if it is not zero; if it is zero, the substitution still provides an even increment). Thus, with regard to parities, (14) and (29), indeed, are equivalent equations.

If the equivalent equation (29) has no solution, then the original equation (14) has no solution too. The proof is as follows. Let us assume that (29) has no solution, while (14) does, so that

$$(t_r-s_r)A_r=A_rc/2^r+m_1^N/2^{r-r_t}$$

Adding $2aA_r$ to the left and right parts of this equation, one obtains an equivalent equation, which also should have a solution.

$(t_r + 2a - s_r)A_r = 2aA_r + A_r c / 2^r + m_1^N / 2^{r-r_t}$

According to the substitution, $t_r = t_{1r} - 2a$, so that $t_r + 2a = t_{1r}$, and the obtained equation transforms to (29), which should also have a solution. However, according to our assumption, it has no solution. The obtained contradiction means that the assumption that (14) has a solution is invalid, and, in fact, it has no solution.

Similarly, we can assume that (29) has a solution, while (14) does not. With the substitution $t_{1r} = t_r + 2a$, (29) then converts to (14), which should have a solution. However, this contradicts the initial assumption.

Substitution $s_r = s_{1r} - 2b$, and both substitutions together, are considered similarly. In case of two substitutions, the condition $(a-b) \neq 0$ should be fulfilled, in order for (29) to have a non-zero left part.

This proves the Lemma.

Although we proved the equivalency of equations with regard to their solution properties in a general case, we need such equivalency only when the left part of equivalent equations is zero (because $(t_r - s_r) = 0$). The proposed substitution then makes the left part of the equivalent equation a non-zero value, and the inference about the absence of solution or its uncertainty can be made based on parities and rationalities of the non-zero left and right parts.

Table 6 shows an example of PPTs for the presentation level $(r_t + 3)$. Four initial PPTs are from cells (1,1)-(1,4) in Table 5. If (28) has no solution for even $(t_{r_t+2} - s_{r_t+2})$, then these are PPTs (1,3), (1,4) in Table 6, which satisfy this condition. Accordingly, PPTs (1,1) and (1,2), for which $(t_{r_t+2} - s_{r_t+2})$ is odd, are "uncertain" PPTs, which should be used as initial PPTs for the next, $(r_t + 4)$, level. If, on the contrary, (28) has no solution for odd $(t_{r_t+2} - s_{r_t+2})$, then (1,1) and (1,2) are the "no solution" PPTs, while (1,3), (1,4) become "uncertain" PPTs, which should be used as initial PPTs for the next level. This way, all new PPTs, four per each initial pair, are divided into two halves as before, so that the common ratio of geometrical progression remains equal to 1/2. The case $(t_{r_t+2} - s_{r_t+2}) = 0$ is addressed by Lemma 9 through equivalent equations.

In the same way, as we considered one PPT above, we should consider the rest of initial PPTs in Table 6 and find out, which two PPTs should be used as initial ones for the next level. Then, the same procedure should be repeated for each initial PPT at level $(r_t + 3)$.

Then, the cycle is repeated for the next two levels $(r_t + 4)$ and $(r_t + 5)$, and so forth, to infinity, since there are no anymore threshold values of *r*, at which the corresponding equations can change their form and properties, as it happened at the level $r = r_t + 1$. The following Lemma generalizes the discovered order.

Table 6. PPTs with a factor of 2^{r_t+3}. Initial PPTs are (1,1)-(1,4) from Table 5. Here, $r = r_t$.

	0	1	2
	t_{r+2} s_{r+2}	$2t_{r+3}$ $2s_{r+3} + 1$	$2t_{r+3} + 1$ $2s_{r+3}$
1	$2^{r+2} t_{r+2} + v_{r1}$ $2^{r+2} s_{r+2} + 2^{r+1} + 2^r + v_{r1}$	$2^{r+3} t_{r+3} + v_{r1}$ $2^{r+3} s_{r+3} + 2^{r+2} + 2^{r+1} + 2^r + v_{r1}$	$2^{r+3} t_{r+3} + 2^{r+2} + v_{r1}$ $2^{r+3} s_{r+3} + 2^{r+1} + 2^r + v_{r1}$
2	$2^{r+2} t_{r+2} + 2^{r+1} + v_{r1}$ $2^{r+2} s_{r+2} + 2^r + v_{r1}$	$2^{r+3} t_{r+3} + 2^{r+1} + v_{r1}$ $2^{r+3} s_{r+3} + 2^{r+2} + 2^r + v_{r1}$	$2^{r+3} t_{r+3} + 2^{r+2} + 2^{r+1} + v_{r1}$ $2^{r+3} s_{r+3} + 2^r + v_{r1}$
3	$2^{r+2} t_{r+2} + v_{r1}$	$2^{r+3} t_{r+3} + v_{r2}$	$2^{r+3} t_{r+3} + 2^{r+2} + v_{r1}$

	$2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r2}$	$2^{r+3}s_{r+3}+2^r+v_{r1}$
4	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+1}+v_{r2}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r2}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$

Table 6 continued

3	4
$2t_{r+3}$ $2s_{r+3}$	$2t_{r+3}+1$ $2s_{r+3}+1$
$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$

Lemma 10: *From the presentation level* (r_t+2), *the "no solution" fraction of PPTs is accumulated across two sequential levels, and then the pattern repeats for each two successive levels, to infinity. Some branches can be completed at levels* (r_t+2L), *where* $L=1,2,..$, *but otherwise such levels provide no explicit division into the "no solution" and "uncertain" groups. Except for the PPTs, corresponding to completed branches, PPTs from such levels become initial "uncertain" PPTs for the next presentation levels* (r_t+2L+1), $L=1,2,..$,*. at which all new PPTs are divided into the "no solution" and "uncertain" groups (according to odd or even parity of* (t_r-s_r) *in equation (14)). The "uncertain" PPTs become initial PPTs for the next presentation level, and the two-level cycle repeats to infinity.*

Proof: Previously, we have seen that the Lemma is true for the paired levels (r_t+2) and (r_t+3). Let us assume that Lemma is true for the (r_t+d-1) level, which then supplies initial "uncertain" PPTs for the next level (r_t+d). We need to prove that Lemma is true for the next two levels (r_t+d) and (r_t+d+1). Initial PPTs may have equal and unequal FCs.

Let us consider an equation for a PPT with FCs *v* and *w*.

$$(2^{r_t+d}t_{r_t+d}+v)^N-(2^{r_t+d}s_{r_t+d}+w)^N=2^{N(\mu+1)}m_1^N \qquad (30)$$

where $d\geq 2$.

According to Lemma 4, it can be transformed to an equation

$$(t_{r_t+d}-s_{r_t+d})A_{r_t+d}=A_{r_t+d}c/2^{r_t+d}+m_1^N/2^d \qquad (31)$$

where $A_{r_t+d}=\sum_{i=0}^{N-1}(2^{r_t+d}t_{r_t+d}+v)^{N-1-i}(2^{r_t+d}s_{r_t+d}+w)^i$, $N = 2n + 1$.

The right part of (31) can be an integer, rational or zero. The left part is an integer (if $(t_{r_t+d}-s_{r_t+d})=0$, the left part can be transformed to an integer, using Lemma 9). When the right part is rational, (31) has no solution for any t_{r_t+d} and s_{r_t+d}, and the branch is completed. If the right part is even or odd, (31) has no solution when $(t_{r_t+d}-s_{r_t+d})$ has the opposite parity. Solution is uncertain for the other parity of $(t_{r_t+d}-s_{r_t+d})$, since both parts of (31) have the same parity in this case. However, at this level, we cannot specify particular parity of $(t_{r_t+d}-s_{r_t+d})$, which should be done at the next presentation level (r_t+d+1). When c = 0, (31) has no solution, since the right part is a rational number, while the left part is an integer or zero, and so the branch is completed.

Even if the branch, corresponding to some PPT, is completed, we still can assume that it is "uncertain", and use it as an initial PPT at the next presentation level. There, the new PPTs, corresponding to this initial one, are then divided into the "no solution" and "uncertain" groups. The fraction of the former goes to the total "no solution" fraction, while the latter is used as initial PPTs for the next level, besides other uncertain PPTs. (Such an arrangement, without completed branches, is more convenient for calculation of the total "no solution" fraction.)

Table 7. New PPTs for the initial PPT [$2^{r_t+d}t_{r_t+d}+v$, $2^{r_t+d}s_{r_t+d}+w$] at the presentation level (r_t+d+1) with a factor of 2^{r_t+d+1}.

	0	1	2	3	4
0	t_{r_t+d} s_{r_t+d}	$2t_{r_t+d+1}$ $2s_{r_t+d+1}+1$	$2t_{r_t+d+1}+1$ $2s_{r_t+d+1}$	$2t_{r+3}$ $2s_{r+3}$	$2t_{r+3}+1$ $2s_{r+3}+1$
1	$2^{r_t+d}t_{r_t+d}+v$ $2^{r_t+d}s_{r_t+d}+w$	$2^{r_t+d+1}t_{r_t+d+1}+v$ $2^{r_t+d+1}s_{r_t+d+1}+2^{r_t+d}+w$	$2^{r_t+d+1}t_{r_t+d+1}+2^{r_t+d}+v$ $2^{r_t+d+1}s_{r_t+d+1}+w$	$2^{r_t+d+1}t_{r_t+d+1}+v$ $2^{r_t+d+1}s_{r_t+d+1}+w$	$2^{r_t+d+1}t_{r_t+d+1}+2^{r_t+d}+v$ $2^{r_t+d+1}s_{r_t+d+1}+2^{r_t+d}+w$

Table 7 shows new PPTs for the next presentation level for the initial PPT from (30). At this level, we can choose the needed parities of PPT's terms t_{r_t+d}, s_{r_t+d}, expressed through t_{r_t+d+1}, s_{r_t+d+1}, in order for (31) to have no solution. For instance, if (31) has no solution for even $(t_{r_t+d}-s_{r_t+d})$, then the "no solution" PPT are (1,3), (1,4). Accordingly, solution is uncertain for PPTs (1,1), (1,2), since both parts of (31) have the same parity in this case. Consequently, these PPTs should be used as initial "uncertain" ones for the next presentation level.

We can see from Table 7 that when a PPT of an actually completed branch is used as an "uncertain" PPT for the next level, it produces no new PPTs with some specific features, which could prevent their corresponding equations to be transformed into a form (31). We still obtain PPTs, satisfying conditions of Lemma 4, to which the same equation (14) is applicable. For instance, when v = w, then c = 0 in (31), and so the branch is completed. However, if we use it as an initial PPT for the next presentation level (r_t+d+1), then we are free to choose new PPTs, corresponding to either even or odd values of $(t_{r_t+d}-s_{r_t+d})$, since the corresponding equations have no solution for both scenarios. Then, the PPTs with the opposite parity $(t_{r_t+d}-s_{r_t+d})$ will proceed to the next level as uncertain initial PPTs. As before, such a division produces two equal groups of PPTs, and so the common ratio of the geometrical progression remains equal to 1/2.

So, with the assumption that Lemma is true for the previous level, we confirmed the same pattern earlier discovered for the coupled levels [$(r_t+2),(r_t+3)$]. According to principle of mathematical induction, this means that Lemma is true for any $d\geq 2$.

The proof also confirmed that in the presented arrangement no new threshold values of r could occur, so that the arrangement, indeed, repeats itself in two-level cycles to infinity.

This proves the Lemma.

In this Lemma, we also studied the useful property, considering completed branches as half-completed ones. This property is formulated below as a Corollary.

Corollary 5: *PPTs with the same parity of* $(t_{r_t+d} - s_{r_t+d})$*, corresponding to completed branches, can be considered as regular "uncertain" PPTs, which can be passed to the next level as initial PPTs, so that such a branch actually could be assigned a half-completed status.*

Lemma 11: *At presentation levels above* $(r_t + 1)$*, and in the absence of completed branches, the number of PPTs in "no solution" and "uncertain" groups are equal, when such a division takes place.*

Proof: According to Lemma 10 and Corollary 5, all PPTs, both regular ones, with "no solution" and "uncertain" components, and the PPTs, which could be completed, but continue to participate in the next levels as "uncertain" PPTs, can be presented in a form of Table 7. The solution properties of equations, corresponding to PPTs in Table 7, are defined by equation (14), or more particular, by equations in a form (31), whose solution properties depend on the term $(t_{r_t+d} - s_{r_t+d})$. (Unless the right part is rational, in which case equation has no solution for all parities, and the branch is completed. However, according to Corollary 5, we can still consider such PPT as a regular non-completed one.)

The division of four PPTs into two equal "no solution" and "uncertain" groups is based solely on the parity of $(t_{r_t+d} - s_{r_t+d})$, as Lemma 10 showed, with one parity corresponding to a "no solution" group, and with the opposite parity corresponding to "uncertain" group. The number of PPTs, corresponding to one parity, is therefore equal to 2π, where π is the number of initial PPTs, number two is the number of parity combinations of t_{r_t+d}, s_{r_t+d}, producing the same parity of $(t_{r_t+d} - s_{r_t+d})$, see Table 7. For the opposite parity of $(t_{r_t+d} - s_{r_t+d})$, the number of produced PPTs is also 2π. Thus, the number of PPTs in "no solution" and "uncertain" groups is the same. This proves the Lemma.

4.8. Calculating the total "no solution" fraction

Using Corollary 5, we consider all levels as if they have no completed branches. Then, according to Lemmas 7 and 8, until the level $(r_t + 2)$, all levels have two equal groups of PPTs. One corresponds to a "no solution" fraction, and the other to "uncertain" fraction, so that the common ratio q = 1/2. Substituting these values into (24), one obtains

$$F_{r_t+1} = f_2(1 - q^{r-1})/(1-q) = 1/2(1-(1/2)^{r_t+1-1})/(1/2) = 1-(1/2)^{r_t} \qquad (32)$$

The "no solution" fraction for the level $(r_t + 1)$ is defined by (23) as follows (the last term of a geometrical progression), taking into account that $f_2 = 1/2$.

$$f_{r_t+1} = f_2 q^{r_t+1-2} = (1/2)^{r_t} \qquad (33)$$

Since in the absence of completed branches the "no solution" and "uncertain" fractions are equal, according to Lemma 8, the "uncertain" fraction of PPTs, which is passed to the level $(r_t + 2)$, is the same as the "no solution" fraction (33). This "uncertain" fraction, according to Lemma 11, is equally divided into "no solution" and "uncertain" fractions at each second level, beginning from level $(r_t + 3)$, so that the first term of the geometrical progression, representing the "no solution" fraction of two following coupled levels, is

$$f_{r_t+3} = f_{r_t+1} \times (1/2) \qquad (34)$$

Then, each next two levels add a half of the previous "uncertain" fraction", which is equal to "no solution" fraction. Let $D = \{2L, 2L+1\}$, $L = 1,2...$ This way, $(r_t + D)$ defines the levels' numbers for $r \geq (r_t + 2)$. Levels, at which PPTs are divided into two groups, are levels $(r_t + 3)$, $(r_t + 5)$, …, $(r_t + 2L + 1)$, so that the total "no solution" fraction, obtained by summation of "no solution" fractions of all levels above the $(r_t + 1)$ level, is equal to

$$F_{r_t+2,D} = (1/2)^{r_t}[1/2 + (1/2)^2 + (1/2)^3 + ...(1/2)^L] = (1/2)^{r_t}\sum_{i=1}^{L}(1/2)^i = (1/2)^{r_t}(1-(1/2)^L) \qquad (35)$$

when $D = 2L + 1$, and

$$F_{r_t+2,D} = (1/2)^{r_t}[1/2 + (1/2)^2 + (1/2)^3 + ...(1/2)^{L-1}] = (1/2)^{r_t}\sum_{i=1}^{L-1}(1/2)^i = (1/2)^{r_t}(1-(1/2)^{L-1})$$

when $D = 2L$. (36)

In the last case, the division into the "no solution" and "uncertain" groups did not happen yet at the first level of coupled levels, since it occurs at the second level of the couple, as it was earlier discussed. This is why the power is (L - 1), but not L.

The total "no solution" fraction, accordingly, is defined as $F_{r_t+1+D} = F_{r_t+1} + F_{r_t+2,D}$. For $D = 2L + 1$, we have

$$F_{r_t+1+D} = F_{r_t+1} + F_{r_t+2,D} = 1-(1/2)^{r_t} + (1/2)^{r_t} - (1/2)^{r_t+L} = 1-(1/2)^{r_t+L} \quad (37)$$

It follows form (37) that in the limit

$$\lim_{L\to\infty} F_{r_t+1+D} = \lim_{L\to\infty}(1-(1/2)^{r_t+L}) = 1 \quad (38)$$

The same is true for (36). So, when we consider all branches as non-completed, in the limit, the "no solution" fraction is equal to one. Of course, it may look awkward, considering completed branches as non-completed, but, as Lemma 10 and Corollary 5 showed, this is a legitimate procedure.

Accounting for completed branches. Let level *r* to have *k* completed branches, to which the "no solution" fraction f_{rk} corresponds. Suppose, these branches were not completed. We can consider the PPTs, corresponding to these branches, as regular ones, with "no solution" and "uncertain" components, to infinity. In other words, we assume that there are no more completed branches except in level *r*. (In real situation, if there are completed PPTs above the level *r*, we can also consider them as non-completed ones, according to Corollary 5.) In this scenario, the fraction f_{rk} would be divided equally (Lemma 11) between the "no solution" and "uncertain" fractions on each subsequent level (or on the second level in coupled levels beyond the value of $r = (r_t + 1)$). So, the total "no solution" fraction, accumulated at level *L*, is as follows.

$$F_{r+L} = f_{rk}\sum_{i=1}^{L}(1/2)^i = f_{rk}[1-(1/2)^L] \quad (39)$$

In the limit, (39) transforms to

$$\lim_{L\to\infty} F_{r+D} = \lim_{L\to\infty} f_{rk}[1-(1/2)^L] = f_{rk} \quad (40)$$

So, in the limit, we obtained in (40) *exactly* the same "no solution" fraction, which was taken by *k* completed branches at level *r*. Since, according to (38), in the scenarios with non-completed branches the total "no solution" fraction is equal to one, the result (40) means that accounting for completed branches, in the limit, produces the same "no solution" fraction of one.

Until this point, a renowned Mathematician Professor Heath-Brown agreed that the limit of one for the "no solution" fraction was found correctly (he wrote to the author, "You successfully prove that the proportion of "no solution" branches tends to 1."). However, he did not agree that this necessarily means that (1) has no solution, as the author asserted in the first version of the article. (That approach is described in [2], v. 10, Appendix.) Such an opinion was rather due to the author's fault to explicitly present a principal difference between the deterministic nature of the used approach (that is summing up *deterministic* values of the "no solution" fractions) and the stochastic basis of the asymptotic densities.

In order to not depend on such arguable approach, an entirely new - this time mathematically very much conventional and straightforward - method was implemented (section 4.9 below). It shows that with the increase of the presentation level *any* pair of odd numbers will inevitably correspond to a certain form of equation (8), for which this pair is not a solution. Since this is valid for any pair of odd integers, that would mean that (8) has no solution on the whole set of pairs of odd numbers.

4.9. A "no solution" equation for any pair of odd numbers

Dr. A. Y. Shestopaloff suggested, and A. A. Tantsur supported the idea to present a conventional proof, showing that any pair of odd numbers will eventually correspond to some form of equation (8), for which this pair is not a solution, when the presentation level r increases. This approach is presented here.

Let's assume that a *certain* pair of odd numbers (g,h) is a solution of (8). Then, at each presentation level r, it can be written in the form $(2^r t_{0r}+v,\ 2^r s_{0r}+w)$, presented by PPT $[(2^r t_r+v),\ (2^r s_r+w)]$ with a corresponding equation (14). In order to be a solution of (8), this pair of odd numbers should be able to pass through all successive presentation levels, to infinity, "within" (meaning, defined by) "uncertain" PPTs. Fig. 2 shows such a transition schematically.

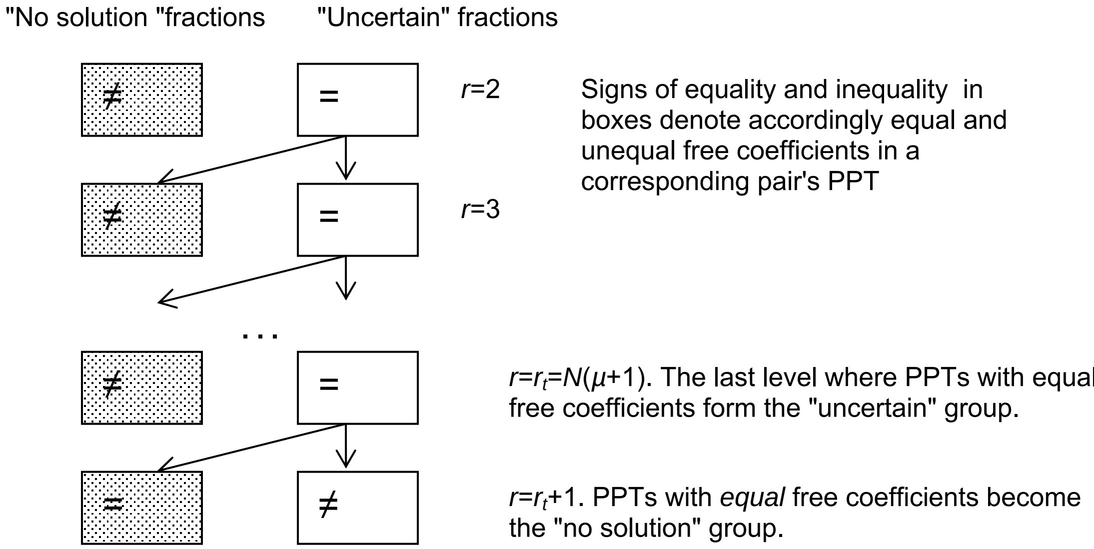

Fig. 2. Transition of a pair of odd numbers through presentation levels. Before the level $r=r_t+1$, PPTs with equal FCs are composing the "uncertain" group. At level $r=r_t+1$, PPTs with equal FCs acquire the "no solution" status.

Since until level $r=r_t$ "uncertain" PPTs have to have *equal* FCs, our pair of numbers should be defined by such PPTs. If the transition to the next presentation level $r\le r_t$ produces unequal FCs, then our pair is represented by a "no solution" PPT (Lemma 6 and Fig. 2). For instance, the pair of numbers $(2^3\times 36+5,\ 2^3\times 7+5)$ is presented at the next level as $(2^4\times 18+5,\ 2^4\times 3+2^3+5)$, so that the PPT $[2^4 t_4+5,\ 2^4 s_4+13]$, which defines this pair, has unequal FCs. If such a transformation happens before the threshold value r_t, then this is a "no solution" PPT, and consequently our pair (g,h), contrary to the assumption, is not a solution of (8). However, if $r_t=3$, then the presentation $r=4$ corresponds to r_t+1, for which "uncertain" PPTs are the ones with *unequal* FCs, which means that our pair preserves the "uncertain" status, and thus the possibility still to be a solution of (8).

The inequality of FCs in PPT at the level r_t+1 could come only as a result of appearance of an additional summand 2^r in the presentation of one number, while the presentation form for the second number should remain the same (otherwise FCs would remain equal). Indeed, the transition from, say, $(2^r s_r+v)$ to the next level r_t+1 can be done only in two ways: $(2^{r+1}s_{r+1}+v)$ or as $(2^{r+1}s_{r+1}+2^r+v)$. Thus, the pair of numbers that manages to pass the threshold level and remain uncertain, should have unequal FCs and be represented by PPTs of the form $[(2^{r_t+1}t_{r_t+1}+v),(2^{r_t+1}s_{r_t+1}+2^{r_t}+v)]$ or $[(2^{r_t+1}t_{r_t+1}+2^{r_t}+v),(2^{r_t+1}s_{r_t+1}+v)]$. There are no other possibilities. The same value of v in both terms is due to the need for our pair to pass the level $r=r_t$ in uncertain status, for which both terms have to have equal FCs.

Now, we can formulate the following Lemma.

Lemma 12: *With increase of presentation level r, for any pair of odd integer numbers there will be eventually an equivalent presentation of equation (8) at some level r, such that this pair will be an explicitly "no solution" pair for this equation.*

Proof: Let us consider equation (8) transformed to (9), which accordingly can be transformed to (14). We assume that the pair (g,h) is a solution of these equations for the fixed power of N. According to Lemmas 4 and 5, an even coefficient c in equation (14) is defined as $c=-\sum_{i=1}^{r-1}2^i K_{ri}$, where $K_{ri}=\{0,1\}$ for $c < 0$, and $K_{ri}=\{-1,0\}$ for $c > 0$, $1\le i\le r-1$, and index 'r' in K_{ri} denotes that it corresponds to presentation level r. However, in our case, for equations corresponding to PPTs at levels $r\ge r_t+1$, the smallest value of index i , where the value $c\ne 0$ appears, is $i=r_t+1$, since unequal FCs for the first time appear at this level (Fig. 2). So, $c=-\sum_{i=r_t+1}^{r-1}2^i K_{ri}=-2^{r_t}\sum_{i=r_t+1}^{r-1}2^{i-r_t}K_{ri}$. Substituting this value into equation (14), one obtains.

$$(t_r-s_r)A_r=(-A_r\sum_{i=r_t+1}^{r-1}2^{i-r_t}K_{ri})/2^{r-r_t}+m_1^N/2^{r-r_t} \tag{41}$$

According to Lemma 4, $A_r=\sum_{i=0}^{N-1}(2^r t_r+v)^{N-1-i}(2^r s_r+w)^i$. However, in our case, $(2^r t_r+v)$ and $(2^r s_r+w)$ are numbers of our concrete pair, which remain the same at each presentation level, and so the value of A_r does not depend on r. Let denote it as $A=A_r$. Since the numbers of our pair are fixed values, the right part of the original equation (9), which is $2^{N(\mu+1)}m_1^N$, is also a fixed value, and so is the value of m_1^N.

We want to show that the range $[R_{\min},R_{\max}]$ of possible values of the right part of equation (41) is finite for all $r\ge r_t+1$, when $r\to\infty$, so that the number of integers in this interval $[R_{\min},R_{\max}]$ is finite. Suppose we will manage to prove that for a given value of r the value of the right part is unique. Since $r\to\infty$, the number of values of the right part of (41) goes to infinity, while the number of integer values, which the right part can take from the interval $[R_{\min},R_{\max}]$, is finite - because the interval is finite.

Since $|K_{ri}|\le 1$, we can write

$$\left|\sum_{i=r_t+1}^{r-1}2^{i-r_t}K_{ri}\right|\le\sum_{i=r_t+1}^{r-1}2^{i-r_t}$$

The sum on the right is a sum of geometrical progression with a common ratio of two.

$$\sum_{i=r_t+1}^{r-1}2^{i-r_t}=2^{r-1-r_t}\times 2-2=2^{r-r_t}-2$$

So that

$$-(2^{r-r_t}-2)\le\sum_{i=r_t+1}^{r-1}2^{i-r_t}K_{ri}\le 2^{r-r_t}-2$$

Substituting the upper limit into the right part of (41), one obtains.

$$-A(2^{r-r_t}-2)/2^{r-r_t}+m_1^N/2^{r-r_t}=-A+(m_1^N+2A)/2^{r-r_t} \tag{42}$$

For the lower value

$$-A(-2^{r-r_t}+2)/2^{r-r_t}+m_1^N/2^{r-r_t}=A+(m_1^N-2A)/2^{r-r_t} \tag{43}$$

The expressions in the right parts of (42) and (43) are strictly monotonic exponential functions of r. Indeed, the first derivatives of exponential functions $(m_1^N - 2A)/2^{r-r_t}$ and $(m_1^N + 2A)/2^{r-r_t}$ remain exponential functions, which do not change the algebraic signs for all $r \geq r_t + 1$, and also cannot be equal to zero. (Also, $(m_1^N - 2A) \neq 0$, since m_1 is odd). This means that the functions themselves are strictly monotonic in this range.

When $r \to \infty$, $(m_1^N + 2A)/2^{r-r_t} \to 0$, $(m_1^N - 2A)/2^{r-r_t} \to 0$, so that the limits of (42) and (43) are accordingly $(-A)$ and A. Since we consider $r \geq r_t + 1$, the other possible boundaries are $[-A + (m_1^N + 2A)/2]$ and $[A + (m_1^N - 2A)/2]$, which are also finite. Thus, the boundaries of the interval for all possible values of the right part of (41) are finite.

$$R_{\min} = \min\{A, -A, [-A + (m_1^N + 2A)/2], [A + (m_1^N - 2A)/2]\}$$
$$R_{\max} = \max\{A, -A, [-A + (m_1^N + 2A)/2], [A + (m_1^N - 2A)/2]\}$$

Consequently, the range of the right part of (41) is finite for all $r \geq r_t + 1$.

Let us denote the right part of (41) as a function $\rho(r)$. We want to show that if $r_1 \neq r_2$, then $\rho(r_1) \neq \rho(r_2)$ for any pair (r_1, r_2). Let us assume that $\rho(r_1) = \rho(r_2)$, that is

$$(-A\sum_{i=r_t+1}^{r_1-1} 2^{i-r_t} K_{r_1 i})/2^{r_1-r_t} + m_1^N/2^{r_1-r_t} = (-A\sum_{i=r_t+1}^{r_2-1} 2^{i-r_t} K_{r_2 i})/2^{r_2-r_t} + m_1^N/2^{r_2-r_t}$$

Without losing generality, one can assume that $r_2 > r_1$. Then, the above equality can be rewritten as follows.

$$(-A\sum_{i=r_t+1}^{r_1-1} 2^{i-r_t} K_{r_1 i}) + m_1^N = (1/2^{r_2-r_1})[(-A\sum_{i=r_t+1}^{r_2-1} 2^{i-r_t} K_{r_2 i}) + m_1^N] \quad (44)$$

The left part of (44) is an integer. The term in the square brackets in the right part is odd, since the sum is even (Lemma 4) and m_1^N is odd. Then, the right part of (44) is rational, because an odd number has no dividers of two. So, contrary to the assumption, there are no such different numbers $r_1 \neq r_2$ that the equality (44) can be fulfilled. Consequently, the assumption is invalid. This means that for each value of r the value of the right part is unique, and so two values of the right part cannot correspond to the same integer number within the interval $[R_{\min}, R_{\max}]$.

Let us denote the number of integers in this interval as N_R. Then, since the number of values of the right part goes to infinity when $r \to \infty$, such a value of r_b always exists that for all $r > r_b$ the number of values of the right part will exceed N_R. Since these values of the right part are unique, some of them have to be *inevitably* rational. The left part of (41) is an integer, and so the equation has no solution in this case. That means that, contrary to the initial assumption, the pair (g,h) is not a solution of (41), and consequently is not a solution of (8). (Of course, equation (41), corresponding to the pair, could take an explicit "no solution" form *before* the number of values of the right part exceeds N_R. It may happen when both parts of (41) have different parities, or when the right part becomes rational.)

Since the obtained conclusions are valid for *any* pair of odd integers, this means that equation (14), and consequently (8), have no solution on the whole set of pairs of odd integer numbers.

This completes the proof of Lemma.

Thus, Lemma 12 proved that, indeed, for *any* pair of odd numbers such a presentation level r always exists, where the corresponding to this pair equation has no solution. Therefore, such a pair is a "no solution" pair of equation (14), and consequently of equation (8). This means that (8) has no solution for any pair of odd integer numbers, and consequently has no integer solution.

Lemma 12 makes unnecessary the content, related to finding a limit of one of the sum of "no solution" fractions. However, the author prefers to keep it for future discussions.

4.10. Cases 2 and 3 as equivalent equations

For the case 3, we have $a = 2n+1$; $x = 2k_1+1$; $y = 2p_1+1$. Then, (1) transforms to (7):

$$(2k_1+1)^{2n+1} + (2p_1+1)^{2n+1} = (2m_1)^{2n+1} \qquad (47)$$

We will show the equivalency of (8) (which is $(2k+1)^{2n+1} - (2p+1)^{2n+1} = (2m)^{2n+1}$) and (47) in terms of solution availability. Since (8) has no integer solution, that would mean that (47) has no solution too.

The notion of equivalent equations. It means that for each set of input variables for one equation there is one and only one matching set of corresponding input variables for the other equation, such that the terms in both equations are the same. For instance, with regard to equations (8) and (47), defined on the set of integer numbers, their equivalency would mean that for any combination of terms $(2k+1)$, $(2p+1)$, $2m$ in (8) there is only one combination of terms $(2k_1+1)$, $(2p_1+1)$, $2m_1$ in (47), such, that $(2k+1) = (2k_1+1)$, $(2p+1) = -(2p_1+1)$, $m = m_1$, so that with such a substitution equation (8) becomes equation (47). Similarly, the substitution $(2k_1+1) = (2k+1)$, $(2p_1+1) = -(2p+1)$, $m_1 = m$ in (47) produces equation (8). It was proved that (8) has no solution in integer numbers, so that it has no solution for any combination of these terms. However, on the set of all possible pairs of odd numbers, on which both equations are defined, these are equivalent equations (as the Lemma below proves). Then, since (8) has no solution, (47) and (7) have no solution too.

Lemma 13: *Equation (8) is equivalent to equation (47) on the set of integer numbers. If one of these equations has no solution in integer numbers, then the other equation also has no solution.*

Proof: Since the odd power does not change the algebraic sign, we can rewrite (8) as follows.

$$(2k+1)^{2n+1} + (-2p-1)^{2n+1} = (2m)^{2n+1} \qquad (48)$$

k, p in (8), and k_1, p_1 in (47) are integers defined on the range $(-\infty, +\infty)$. So, we can do a substitution $p = -p_1 - 1$ in (48). This substitution is an equivalent one, because (i) it does not change the range of the substituted parameter, neither it changes the ranges of the terms, defined by these parameters; (ii) this is a one-to-one substitution.

$$(2k_1+1)^{2n+1} + (2p_1+1)^{2n+1} = (2m_1)^{2n+1} \qquad (49)$$

where $k_1 = k$, $m_1 = m$. In this transformation, the range of parameters and equations' terms remains the same, that is $(-\infty < p < \infty)$, $(-\infty < p_1 < \infty)$, and so $(-\infty < (2p+1) < \infty)$, $(-\infty < (2p_1+1) < \infty)$. Thus, equation (48), which is (8), became equation (49).

Similarly, we can obtain equation (8) from (47), using substitution $p_1 = -p - 1$ in (47).

$$(2k_1+1)^{2n+1} + (2(-p-1)+1)^{2n+1} = (2m)^{2n+1} \qquad (50)$$

This transforms into equation (8).

$$(2k+1)^{2n+1} - (2p+1)^{2n+1} = (2m)^{2n+1} \qquad (51)$$

where $k_1 = k$.

Thus, (8) and (47), indeed, are equivalent equations.

Now, we should prove that if one of these equations has no solution, then the other equation also has no solution. For that, let us assume that equation (47) has no solution, while the equivalent equation (8) has a solution for the parameters (k_0, p_0, m_0), that is

$$(2k_0+1)^{2n+1} - (2p_0+1)^{2n+1} = (2m_0)^{2n+1} \qquad (52)$$

Doing an equivalent substitution $p_0 = -p_1 - 1$, one obtains

$$(2k_0+1)^{2n+1} + (2p_1+1)^{2n+1} = (2m_0)^{2n+1} \qquad (53)$$

Equation (53) (which is the original equation (47)), accordingly, has a solution for the parameters (k_0, p_1, m_0). However, this contradicts to the assumption that (47) has no solution. So, the equivalent equation (8) also has no solution. Similarly, we can assume that (8) has no solution, while (47) has a solution, and find through similar contradiction that (47) has no solution.

This completes the proof of Lemma.

It follows from Lemma 13 that it is suffice to prove that only one of the equations, (8) or (47), has no solution, in order to prove that both equations have no solution. Previously, we found that (8) has no solution in integers numbers. So, according to Lemma 13, (47), which presents case 3 for equation (1), also has no solution.

5. Conclusion

We found that cases 1, 3 and 4 can be converged to case 2. We proved that the corresponding to case 2 equation (8) has no solution in integer numbers. This means that (1) has no solution in integer numbers for all four cases, which proves FLT.

Introduced concepts and approaches can be applied to other problems of number theory.

6. Acknowledgements

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