Proof of Fermat Last Theorem based on successive presentations of pairs of odd numbers

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A simpler proof of Fermat Last Theorem (FLT), based mostly on new concepts, is suggested. FLT was formulated by Fermat in 1637, and proved by A. Wiles in 1995. The initial equation *x^n + y^n = z^n* is considered not in natural, but in integer numbers. It is subdivided into four equations based on parity of terms and their powers. Cases 1, 3 and 4 can be converted to case 2. It is studied using presentations of pairs of odd numbers with a successively increasing factor of 2^*r*. The proposed methods and ideas can be used for studying other problems in number theory.

2010 Mathematics Subject Classification (MSC) 11D41 (Primary)
Keywords: Diophantine equations; integer numbers; binomial expansion; parity

1. Introduction

One of the reasons that FLT still attracts people is that the known solution [1], in their view, is too complicated for the problem. Here, we introduce a simpler approach based on earlier work [2].

2. FLT sub-equations

Let us consider an equation.

$$x^a + y^a = z^a \tag{1}$$

The power *a* is a natural number $a \ge 3$. Unlike in the original FLT equation, here, *x*, *y*, *z* belong to the set of integer numbers **Z**. Combinations with zero values are not considered as solutions. We assume that variables *x*, *y*, *z* have no common divisor. Indeed, if they have such a divisor *d*, both parts of equation can be divided by d^a, so that the new variables $x_1 = x/d$, $y_1 = y/d$, $z_1 = z/d$ will have no common divisor. We will call such a solution, without a common divisor, a *primitive solution*. From the formulas above, it is clear that any non-primitive solution can be reduced to a primitive solution by dividing by the greatest common divisor. The reverse is also true, that is any non-primitive solution can be obtained from a primitive solution by multiplying the primitive solution by a certain number. So, it is suffice to consider primitive solutions only.

Values x, y, z in (1) cannot be all even. Indeed, if this is so, this means that the solution is not primitive. By dividing it by the greatest common divisor, it can be reduced to a primitive solution. Obviously, x, y, z cannot be all odd. So, the only possible combinations left are when x and y are both odd, then z is even, or when one of the variables, x or y, is even, and the other is odd. In this case, z is odd. Thus, equation (1) can be subdivided into the following cases, which cover all permissible permutations of equation's parameters.

1. $a = 2n$; $x = 2k + 1$; $y = 2p + 1$. Then, z is even, $z = 2m$. (2)
2. $a = 2n + 1$; $x = 2p + 1$; $y = 2m$. Then, z is odd, $z = 2k + 1$. (3)
3. $a = 2n + 1$; $x = 2k + 1$; $y = 2p + 1$. Then, z is even, $z = 2m$. (4)
4. $a = 2n$; $x = 2p + 1$; $y = 2m$. Then, z is odd, $z = 2k + 1$. (5)

3. Conversion of cases 1, 3 and 4 to case 2

It will be shown later that case 3 is equivalent to case 2. Regarding cases 1 and 4, as Dr. M. J. Leamer noted in his comment, there is a well known way to show that considering equation (1) is equivalent (in terms of existence of solution) to the case when exponent a is represented as a product of number four and (or) odd prime numbers. Indeed, we can assume that $a = fp$, where $f \geq 1$ is a natural number, p is a product of one or more prime factors, so that p is odd. (Certainly, prime factors of a can be distributed between f and p.) We can rewrite (1) as

$$(x^f)^p + (y^f)^p = (z^f)^p$$

Then, if there is no integer solution for the odd exponent p>1, then (1) has no integer solution too. Indeed, if one assumes that {x,y,z} is a solution of (1), then { x^f , y^f , z^f } would be integers representing a solution for the above equation. However, by assumption, it has no solution.

When a has no prime factors, $p = 1$, f is even. Since $a \geq 3$, $f \geq 4$. When f is divisible by four, we can use Corollary 2 on p. 53 in [3] that (1) has no solution for $a = 4$, representing the terms of (1) - say, the first one, as $(x^{f/4})^4$. When f is divisible by two, but not four, that would mean that $p \geq 3$ (since $a \geq 3$), and we again can convert (1) to an equation with an odd power.

Thus, cases 1 and 4 with even powers can be converted to cases 3 and 2 accordingly. Since case 3 is equivalent to case 2, this means that all four cases converge to case 2. So, it is suffice to only prove that there is no integer solution for case 2. Independent solutions for cases 1 and 4 are presented in [2].

4. Cases 2 and 3

We will need several Lemmas for these cases.

4.1. Presentation of numbers in a binary form

Lemma 1: *Each non-negative integer number n can be presented in a form*

$$n = \sum_{i=0}^{r} 2^i K_i \tag{6}$$

where $K_i = \{0,1\}$.

Proof: Effectively, this Lemma states the fact that any number can be written in a binary presentation.

From Lemma 1, the following Corollary follows.

Corollary 1: Any negative integer number n can be presented as $n = \sum_{i=0}^{r} 2^i B_i$, where $B_i = \{-1,0\}$.

4.2. Presentation of equation (1) for cases 2 and 3

For the case 3, we have $a = 2n + 1$; $x = 2k_1 + 1$; $y = 2p_1 + 1$. Then, (1) transforms to

$$(2k_1 + 1)^{2n+1} + (2p_1 + 1)^{2n+1} = (2m)^{2n+1} \qquad (7)$$

For the case 2, the power $a = 2n + 1$; $x = 2p + 1$; $y = 2m$. Then, *z* is odd, $z = 2k + 1$.

$$(2p + 1)^{2n+1} + (2m)^{2n+1} = (2k + 1)^{2n+1}$$

It can be rewritten in a form

$$(2k + 1)^{2n+1} - (2p + 1)^{2n+1} = (2m)^{2n+1} \qquad (8)$$

We can present *m* as $m = 2^{\mu} m_1$, where $\mu \geq 0$, and m_1 is an odd number. Then, (8) transforms to

$$(2k + 1)^N - (2p + 1)^N = 2^{N(\mu+1)} m_1^N \qquad (9)$$

where $N = 2n + 1$. Note that the value $r_t = N(\mu + 1)$ is a threshold one. If we divide both parts of the equation by 2^r, then for $r < r_t$ the right part is even, for $r = r_t$ it is odd, and for $r > r_t$ it is rational.

In the following, we will use a presentation of pairs of odd numbers with a factor of 2^r, where $r \geq 1$, whose properties are considered below.

4.3. Presentation of pairs of odd numbers with a factor of 2^r

4.3.1. Introducing pairs of presentation terms

Let us consider an infinite set of *pairs* of odd integers produced by a pair of terms $[(2k+1), (2p+1)]$, *k* and *p* are *integer* variables without duplicate values. The set, produced by a term $(2k+1)$, can be presented by two terms with a factor of four (for the even and odd *k* (*k*=2*t*, *k*=2*t*+1)), which are $(4t+1)$ and $(4t+3)$. Let us call the latter as subterms. Each such subterm represents a subset of odd numbers. Similarly, the set of odd numbers $\{(2p+1)\}$, produced by a term $(2p+1)$, can be presented by two subterms $(4s+1)$ and $(4s+3)$. (Here and in the entire paper such parameters as *k*, *p*, *t* and *s*, presenting in terms defining sets of numbers, are *integer* variables without duplicate values.) Thus, the original set, produced by a pair of terms $[(2k+1), (2p+1)]$, can be presented by four possible pairs of the above subterms with a presentation factor of four (2^2). In the following, such a pair of terms (for an arbitrary presentation layer) will be called a *pair of presentation terms* (PPT). Note that in this paper, we will consider only the terms presenting odd numbers. PPT *defines* a set of pairs of odd numbers. Such, each *one* PPT, presented above, defines an *infinite* set of pairs of odd numbers.

In essence, PPT *is* a set of numbers it produces. The distinction between PPTs and sets of pairs of odd numbers they produce is subtle; it emerges only when one begins splitting presentation terms (and consequently splitting the corresponding sets). Splitting and assembling infinite sets is not easy. The task could be facilitated, if instead of operating on infinite sets one could operate on finite or easier countable proxies of such sets (say, instead of dealing with infinite number of infinite sets to deal with one infinite set only). Of course, such proxies have to have certain properties, which make operations on them equivalent to operations on infinite sets (in a desirable sense). In our case, the introduced PPTs are such proxies.

Table 1 shows four possible PPTs, expressed with a factor of four. Such a presentation produces a *complete set* of pairs of odd integer numbers, since we considered all possible combinations of parities of *k* and *p*. (The completeness of such a presentation will be proved later for a general case of presentation with a factor 2^r).

We can continue presenting sets of pairs of odd numbers by PPTs using a successively increasing factor of 2^r. Initial PPTs for the next presentation level with a factor of 2^3 are in cells (2,1)-(2,4). Table 2 shows the presentation with a factor of 2^3 for two PPTs - from cells (2,3), (2,4) in Table 1. Note that index '3' for variables *t*, *s* corresponds to power *r*=3 in a presentation factor 2^r. Such correspondence of the index to the power of two in a presentation factor will be used throughout the paper.

Table 1. All possible PPTs, defining sets of pairs of odd numbers, expressed with a factor of four.

	0	1	2	3	4
1	k p	$2t_2$ $2s_2+1$	$2t_2+1$ $2s_2$	$2t_2$ $2s_2$	$2t_2+1$ $2s_2+1$
2	$2k+1$ $2p+1$	$\mathbf{4t_2+1}$ $\mathbf{4s_2+3}$	$\mathbf{4t_2+3}$ $\mathbf{4s_2+1}$	$4t_2+1$ $4s_2+1$	$4t_2+3$ $4s_2+3$

Table 2. PPTs, expressed with a factor of eight (2^3), corresponding to initial PPTs $[4t_2+1, 4s_2+1]$, $[4t_2+3, 4s_2+3]$ from Table 1.

	0	1	2	3	4
1	t_2 s_2	$2t_3$ $2s_3+1$	$2t_3+1$ $2s_3$	$2t_3$ $2s_3$	$2t_3+1$ $2s_3+1$
2	$4t_2+1$ $4s_2+1$	$\mathbf{8t_3+1}$ $\mathbf{8s_3+5}$	$\mathbf{8t_3+5}$ $\mathbf{8s_3+1}$	$8t_3+1$ $8s_3+1$	$8t_3+5$ $8s_3+5$
3	$4t_2+3$ $4s_2+3$	$\mathbf{8t_3+3}$ $\mathbf{8s_3+7}$	$\mathbf{8t_3+7}$ $\mathbf{8s_3+3}$	$8t_3+3$ $8s_3+3$	$8t_3+7$ $8s_3+7$

4.3.2. Presentation terms and infinite subsets of integer numbers

So, we have two tightly related entities: PPTs and the corresponding infinite sets of pairs of odd numbers they produce. Eventually, we need to prove that (1) has no solution for all possible *pairs of odd numbers*. However, the proof is based on consideration of PPTs, which are producers of infinite sets and subsets of pairs of odd numbers. On the surface, one observes that the number of PPTs at each presentation level is finite, while they produce subsets composed of infinite number of pairs of odd numbers, which could be perceived as an issue.

In fact, the same issue is implicitly present in all problems, dealing with infinite sets of numbers. However, in those problems, the sets and the appropriate terms, producers of these sets, relate to one presentation level, so that one even doesn't think about such an issue, taking for granted that the expression-producer, indeed, represents the infinite set; in fact, *is* this set. For instance, if one considers an infinite set of odd integer numbers, then a term, producing this set, is $(2k+1)$, where k is an integer. Once one proves that a certain problem has no solution for $(2k+1)$, that is for the *term*, this implicitly assumes that the problem has no solution for the *set* of all odd integers. In this case, there is no question about the legitimacy of the used approach.

The issue emerges when we represent the term $(2k+1)$ as a union of two terms $(4t+1)$ and $(4t+3)$, t is an integer. Since k can only be odd or even, these two terms describe all possible odd integers, the same as the initial term $(2k+1)$. Each of these terms is a producer of the associated infinite subset of odd integers. Then, can one say that the union of these two infinite subsets represents the *whole* set of odd integers, same as the term $(2k+1)$ does? Intuitively, this is obviously so. We have two non-intersecting subsets, which together comprise *all possible* odd integers. We just have to prove that such *subsets*, indeed, are non-intersecting, and the operations of splitting them and assembling back - by appropriate splitting and assembling their presentation terms - are unique and reversible. The uniqueness is understood as follows. Suppose subset S is produced by a term T, which then is split into several subterms $t_1, t_2, ... t_m$ for another presentation level. Each of these subterms accordingly produces subsets $s_1, s_2, ... s_m$. Then, such a splitting of the term T is unique (and the appropriate operation of splitting subset S into subsets $s_1, s_2, ... s_m$ is unique too), if for any element in S there is one and only one such element in one of the subsets $s_1, s_2, ... s_m$. The reverse operation, assembling subterms $t_1, t_2, ... t_m$ into the term T (with appropriate assembling of subsets $s_1, s_2, ... s_m$ into set S) is defined similarly.

Infinite subsets of numbers are produced by the subterms. There is no infinite subset of numbers without its producer, the term, which *entirely* defines the properties of the subset. Operating on the terms, we operate on the produced by them subsets of numbers. If subterms uniquely add to a term, generating a certain set of numbers, this is the same as adding corresponding subsets of numbers to produce such a set. We just have to make sure that the operations of adding subterms are unique.

What embarrasses people in the above procedure though, is splitting and assembling *infinite* sets, which they mentally detach from the presentation terms, quickly forgetting that these terms in essence *are* the sets in question. In fact, there is no difference in this regard compared to considering the term $(2k+1)$ and the appropriate set of odd integers as *identical* things - which they truly are. Such confusion might come from the notion of asymptotic density [4,5], which is used for characterization of infinite sets with relation to the set of integer numbers. Once people come across the aforementioned splitting of infinite subsets, they begin to think of their characterization in terms of *densities*, which is *absolutely not the case* in our situation. By definition, density is rather a *stochastic* notion, while we consider *only deterministic* values. The fact that for a certain infinite set the density converges to value d_s does not mean that *all* elements of the set satisfy to a certain criterion, say, not being a solution of some equation. Any finite number of such elements-exceptions, for which the equation does have a solution, won't change the value of d_s, since the number of elements in the set is infinite. In our case, we need *all* elements of the considered sets and subsets to satisfy the same criterion, that is to not be a solution of equation (1), with absolutely no possibility of any exception. The notion of density is certainly unsuitable for such a purpose, and so *it is not used* in this proof, as some commenters wrongly assumed.

4.4. The concept of the proof

Each PPT in Tables 1 and 2, and in subsequent presentations, defines an *infinite* set of pairs of odd numbers. However, the number of PPTs at each presentation level is *finite*. All PPTs, belonging to the same presentation level, together produce the ***whole** set of pairs of odd numbers*.

A subset of PPTs from one level can be uniquely transformed to a subset of PPTs at another presentation level. The subset of pairs of odd numbers, associated with the initial PPTs, will remain *the same* in such a transition. Using such transformations of PPTs, we can distribute the initial set of PPTs (from the initial level) across different presentation levels, and vice versa (that is to combine PPTs from different upper presentation levels back to initial or other lower level). Accordingly, the pairs of odd numbers, associated with PPTs at different presentation levels, will be associated with one presentation level again. Such transformations are unique, that is they are one-to-one transformations in both directions, meaning that for each pair of odd numbers at one presentation level there is one and only one such pair at any other presentation level. The properties of a subset of pairs of odd numbers, acquired at other presentation levels (say, that (1) has no solution on this subset), certainly remain with this subset at another presentation level (because these are just the same combinations of numbers).

It will be proved in Lemma 3 that the infinite sets, defined by PPTs at the same level, indeed, are unique and do not intersect. At each presentation level, equation (8) has no solution for a certain fraction of PPTs. Such "no solution" fractions accumulate through subsequent presentation levels, producing a greater and greater total fraction of PPTs, for which (8) has no solution. In the limit, this total "no solution" fraction becomes equal to one, which can be considered as if all "no solution" PPTs sums up to the initial PPT [$(2k+1),(2p+1)$], which defines a set of all possible pairs of odd numbers. Accordingly, this will mean that (8) has no solution for all possible pairs of odd numbers - the result we aim for. (This is one possible - to some extent subtle - approach to connect the limit of one with the whole set of pairs of odd numbers. An independent, more conventional, approach is also presented in section 4.9.)

At this point, it is important to understand that we deal with purely *deterministic* (but not stochastic!) values and relationships. The "no solution" fractions, associated with finite number of PPTs at different presentation levels, are such deterministic values.

The illustration of the concept of the proof below *is not* an actual proof, which is different. It only illustrates *one* of the possible considerations, related to parity, while the actual proof uses also rationality and zero values of equations' terms (without which the proof would be unlikely possible).

We begin with the *whole set* of all possible pairs of odd integer numbers, defined by a PPT $[(2k+1),(2p+1)]$, k and p are integers. At the next presentation level (r=2) we have four PPTs listed in Table 1. Equations, corresponding to a *half* of these PPTs, have no solution. Then, the half of PPTs (that is two of them), for which equations have no solution, is set aside (they constitute the "no solution" fraction f_{2ns}). The remaining half of PPTs compose an "uncertain" fraction of PPTs, for which solution is uncertain. The "uncertain" fraction is equal to $f_{2u}=1-f_{2ns}=1/2$. The following example illustrates the approach.

Equation (8) can be transformed as a difference of two numbers in odd powers.

$$2(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^{i}=(2m)^{2n+1}$$

Dividing both parts by two, one obtains

$$(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^{i}=m(2m)^{2n}$$

Here, the sum is odd as an odd quantity of odd numbers. If the factor $(k-p)$ is odd, then the left part is odd, while the right part is even (since $n > 0$). This means that there is no solution in this case. The value of $(k-p)$ is odd when one of the terms is odd and the other is even, which are the values of k and p in cells (1,1), (1,2) in Table 1, corresponding to PPTs $[4t+1, 4s+3]$ and $[4t+3, 4s+1]$ with a presentation factor of 2^2. The change of algebraic signs of k and p does not change the parity of the left part. So, the result is valid for *integer* numbers k and p. When $(k-p)=0$, the left part is zero, while the right part is an integer. So, there is no solution in this case.

When $(k-p)$ is even, both parts of equation are even, and solution is uncertain. This corresponds to values of k and p in cells (1,3), (1,4) in Table 1, with corresponding PPTs $[4t+1, 4s+1]$ and $[4t+3, 4s+3]$. These "uncertain" PPTs should be used as initial ones for the next presentation level with a factor of 2^3 (Table 2).

At the presentation level with r=3, we again find that a half of new PPTs (four PPTs in bold in Table 2) correspond to a "no solution" fraction, which is found as $f_{3ns}=f_{2u}\times 1/2=1/4$. The fraction of remaining uncertain PPTs is accordingly $f_{3u}=f_{2u}-f_{3ns}=1/2-1/4=1/4$. Therefore, two presentation levels produce the following total fraction of PPTs, for which (8) has no solution, $F_{3NS}=f_{2ns}+f_{3ns}=1/2+1/4=3/4$. The "uncertain" fraction $f_{3u}=1-3/4=1/4$, gives initial PPTs for the next presentation level (with r=4), and so forth, until in infinity the "no solution" fraction accumulates to one. In other words, all "no solution" fractions sum up to the initial term $[(2k+1),(2p+1)]$, which produces the set of all possible pairs of odd numbers. Due to uniqueness of used transformations, this would mean that (8) has no solution. (The real situation with the "no solution" fractions is more complicated, since some PPTs may have no solution for the entire PPT, and such a branch is closed. However, the total "no solution" fraction is still equal to one in the limit.) Fig. 1 illustrates the concept in more detail.

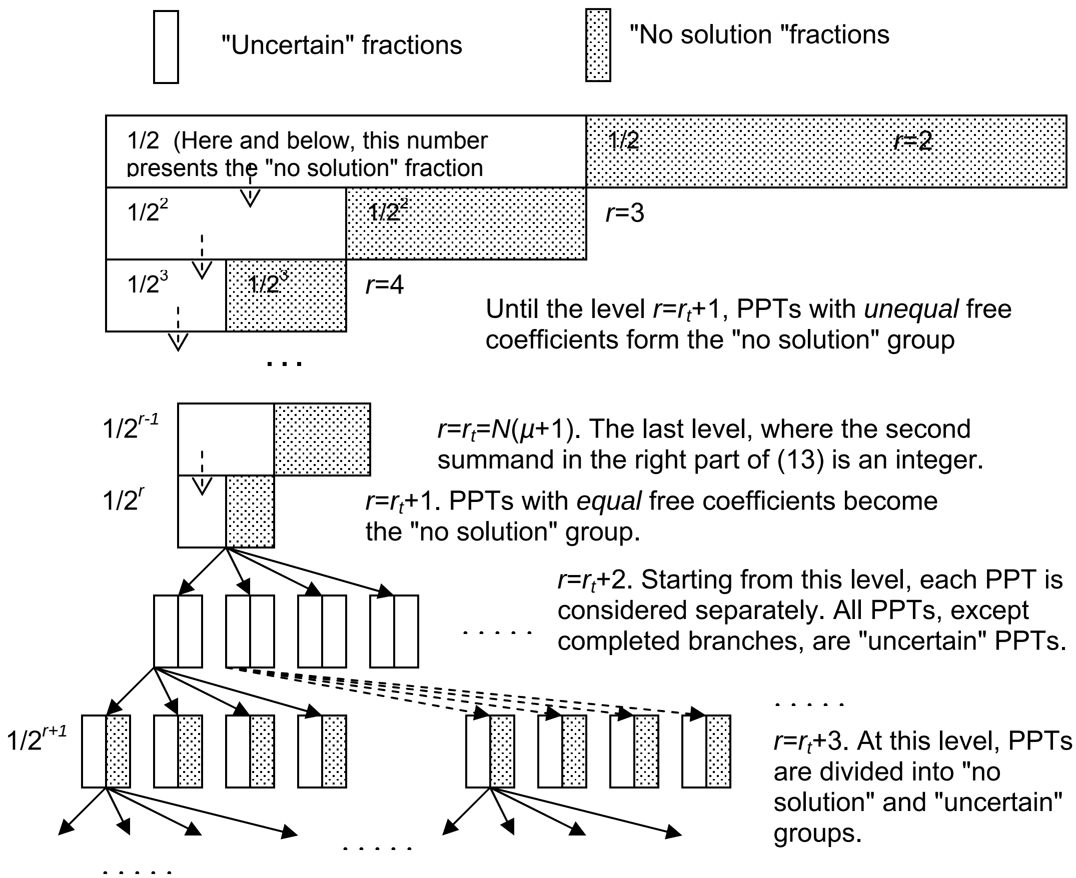

Fig. 1. Graphical presentation of how the "No solution" fraction accumulates through presentation levels, and the appropriate decrease of "Uncertain" fraction. The value of $r=r_t=N(\mu+1)$ is a threshold value, where transition begins from even right parts of equations to integer or rational ones.

4.4.1. Properties of PPTs with a factor of 2^r

Lemma 2: *Successive presentations of odd numbers with a factor of* 2^r *cannot contain a free coefficient, whose module is greater or equal to* 2^r.

Proof: Presentations of odd numbers with factors 2^2 and 2^3 satisfy this requirement. Let us assume that this is true for a presentation level *r*, that is the free coefficient *v* in a term $(2^r t_r + v)$ satisfies the condition $|v| < 2^r$. At a presentation level (*r* + 1), this term is presented as $(2^{r+1} t_{r+1} + 2^r + v)$ or $(2^{r+1} t_{r+1} + v)$. In the latter term, the condition is already fulfilled. In the first term, $0 < 2^r + v < 2^r + 2^r = 2^{r+1}$, since $|v| < 2^r$ is true for level *r* by assumption (zero is obtained for negative *v*). So, assuming that the condition is fulfilled at the level *r*, we obtained that it is also fulfilled at the level (*r* + 1). According to principle of mathematical induction, this means the validity of the assumption. This proves the Lemma.

Note: PPT with a negative free coefficient can be always transformed to a PPT with a positive one. Let *v* > 0. Then one can write $(2^r t_r - v) = 2^r (t_r - 1) + (2^r - v)$. According to Lemma 2, the free coefficient $(2^r - v)$ is positive. So, without restricting generality, one can consider only PPTs with positive free coefficients.

The number of PPTs grows for successive *complete* presentations in a geometrical progression with a common ratio of *four*, since each initial PPT produces four new PPTs at the next presentation level. (Each new PPT corresponds to one of the four possible parity combinations of input parameters, like t_2, s_2 in Table 2, whose parity is expressed through t_3, s_3.)

For the following, we need to prove that (a) such a presentation produces the whole set of pairs of odd numbers at each level; (b) the presentation is unique, that is for each pair of odd integers at one level there is one and only one such pair at any other presentation level.

Lemma 3: *Successive PPTs with a factor of* 2^r, $r \geq 2$, *produce the same whole set of pairs of odd numbers at each presentation level. Such presentations are unique, that is for any pair of odd numbers there is one and only one the same pair at any other presentation level. Each PPT at the same presentation level produces a unique set of pairs of odd numbers.*

Proof: First let us note that each next presentation level (r+1) is obtained prom the previous one through branching of each initial pair (from level r) into *all* four possible combinations of parities of parameters t_r and s_r, so that there are no any other possible combinations of parities, and so no element of the original set, defined by the initial PPT $[(2^r t_r + v), (2^r s_r + w)]$ could be missing at the next presentation level. This means that any PPT from an arbitrary level r is fully represented at level (r+1), although in the form of four new PPTs. In fact, this is just a different form of presentation of the same PPT.

In the initial PPT, the term $(2^r t_r + v)$ can be presented at level (r+1) only in two forms (for even and odd values of t_r), that is as $2^r(2t_{r+1}) + v = 2^{r+1} t_{r+1} + v$, or $2^r(2t_{r+1} + 1) + v = 2^{r+1} t_{r+1} + 2^r + v$. Similarly, the term $(2^r s_r + w)$ can be represented in the same two forms only. So, only four combinations of these terms, containing both t and s parameters, are possible. These combinations are unique, because the combinations of free coefficients are unique. Indeed, the combinations are as follows: [v,w], $[2^r + v, w]$, $[v, 2^r + w]$, $[2^r + v, 2^r + w]$. According to Lemma 2, free coefficients by module are less than 2^r, and so these combinations cannot produce the same pairs of odd integers. Thus, each of the four new PPTs produces unique set of pairs of odd integers, and these sets, defined by four PPTs, do not intersect, while together they contain the whole set of pairs of odd numbers, produced by initial PPT.

Now, we should show that there are no duplicate elements in each set defined by new PPTs. Suppose there are such duplicate pairs of odd numbers ($2^{r+1} t_{r+1,d} + v_d$, $2^{r+1} s_{r+1,d} + w_d$) in the set defined by PPT $[2^{r+1} t_{r+1} + v, \ 2^{r+1} s_{r+1} + w]$. Transformation from level r to level (r+1) is a one-to-one transformation, that is one value of $t_{r,1}$ can produce only one value at level (r+1), and the same is true for $s_{r,1}$. So, if we have duplicate values, then they should be produced by different values of t_r and s_r, say by $t_{r,2}$ and $s_{r,2}$. So, one can write the following:

$2^r(t_{r,1}) + v = 2^r(t_{r,2}) + v = 2^{r+1} t_{r+1} + v$, or $2^r(t_{r,1}) + v = 2^r(t_{r,2}) + v$, from which follows $t_{r,1} = t_{r,2}$. So, we obtained an equality that is contrary to our assumption that $t_{r,1} \neq t_{r,2}$, which means that it is invalid. Similarly, we can prove that, contrary to the assumption, $s_{r,1} = s_{r,2}$. So, it is impossible to have duplicate elements at any two adjacent presentation levels, and so at any level (provided the initial level has no duplicates, as it was earlier discussed). This means that contrary to our assumption the subset does not have duplicate elements.

Then, the above considerations can be repeated for three other PPTs, which also have no duplicates.

So, we found that (a) the subsets defined by new PPTs at level (r+1) are unique in that regard that each of them contains unique elements; (b) each of these subsets has no duplicate elements; (c) four new PPTs at level (r+1) together define the same set of pairs of odd numbers as the initial PPT at level r.

The reverse is also true, that is four PPTs at presentation level (r+1) uniquely converge to one initial PPT at lower level r. Indeed, two terms with parameter t converge to the same term $(2^r t_r + v)$.

$$2^{r+1}t_{r+1} + v = 2^r(2t_{r+1}) + v = 2^r t_r + v \tag{10}$$

$$2^{r+1}t_{r+1} + 2^{r+1} + v = 2^r(2t_{r+1} + 1) + v = 2^r t_r + v \tag{11}$$

where $t_r = \{2t_{r+1}, 2t_{r+1} + 1\}$.

The same convergence to a single term can be obtained for a general case of presenting two terms at level (r+1) using Lemma 1, and then transforming them to level r.

$$2^{r+1}t_{r+1} + \sum_{i=1}^{r} 2^i K_i + 1 = 2^r(2t_{r+1} + K_r) + \sum_{i=1}^{r-1} 2^i K_i + 1 = 2^r t_r + \sum_{i=1}^{r-1} 2^i K_i + 1 \tag{12}$$

For a positive number, $K_r = \{0,1\}$, so $t_r = \{2t_{r+1}, 2t_{r+1} + 1\}$, that is the same set of integer numbers, on which t_r was defined originally. The same is true for negative numbers.

Similarly, one can convert two terms with parameter s_{r+1} at level (r+1) to a single term with parameter s_r at level r. So, four PPTs at level (r+1), indeed, converge to one PPT [$2^r t_r + v$, $2^r s_r + w$] at level r, which, accordingly, defines the same initial set of pairs of odd integers.

Therefore, such transformations of presentation forms from level r to level (r+1) and backward are unique; they neither remove nor add any new elements compared to the initial set. The initial set does not have duplicate entries, and so four sets at the next presentation level do not have duplicate entries too. So, for each pair of odd numbers at level r there is one and only one pair of odd numbers at level (r+1), and vice versa. This proves the Lemma.

It follows from Lemma that all PPTs of each presentation level together define the same set of pairs of odd integers, as the initial PPT [$(2k+1)$, $(2p+1)$] does, that is the *whole* set of pairs of odd integers. This result can be formulated as a Corollary.

Corollary 2: All PPTs of each presentation layer together define the whole set of pairs of odd integers. This set has no duplicate entries.

4.5. Properties of equations, corresponding to pairs of odd numbers with a factor of 2^r

This section introduces an equation, to which all equations, corresponding to pairs of odd numbers, can be transformed, and explores its properties.

Lemma 4: *Let us consider an equation*

$$(2^r t_r + v)^N - (2^r s_r + w)^N = 2^{N(\mu+1)} m_1^N \tag{13}$$

where t_r and s_r are integers; N=2n+1; m_1 is odd; v, w are positive odd (possibly equal) numbers, obtained through successive PPTs (and consequently through successive presentations of pairs of odd numbers). Then, for any $r \geq 3$, such equations can be transformed to the following form

$$(t_r - s_r)A_r = A_r c / 2^r + m_1^N / 2^{r-r_t} \tag{14}$$

where $A_r = \sum_{i=0}^{N-1} (2^r t_r + v)^{N-1-i} (2^r s_r + w)^i$ *is an odd integer; c is an integer;* $r_t = N(\mu+1)$.

Proof: Equation (13) is equation (8), rewritten for a presentation with a factor of 2^r.

$$[2^r(t_r - s_r) + (v - w)] \sum_{i=0}^{N-1} (2^r t_r + v)^{N-1-i} (2^r s_r + w)^i = 2^{N(\mu+1)} m_1^N \tag{15}$$

The sum in (15) is odd, because it presents the sum of odd quantity of odd values. Let us denote it

$$A_r = \sum_{i=0}^{N-1} (2^r t_r + v)^{N-1-i} (2^r s_r + w)^i$$

Since v and w are odd, their difference is even. Also, in successive presentation of odd numbers, according to Lemma 2, $v < 2^r$, $w < 2^r$. Since both values are positive (see a note after Lemma 2), their absolute difference is also less than 2^r. According to Lemma 1 and Corollary 1, (*v* - *w*) can be presented as a sum of powers of two with coefficients, having the same algebraic sign. Since $|v-w|<2^r$, such a sum cannot contain a summand with a power greater than 2^{r-1}, when all coefficients K_i have the same algebraic sign.

$$[2^r(t_r-s_r)+\sum_{i=1}^{r-1}2^iK_i]A_r=2^{N(\mu+1)}m_1^N \qquad (16)$$

Then, (16) can be rewritten as follows.

$$2^r(t_r-s_r)A_r=-\left(\sum_{i=1}^{r-1}2^iK_i\right)A_r+2^{N(\mu+1)}m_1^N \qquad (17)$$

Let us denote $c=-\sum_{i=1}^{r-1}2^iK_i$. Since $c=w-v$, when $w=v$ (that is free coefficients are equal), $c=0$. When $w\neq v$, the value of $c\neq 0$. Dividing both parts of (17) by 2^r, and taking into account that $r_t=N(\mu+1)$, we obtain

$$(t_r-s_r)A_r=A_rc/2^r+m_1^N/2^{r-r_t} \qquad (18)$$

This proves the Lemma.

Lemma 5: *If $c\neq 0$ in (18), then $A_rc/2^r$ is a rational number.*

Proof: It was indicated in Lemma 4 that when free coefficients *w* and *v* are unequal, $c\neq 0$. According to Lemma 1, we can always use a presentation $\sum_{i=1}^{r-1}2^iK_i$ with the range of values $K_i=\{0,1\}$, $1\le i\le r-1$, when $c>0$, and $K_i=\{-1,0\}$ when $c<0$. Then

$$|c|=\left|\sum_{i=1}^{r-1}2^iK_i\right|\le\sum_{i=1}^{r-1}2^i=2(2^{r-1}-1)/(2-1)=2^r-2 \qquad (19)$$

(Here, we substituted the sum of a geometrical progression with a common ratio of two and the first term of two.) Accordingly

$$\left|A_rc/2^r\right|\le|A_r|(1-1/2^{r-1}) \qquad (20)$$

Dividing inequality (20) by a positive number $|A_r|$, one obtains

$$\left|c/2^r\right|\le(1-1/2^{r-1}) \qquad (21)$$

Thus, $c/2^r$ is a rational number. The term A_r is an odd number, which, consequently, contains no dividers of two. In turn, this means that $A_rc/2^r$ is a rational number. This proves the Lemma.

Lemma 6: *Equation (14) has no solution for PPTs with unequal free coefficients when $r\le N(\mu+1)$, while solution is uncertain for PPTs with equal free coefficients.*

Proof: For $r\le N(\mu+1)=r_t$, the term $2^{r_t-r}m_1^N$ in (14) is an integer. According to Lemma 5, the summand $A_rc/2^r$ is rational for PPTs with unequal free coefficients. So, the right part of (14) is rational. On the other hand, the left part is an integer when $(t_r-s_r)\neq 0$. This means that (14) has no solution in this case. When $(t_r-s_r)=0$, (14) presents equality of zero (in the left part), and of a rational number, which is impossible too. So, (14) has no solution for PPTs with unequal free coefficients.

When free coefficients are equal, *c* = 0, and (14) transforms to

$$(t_r-s_r)A_r=2^{r_t-r}m_1^N \qquad (22)$$

For $r < r_t$, the right part is even, for $r = r_t$ it is odd. The left part can be odd, or even, or zero. So, the solution of this equation is uncertain. Consequently, the PPTs, whose terms have equal free coefficients, should be used as initial PPTs for the next presentation level. This proves the Lemma.

Now, we should establish relationships between the sizes of groups, corresponding to PPTs with equal and unequal free coefficients, and the parity of the term $(t_r - s_r)$ in (14).

Lemma 7: *When initial PPTs, obtained from r-level of presentation for level (r+1), have equal free coefficients, the number of PPTs with equal and unequal free coefficients at level (r+1) is the same and is equal to 1/2 of the total number of PPTs. The group of PPTs with equal free coefficients correspond to even values of* $(t_r - s_r)$*, while PPTs with unequal free coefficients correspond to odd* $(t_r - s_r)$*, so that it is equivalent subdividing the PPTs based on parity of* $(t_r - s_r)$*, or on the basis of equal and unequal free coefficients.*

Proof: It follows from Table 1 that for $r_2 = 2$ the quantities of PPTs with equal and unequal free coefficients are equal. Consequently, each group constitutes a half of all PPTs. Odd values of $(t_{r_2} - s_{r_2})$ correspond to PPTs at level r = 3 with unequal free coefficients. Accordingly, even values of $(t_{r_2} - s_{r_2})$ correspond to PPTs with equal free coefficients. Let us assume that the same is true for an initial PPT with equal free coefficients at a greater level r, $r \geq 2$. The presentation for all possible parity combinations of t_r and s_r at level (r+1) is shown in Table 3 for one generic PPT with equal free coefficients.

It follows from Table 3 that the number of PPTs with equal and unequal free coefficients is the same, and is equal to 1/2 of quantity of all PPTs. Unequal free coefficients correspond to odd values of $(t_r - s_r)$, while even values $(t_r - s_r)$ correspond to PPTs with equal free coefficients. So, we obtained the same results as for $r = 2$. Since the rest of initial PPTs have the same form (in all of them free coefficients are equal), depending on the parity of $(t_r - s_r)$, they also produce a half of PPTs with equal free coefficients, and a half with unequal ones. According to principle of mathematical induction, this means that the found properties are valid for any presentation level $r \geq 2$. This proves the Lemma.

Table 3. Presentation with a factor 2^r for a PPT with equal free coefficients.

	0	1	2	3	4
1	t_r s_r	$2t_{r+1}$ $2s_{r+1}+1$	$2t_{r+1}+1$ $2s_{r+1}$	$2t_{r+1}$ $2s_{r+1}$	$2t_{r+1}+1$ $2s_{r+1}+1$
2	$2^r t_r+v_i$ $2^r s_r+v_i$	$2^{r+1}t_{r+1}+v_i$ $2^{r+1}s_{r+1}+2^r+v_i$	$2^{r+1}t_{r+1}+2^r+v_i$ $2^{r+1}s_{r+1}+v_i$	$2^{r+1}t_{r+1}+v_i$ $2^{r+1}s_{r+1}+v_i$	$2^{r+1}t_{r+1}+2^r+v_i$ $2^{r+1}s_{r+1}+2^r+v_i$

Corollary 3: *Consider successive presentations of pairs of odd integers by PPTs having a factor of* 2^r*, which use initial PPTs with equal free coefficients from the previous level, beginning with one PPT. Then, the number of initial PPTs at level r is equal to* 2^{r-1}.

Proof: For a factor of two, we have one PPT; for a factor of 2^2 there are two PPTs with equal free coefficients (Table 1); for a factor of 2^3 there are 2^2 such PPTs (Table 2), and so forth. The total number of PPTs increases by four times for the next presentation level (since each initial PPT produces four new pairs, one per parity combination of t_r, s_r). From this amount, a half of PPTs correspond to PPTs with equal free coefficients, according to Lemma 7. The value of 2^{r-1} reflects on the fact that at each presentation level the number of PPTs with equal free coefficients doubles. This proves the Corollary.

Corollary 4: *For* $r \le r_t = N(\mu+1)$ *, the fraction of PPTs, for which equation (8) has no solution at a presentation level r, is equal to*

$$f_r = (1/2)^{r-1} \tag{23}$$

Proof: It was shown in Lemma 6 that in this case (13) has no solution for PPTs with unequal free coefficients, while, according to Lemma 7, these PPTs constitute half of all PPTs at a given presentation level. Thus, (23) is true for $r = 2$. Let us assume that Lemma is valid for the value of $r > 2$. According to Lemma 6, for $r \le r_t$, the corresponding equations have no solution for PPTs with unequal free coefficients, so that initial PPTs for the next level are always PPTs with equal free coefficients. Then, the fraction f_{ru} of PPTs, for which solution is uncertain, is the same, as the fraction of "no solution" PPTs, that is $f_{ru} = (1/2)^{r-1}$. This fraction contains initial PPTs for the presentation level (*r*+1). At this level, all PPTs are again divided into two equal groups of "no solution" and "uncertain" PPTs, so that the "no solution" fraction is

$$f_{r+1} = f_{ru} \times (1/2) = (1/2)^{r-1}/2 = (1/2)^r ,$$

which is formula (23) for the level (*r*+1). According to principle of mathematical induction, this means validity of (23). This proves the Corollary.

Lemma 8: *At each next presentation level (r+1), the number of PPTs, corresponding to odd and even values of* $(t_r - s_r)$ *, are equal.*

Proof: Suppose we have p_{r+1} initial PPTs at a presentation level (*r*+1). Each initial PPT produces four PPTs at level (*r*+1), one PPT per each possible parity combination of terms t_r, s_r, listed in the first row of Table 3. These parity combinations do not depend, whether the initial PPTs have equal or unequal free terms, and also do not depend on the value of *r* compared to r_t. Two of these parity combinations (in cells (1,1), (1,2) in Table 3) produce odd values of $(t_r - s_r)$, namely when t_r, s_r are equal to $[2t_{r+1}, 2s_{r+1}+1]$, $[2t_{r+1}+1, 2s_{r+1}]$. Two other combinations, in cells (1,3), (1,4), produce even values of $(t_r - s_r)$ for PPTs $[2t_{r+1}, 2s_{r+1}]$, $[2t_{r+1}+1, 2s_{r+1}+1]$. So, the number of PPTs, for which $(t_r - s_r)$ is odd is equal to $2p_{r+1}$. The number of PPTs, for which $(t_r - s_r)$ is even, is also $2p_{r+1}$. So, quantities of PPTs, corresponding to odd and even values of $(t_r - s_r)$, are equal. This proves the Lemma.

Note: At the presentation level (*r*+1), odd values $(t_r - s_r)$ cannot be zero, given the presentation of t_r and s_r through t_{r+1} and s_{r+1} in Table 3. Even values of $(t_r - s_r)$ can be zero. However, from the perspective of existence of a solution, such a zero term can be transformed to a non-zero even term (such a transition is addressed by Lemma 9).

4.6. Finding fraction of "no solution" PPTs for presentation levels with $r \le r_t = N(\mu+1)$

We found so far that for $r \le r_t = N(\mu+1)$ the following is true:

(a) Initial PPTs with equal free coefficients, taken from level *r*, produce equal number of PPTs with equal and unequal free coefficients at a presentation level (*r*+1), Lemma 7;

(b) Corresponding to PPTs equations have no solution for PPTs with unequal free coefficients, while solution is uncertain for PPTs with equal free coefficients, Lemma 6;

(c) Each presentation level adds a "no solution" fraction of PPTs equal to $f_r = (1/2)^{r-1}$;

(d) Sets of pairs of odd integers, defined by "no solution" PPTs, do not intersect. This follows from Lemma 3, since each PPT of the next level defines unique set of pairs of odd integers compared to sets defined by other PPTs of the same level.

So, each previous level supplies to the next presentation level "uncertain" PPTs, which constitutes half of all PPTs of the previous level. These initial PPTs have equal free coefficients. This allows finding a "no solution" fraction of PPTs from successive presentations with a factor of 2^r. Since each level adds 1/2 of PPTs to a "no solution" fraction, the total such fraction F_r is equal to a sum of geometrical progression with a common ratio $q = 1/2$, and the first term $f_2 = 1/2$ (the "no solution" fraction at level *r*=2). Fig. 1 illustrates this consideration.

So, we can write

$$F_r = \sum_{i=2}^{r} f_i = f_2 \sum_{i=2}^{r} q^{i-2} = f_2 (1-q^{r-1})/(1-q) \qquad (24)$$

For example, for *r*=5, $F_r = 15/16$. Note that if such a progression is valid to infinity, the total fraction in the limit would be

$$\lim_{r\to\infty} F_r = f_2/(1-q) = (1/2)/(1/2) = 1 \qquad (25)$$

(Here, the limit is understood as an ordinary Cauchy's limit.) The obtained limit of one would mean that the union of all "no solution" PPTs converges to initial PPT $[(2k+1),(2p+1)]$, from which the presentation of pairs of odd integer numbers with a factor of 2^r began, and whose fraction was taken as a reference value of one. However, in order to realize such considerations, one needs to confirm that such a progression is true for $r > r_t = N(\mu+1)$.

4.7. Transcending the threshold level $r = r_t$

Presentation level $(r_t + 1)$

Table 4 shows PPTs for level $(r_t + 1)$. The number of initial PPTs is defined by Corollary 3, and is equal to 2^{r_t} for this level. For PPTs with *equal* free coefficients (columns 3 and 4 in Table 4), (14) transform to

$$(t_{r_t+1} - s_{r_t+1}) A_{r_t+1,ij} = m_1^N / 2 \qquad (26)$$

The right part of (26) is rational (m_1 is an odd number). The left part is an integer. So, (26) has no solution for PPTs with equal free coefficients (and, consequently, for even $(t_{r_t} - s_{r_t})$, according to Lemmas 7 and 8). When $(t_{r_t} - s_{r_t}) = 0$, the left part is zero, while the right part is rational. So, (26) has no solution too. This group of PPTs constitutes 1/2 of all PPTs (Lemma 8), so that the common ratio remains equal to 1/2, and formula (24) stays valid.

Table 4. PPTs with a factor of 2^{r_t+1}. It is assumed that $r = r_t$.

	0	1	2	3	4
	t_r s_r	$2t_{r+1}$ $2s_{r+1}+1$	$2t_{r+1}+1$ $2s_{r+1}$	$2t_{r+1}$ $2s_{r+1}$	$2t_{r+1}+1$ $2s_{r+1}+1$
1	$2^r t_r + v_{r1}$ $2^r s_r + v_{r1}$	$2^{r+1} t_{r+1} + v_{r1}$ $2^{r+1} s_{r+1} + 2^r + v_{r1}$	$2^{r+1} t_{r+1} + 2^r + v_{r1}$ $2^{r+1} s_{r+1} + v_{r1}$	$2^{r+1} t_{r+1} + v_{r1}$ $2^{r+1} s_{r+1} + v_{r1}$	$2^{r+1} t_{r+1} + 2^r + v_{r1}$ $2^{r+1} s_{r+1} + 2^r + v_{r1}$
2	$2^r t_r + v_{r2}$ $2^r s_r + v_{r2}$	$2^{r+1} t_{r+1} + v_{r2}$ $2^{r+1} s_{r+1} + 2^r + v_{r2}$	$2^{r+1} t_{r+1} + 2^r + v_{r2}$ $2^{r+1} s_{r+1} + v_{r2}$	$2^{r+1} t_{r+1} + v_{r2}$ $2^{r+1} s_{r+1} + v_{r2}$	$2^{r+1} t_{r+1} + 2^r + v_{r2}$ $2^{r+1} s_{r+1} + 2^r + v_{r2}$
…					
2^r	$2^r t_r + v_{rR}$ $2^r s_r + v_{rR}$	$2^{r+1} t_{r+1} + v_{rR}$ $2^{r+1} s_{r+1} + 2^r + v_{rR}$	$2^{r+1} t_{r+1} + 2^r + v_{rR}$ $2^{r+1} s_{r+1} + v_{rR}$	$2^{r+1} t_{r+1} + v_{rR}$ $2^{r+1} s_{r+1} + v_{rR}$	$2^{r+1} t_{r+1} + 2^r + v_{rR}$ $2^{r+1} s_{r+1} + 2^r + v_{rR}$

For PPTs with unequal free coefficients (and consequently odd $(t_{r_t} - s_{r_t})$, Lemma 8), (14) transforms to

$$(t_{r_t+1} - s_{r_t+1})A_{r_t+1,j} = A_{r_t+1,j}c/2^{r_t+1} + m_1^N/2 \quad (27)$$

The right part can be rational, an integer or zero. Since the sums $A_{r+1,j}$ are all odd, parity of the left part in (27) is defined by the term $(t_{r_t+1} - s_{r_t+1})$, which can be odd, even or zero. So, solution of (27) for odd $(t_{r_t} - s_{r_t})$ is uncertain, and such PPTs should be used as initial PPTs for the next presentation level $(r_t + 2)$. As it was mentioned (a note after Lemma 8), for odd $(t_{r_t} - s_{r_t})$, the term $(t_{r_t+1} - s_{r_t+1}) \neq 0$.

Recall that before the level $(r_t + 1)$, PPTs with *unequal* free coefficients had no solution, while (26) has no solution for *even* $(t_{r_t+1} - s_{r_t+1})$, corresponding to PPTs with *equal* free coefficients. In this regard, the level $(r_t + 1)$ reverses the groups of PPTs. The "uncertain" group of PPTs is now composed of PPTs with *unequal* free coefficients (and accordingly with *odd* $(t_{r_t} - s_{r_t})$). These PPTs (in columns 1 and 2 in Table 4) should be used as initial PPTs at the next presentation level $(r_t + 2)$.

Transition in the presentation level $(r_t + 2)$

Level $(r_t + 1)$ supplied initial PPTs with unequal free coefficients. This means that we do not have anymore distinct groups with equal and unequal free coefficients at level $(r_t + 2)$, as before, since the initial PPTs with unequal free coefficients produce mostly PPTs with unequal free coefficients, with occasional inclusion of PPTs with equal ones. Previously, we have seen that the parity of parameter $(t_r - s_r)$ defined the absence or uncertainty of solution. However, beginning from level $(r_t + 2)$, this parameter lost association with groups of PPTs with equal and unequal free coefficients. This is due to the fact that the right part of equation (27) can be an integer, a rational number, or zero *per PPT basis*, and so we should consider the use of parameter $(t_r - s_r)$ this way. We will still have a half of "no solution" and a half of "uncertain" PPTs, but only for a block of four PPTs, corresponding to each initial PPT. This is the assembly of such "uncertain" PPTs from each block, which goes to the next level (Fig. 1). Table 5 shows PPTs for level $(r_t + 2)$.

When $r=(r_t+2)$, (14) transforms to

$$(t_{r_t+2} - s_{r_t+2})A_{r+2,ij} = A_{r_t+2,ij}c/2^{r_t+2} + m_1^N/4 \quad (28)$$

where index '*ij*' denotes the cell number. The right part of (28) can be rational, an integer, or zero. When the left part is an integer (the case, when it's zero, will be considered later), (28) has no solution for any $(t_{r_t+2} - s_{r_t+2})$ for the rational or zero right part, and, consequently, this branch is completed. (Compared to continuing branches, the completed branch delivers *double* fraction of PPTs, for which (8) has no solution, since in this case two equal fractions of PPTs compose one "no solution" fraction.) If the right part is an integer, (28) has no solution when $(t_{r_t+2} - s_{r_t+2})$ has the opposite parity, and the solution is uncertain for another parity of $(t_{r_t+2} - s_{r_t+2})$. The number of combinations of parameters t_{r_t+2} and s_{r_t+2}, corresponding to each parity, is equal to two from four in this case, and so we still have equal division between the "no solution" and "uncertain" PPTs. However, at this level, we have no distinction between the odd and even values of t_{r_t+2} and s_{r_t+2} in the same way, as before, when there was an association with equal and unequal free coefficients. Such distinction can be done *only* at the next presentation level $(r_t + 3)$. All PPTs at level $r_t + 2$ correspond to "uncertain" equations, except for the cases when the PPT's branch is completed.

Table 5. PPTs with a factor of 2^{r_t+2}, obtained from initial PPTs in Table 4, for which $(t_r - s_r)$ is odd. First two rows correspond to cells (1,1), (1,2) in Table 4. It is assumed that $r = r_t$.

	0	1	2
	t_{r+1} s_{r+1}	$2t_{r+2}$ $2s_{r+2}+1$	$2t_{r+2}+1$ $2s_{r+2}$
1	$2^{r+1}t_{r+1}+v_{r1}$ $2^{r+1}s_{r+1}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$
2	$2^{r+1}t_{r+1}+2^r+v_{r1}$ $2^{r+1}s_{r+1}+v_{r1}$	$2^{r+2}t_{r+2}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+v_{r1}$
. . .	. . .	. . .	. . .
2^{r+1}	. . .	. . .	. . .

Table 5 continued

3	4
$2t_{r+2}$ $2s_{r+2}$	$2t_{r+2}+1$ $2s_{r+2}+1$
$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$
$2^{r+2}t_{r+2}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+v_{r1}$
. . .	. . .
. . .	. . .

The case of $(t_{r_t+2} - s_{r_t+2}) = 0$ is also an "uncertain" one, since there is a possibility that two terms in the right part are equal in absolute values and have the opposite algebraic signs.

Note that values $A_{r_t+2,ij}$ are different, so that the right parts of corresponding equations, transformed to a form (14), may have dissimilar parities (as well as may be rational or zeros) for different PPTs. (The right part can be an integer, provided $c \neq 0$ in (14), otherwise the right part is equal to $m_1^N / 2^{r-r_t}$, which is always rational for $r > r_t$, so that such a branch is completed.) This is why, starting from this level, one should consider each PPT *separately* (Fig. 1). (In fact, it is possible to show that at level $(r_t + 2)$, when $c \neq 0$, integer right parts of these equations have the same parity. However, this is not necessarily true for the next levels, so we use the same generic approach for this level and above.)

With regard to accumulation of a total "no solution" fraction, we have the same common ratio of 1/2, although it is obtained differently - not per group, as previously, but per PPT, and then such "per PPT" fractions are summed up, in order to obtain the total "no solution" fraction. We will consider this assembling process in detail later.

So, we found that the corresponding equations for PPTs in both groups (meaning groups of PPTs, having either even or odd values of $(t_{r_t+2} - s_{r_t+2})$) converge to equations, which have no solution for one parity of $(t_{r_t+2} - s_{r_t+2})$, and accordingly for one half of PPTs (according to Lemma 8), while solution is uncertain for the other parity, corresponding to the second half of PPTs. So, the common ratio for a geometric progression, defining fractions of "no solution" PPTs, will remain equal to 1/2. However, because we can specify particular PPTs, corresponding to odd or even $(t_{r_t+2} - s_{r_t+2})$, at the next level only, this common ratio accordingly should be assigned to a presentation level, where such a specification actually happens; in this case, this is the next level $(r_t + 3)$. At level $r_t + 2$, all equations, corresponding to initial PPTs, have the same form (14), and consequently, the same "uncertain" status. All PPTs (except for completed ones) are "uncertain" PPTs.

Presentation level $(r_t + 3)$

We will need the following Lemma to address zero values of $(t_r - s_r) = 0$ in equation (14). Note that $(t_r - s_r) = 0$ only when both parameters are equal (and, of course, have the same parity), including when both are equal to zero. When $(t_r - s_r)$ is odd (parameters have different parity), $(t_r - s_r) \neq 0$.

Lemma 9: *Equation (14), that is* $(t_r - s_r)A_r = A_r c/2^r + m_1^N / 2^{r-r_t}$, *is equivalent to equation* $(t_{1r} - s_{1r})A_r = A_r c/2^r + m_1^N / 2^{r-r_t}$ *in terms of parities of both parts, with the substitutions* $t_r = t_{1r} - 2a$ *and* $s_r = s_{1r} - 2b$, *where a and b are integers. If the second equation has no solution based on parity or rationality considerations, then the first equation also has no solution, and vice versa.*

Proof: According to the notion of presentation of odd numbers with a factor of 2^r, the terms t_r and s_r are integers, having ranges of definition $(-\infty < t_r < \infty)$ and $(-\infty < s_r < \infty)$. The only property, which is of importance with regard to such a presentation, is that these parameters should be defined on the whole set of integer numbers, in order to include *all* possible numbers, corresponding to a particular presentation; for instance, the term $(2^r t_r + v_r)$ should produce the whole set of the appropriate "stroboscopic" numbers in the range $(-\infty, \infty)$, located at the distance 2^r from each other. As long as this condition is fulfilled, that is such a set can be reproduced, we can make an equivalent substitution for parameters t_r, s_r. For instance, the substitution $t_r = t_{1r} - 2a$ is an equivalent one. Indeed, it preserves the range of definition $(-\infty < t_{1r} < \infty)$, and accordingly produces all numbers, which parameter t_r produces (only with a shift of $(-2a \times 2^r)$ for the same values of t_r and t_{1r}). However, this shift makes no difference with regard to the range of produced numbers, since our range $(-\infty, \infty)$ is infinite in both directions. On the other hand, when $(t_r - s_r) = 0$, we have $(t_{1r} - s_r) \neq 0$, and vice versa. So, for $(t_r - s_r) = 0$, such a substitution produces an equation with a non-zero left part.

Substituting $t_r = t_{1r} - 2a$ into (14), one obtains the equation

$$(t_{1r} - s_r)A_r = 2aA_r + A_r c/2^r + m_1^N / 2^{r-r_t} \quad (29)$$

When $(t_r - s_r) = 0$, we have $(t_{1r} - s_r) = 2a \neq 0$. Also, the appearance of the even term $2aA_r$ does not change the parity of the right part, nor the substitution $t_r = t_{1r} - 2a$ changes the parity of the left part (if it is not zero; if it is zero, the substitution still provides an even increment). Thus, with regard to parities, (14) and (29), indeed, are equivalent equations.

If the equivalent equation (29) has no solution, then the original equation (14) has no solution too. The proof is as follows. Let us assume that (29) has no solution, while (14) does, so that

$$(t_r - s_r)A_r = A_r c/2^r + m_1^N / 2^{r-r_t}$$

Adding $2aA_r$ to the left and right parts of this equation, one obtains an equivalent equation, which also should have a solution.

$$(t_r + 2a - s_r)A_r = 2aA_r + A_r c/2^r + m_1^N / 2^{r-r_t}$$

According to the substitution, $t_r = t_{1r} - 2a$, so that $t_r + 2a = t_{1r}$, and the obtained equation transforms to (29), which should also have a solution. However, according to our assumption, it has no solution. The obtained contradiction means that the assumption that (14) has a solution is invalid, and, in fact, it has no solution.

Similarly, we can assume that (29) has a solution, while (14) does not, and show that then (14) should have a solution, which would contradict to the initial assumption.

Although we proved the equivalency of equations with regard to their solution properties in a general case, we need such equivalency only for the case when the left part of equivalent equations is zero (because $(t_r - s_r) = 0$ or $(t_{1r} - s_r) = 0$). The proposed substitution then makes the left part of the equivalent equation a non-zero value, and the inference about the absence of solution or its uncertainty can be made based on parities of the left and right parts. Certainly, one can do an analogous substitution for s_r, or both parameters. This proves the Lemma.

Table 6. PPTs with a factor of 2^{r_t+3}. Initial PPTs are (1,1)-(1,4) from Table 5. Here, $r = r_t$.

	0	1	2
	t_{r+2} s_{r+2}	$2t_{r+3}$ $2s_{r+3}+1$	$2t_{r+3}+1$ $2s_{r+3}$
1	$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$
2	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$
3	$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+v_{r2}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r2}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$
4	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+1}+v_{r2}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r2}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$

Table 6 continued

3	4
$2t_{r+3}$ $2s_{r+3}$	$2t_{r+3}+1$ $2s_{r+3}+1$
$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$

Table 6 shows an example of PPTs for the presentation level (r_t+3). Four initial PPTs are from cells (1,1)-(1,4) in Table 5. If (28) has no solution for even $(t_{r_t+2} - s_{r_t+2})$, then these are PPTs (1,3), (1,4) in Table 6, which satisfy this condition. Accordingly, PPTs (1,1) and (1,2), for which $(t_{r_t+2} - s_{r_t+2})$ is odd, are "uncertain" PPTs, which should be used as initial PPTs for the next, (r_t+4), level. If, on the contrary, (28) has no solution for odd $(t_{r_t+2} - s_{r_t+2})$, then (1,1) and (1,2) are the "no solution" PPTs, while (1,3), (1,4) become "uncertain" PPTs, which should be used as initial PPTs for the next level. This way, all new PPTs, four per each initial pair, are divided into two halves as before, so that the common ratio of geometrical progression remains equal to 1/2. The case $(t_{r_t+2} - s_{r_t+2}) = 0$ is addressed by Lemma 9 through equivalent equations.

In the same way, as we considered one PPT above, we should consider the rest of initial PPTs in Table 6 and find out, which two PPTs should be used as initial ones for the next level. Then, the same procedure should be repeated for each initial PPT at level $(r_t + 3)$.

Then, the cycle is repeated for the next two levels $(r_t + 4)$ and $(r_t + 5)$, and so forth, to infinity, since there are no anymore threshold values of *r*, at which the right part could change the parity (if it's an integer), and the corresponding equations - their form and properties. The following Lemma generalizes the discovered order.

Lemma 10: *From the presentation level $(r_t + 2)$, the "no solution" fraction of PPTs is accumulated across two sequential levels, and then the pattern repeats for each two successive levels, to infinity. Some branches can be completed at levels $(r_t + 2L)$, where $L = 1,2,..$, but otherwise such levels provide no explicit division into the "no solution" and "uncertain" groups. Except for the PPTs, corresponding to completed branches, PPTs from such levels become initial "uncertain" PPTs for the next presentation levels $(r_t + 2L + 1)$, $L = 1,2,..$,. at which all new PPTs are divided into the "no solution" and "uncertain" groups (according to odd or even parity of $(t_r - s_r)$ in equation (14)). The "uncertain" PPTs become initial PPTs for the next presentation level, and the two-level cycle repeats to infinity.*

Proof: Previously, we have seen that the Lemma is true for the paired levels $(r_t + 2)$ and $(r_t + 3)$. Let us assume that Lemma is true for the $(r_t + d - 1)$ level, which then supplies initial "uncertain" PPTs for the next level $(r_t + d)$. We need to prove that Lemma is true for the next two levels $(r_t + d)$ and $(r_t + d + 1)$. Initial PPTs may have equal and unequal free coefficients.

Let us consider an equation for a PPT with free coefficients *v* and *w*.

$$(2^{r_t+d} t_{r_t+d} + v)^N - (2^{r_t+d} s_{r_t+d} + w)^N = 2^{N(\mu+1)} m_1^N \qquad (30)$$

where $d \geq 2$.

According to Lemma 4, it can be transformed to an equation

$$(t_{r_t+d} - s_{r_t+d}) A_{r_t+d} = A_{r_t+d} c / 2^{r_t+d} + m_1^N / 2^d \qquad (31)$$

where $A_{r_t+d} = \sum_{i=0}^{N-1} (2^{r_t+d} t_{r_t+d} + v)^{N-1-i} (2^{r_t+d} s_{r_t+d} + w)^i$, $N = 2n + 1$.

The right part of (31) can be an integer, rational or zero. The left part is an integer (if $(t_{r_t+d} - s_{r_t+d}) = 0$, the left part can be transformed to an integer, using Lemma 9). When the right part is rational, (31) has no solution for any t_{r_t+d} and s_{r_t+d}, and the branch is completed. If the right part is even or odd, (31) has no solution when $(t_{r_t+d} - s_{r_t+d})$ has the opposite parity. Solution is uncertain for the other parity of $(t_{r_t+d} - s_{r_t+d})$, since both parts of (31) have the same parity in this case. However, at this level, we cannot specify particular parity of $(t_{r_t+d} - s_{r_t+d})$, which should be done at the next presentation level $(r_t + d + 1)$. When *c* = 0, (31) has no solution, since the right part is a rational number, while the left part is an integer or zero, and so the branch is completed.

Even if the branch, corresponding to some PPT, is completed, we still can assume that it is "uncertain", and use it as an initial PPT at the next presentation level. There, the new PPTs, corresponding to this initial one, are then divided into the "no solution" and "uncertain" groups. The fraction of the former goes to the total "no solution" fraction, while the latter is used as initial PPTs for the next level, besides other uncertain PPTs. (Such an arrangement, without completed branches, is more convenient for calculation of the total "no solution" fraction.)

Table 7. New PPTs for the initial PPT $[\,2^{r_t+d}t_{r_t+d}+v\,,\;2^{r_t+d}s_{r_t+d}+w\,]$ at the presentation level (r_t+d+1) with a factor of 2^{r_t+d+1}.

	0	1	2	3	4
0	t_{r_t+d} s_{r_t+d}	$2t_{r_t+d+1}$ $2s_{r_t+d+1}+1$	$2t_{r_t+d+1}+1$ $2s_{r_t+d+1}$	$2t_{r+3}$ $2s_{r+3}$	$2t_{r+3}+1$ $2s_{r+3}+1$
1	$2^{r_t+d}t_{r_t+d}+v$ $2^{r_t+d}s_{r_t+d}+w$	$2^{r_t+d+1}t_{r_t+d+1}+v$ $2^{r_t+d+1}s_{r_t+d+1}+2^{r_t+d}+w$	$2^{r_t+d+1}t_{r_t+d+1}+2^{r_t+d}+v$ $2^{r_t+d+1}s_{r_t+d+1}+w$	$2^{r_t+d+1}t_{r_t+d+1}+v$ $2^{r_t+d+1}s_{r_t+d+1}+w$	$2^{r_t+d+1}t_{r_t+d+1}+2^{r_t+d}+v$ $2^{r_t+d+1}s_{r_t+d+1}+2^{r_t+d}+w$

Table 7 shows new PPTs for the next presentation level for the initial PPT from (30). At this level, we can choose the needed parities of PPT's terms t_{r_t+d}, s_{r_t+d}, expressed through t_{r_t+d+1}, s_{r_t+d+1}, in order for (31) to have no solution. For instance, if (31) has no solution for even $(t_{r_t+d}-s_{r_t+d})$, then the "no solution" PPT are (1,3), (1,4). Accordingly, solution is uncertain for PPTs (1,1), (1,2), since both parts of (31) have the same parity in this case. Consequently, these PPTs should be used as initial "uncertain" ones for the next presentation level.

We can see from Table 7 that when a PPT of an actually completed branch is used as an "uncertain" PPT for the next level, it produces no new PPTs with some specific features, which could prevent their corresponding equations to be transformed into a form (31). We still obtain PPTs, satisfying conditions of Lemma 4, to which the same equation (14) is applicable. For instance, when $v = w$, then $c = 0$ in (31), and so the branch is completed. However, if we use it as an initial PPT for the next presentation level (r_t+d+1), then we are free to choose new PPTs, corresponding to either even or odd values of $(t_{r_t+d}-s_{r_t+d})$, since the corresponding equations have no solution for both scenarios. Then, the PPTs with the opposite parity $(t_{r_t+d}-s_{r_t+d})$ will proceed to the next level as uncertain initial PPTs. As before, such a division produces two equal groups of PPTs, and so the common ratio of the geometrical progression remains equal to 1/2.

So, with the assumption that Lemma is true for the previous level, we confirmed the same pattern earlier discovered for the coupled levels $[(r_t+2),(r_t+3)]$. According to principle of mathematical induction, this means that Lemma is true for any $d \ge 2$. This proves the Lemma.

In this Lemma, we also studied the useful property, considering completed branches as non-completed ones. This property is formulated below as a Corollary.

Corollary 5: *PPTs, corresponding to completed branches, can be considered as regular "uncertain" PPTs, which can be passed to the next level as initial PPTs, so that such a branch is actually assigned a non-completed status.*

Lemma 11: *At presentation levels above (r_t+1), and in the absence of completed branches, the number of PPTs in "no solution" and "uncertain" groups are equal, when such a division takes place.*

Proof: According to Lemma 10 and Corollary 5, all PPTs, both regular ones, with "no solution" and "uncertain" components, and the PPTs, which could be completed, but continue to participate in the next levels as non-completed PPTs, can be presented in a form of Table 7. The solution properties of equations, corresponding to PPTs in Table 7, are defined by equation (14), or more particular, by equations in a form (31), whose solution properties depend on the term $(t_{r_t+d}-s_{r_t+d})$. (Unless the right part is rational, in which case equation has no solution for all parities, and the branch is completed. However, according to Corollary 5, we can still consider such PPT as a regular non-completed one.)

The division of four PPTs into two equal "no solution" and "uncertain" groups is based solely on the parity of $(t_{r_t+d} - s_{r_t+d})$, as Lemma 10 showed, with one parity corresponding to a "no solution" group, and with the opposite parity corresponding to "uncertain" group. The number of PPTs, corresponding to one parity, is therefore equal to 2π, where π is the number of initial PPTs, number two is the number of parity combinations of t_{r_t+d}, s_{r_t+d}, producing the same parity of $(t_{r_t+d} - s_{r_t+d})$, see Table 7. For the opposite parity of $(t_{r_t+d} - s_{r_t+d})$, the number of produced PPTs is also 2π. Thus, the number of PPTs in "no solution" and "uncertain" groups is the same. This proves the Lemma.

4.8. Calculating the total "no solution" fraction

Using Corollary 5, we consider all levels as if they have no completed branches. Then, according to Lemmas 7 and 8, until the level $(r_t + 2)$, all levels have two equal groups of PPTs. One corresponds to a "no solution" fraction, and the other to "uncertain" fraction, so that the common ratio q = 1/2. Substituting these values into (24), one obtains

$$F_{r_t+1} = f_2(1-q^{r-1})/(1-q) = 1/2(1-(1/2)^{r_t+1-1})/(1/2) = 1-(1/2)^{r_t} \qquad (32)$$

The "no solution" fraction for the level $(r_t + 1)$ is defined by (23) as follows (the last term of a geometrical progression), taking into account that $f_2 = 1/2$.

$$f_{r_t+1} = f_2 q^{r_t+1-2} = (1/2)^{r_t} \qquad (33)$$

Since in the absence of completed branches the "no solution" and "uncertain" fractions are equal, according to Lemma 8, the "uncertain" fraction of pairs, which is passed to the level $(r_t + 2)$, is the same as the "no solution" fraction (33). This "uncertain" fraction, according to Lemma 11, is equally divided into "no solution" and "uncertain" fractions at each second level, beginning from level $(r_t + 3)$, so that the first term of the geometrical progression, representing the "no solution" fraction of two following coupled levels, is

$$f_{r_t+3} = f_{r_t+1} \times (1/2) \qquad (34)$$

Then, each next two levels add a half of the previous "uncertain" fraction", which is equal to "no solution" fraction. Let $D = \{2L, 2L+1\}$, $L = 1,2...$ This way, $(r_t + D)$ defines the levels' numbers for $r \geq (r_t + 2)$. Levels, at which PPTs are divided into two groups, are levels $(r_t + 3)$, $(r_t + 5)$, …, $(r_t + 2L + 1)$, so that the total "no solution" fraction, obtained by summation of "no solution" fractions of all levels above the $(r_t + 1)$ level, is equal to

$$F_{r_t+2,D} = (1/2)^{r_t}[1/2 + (1/2)^2 + (1/2)^3 + ...(1/2)^L] = (1/2)^{r_t}\sum_{i=1}^{L}(1/2)^i = (1/2)^{r_t}(1-(1/2)^L) \qquad (35)$$

when $D = 2L + 1$, and

$$F_{r_t+2,D} = (1/2)^{r_t}[1/2 + (1/2)^2 + (1/2)^3 + ...(1/2)^{L-1}] = (1/2)^{r_t}\sum_{i=1}^{L-1}(1/2)^i = (1/2)^{r_t}(1-(1/2)^{L-1})$$

when $D = 2L$. (36)

In the last case, the division into the "no solution" and "uncertain" groups did not happen yet at the first level of coupled levels, since it occurs at the second level of the couple, as it was earlier discussed. This is why the power is (L - 1), but not L.

The total "no solution" fraction, accordingly, is defined as $F_{r_t+1+D} = F_{r_t+1} + F_{r_t+2,D}$. For $D = 2L + 1$, we have

$$F_{r_t+1+D} = F_{r_t+1} + F_{r_t+2,D} = 1-(1/2)^{r_t} + (1/2)^{r_t} - (1/2)^{r_t+L} = 1-(1/2)^{r_t+L} \qquad (37)$$

It follows form (37) that in the limit

$$\lim_{L->\infty} F_{r_t+1+D} = \lim_{L->\infty}(1-(1/2)^{r_t+L}) = 1 \qquad (38)$$

The same is true for (36). So, when we consider all branches as non-completed, in the limit, the "no solution" fraction is equal to one. Of course, it may look awkward, considering completed branches as non-completed, but, as Lemma 10 and Corollary 5 showed, this is a legitimate procedure.

Accounting for completed branches. Let us assume that level *r* has *k* completed branches, to which the "no solution" fraction f_{rk} corresponds. Suppose, these branches were not completed. We can consider the PPTs, corresponding to these branches, as regular ones, with "no solution" and "uncertain" components, to infinity. In other words, we assume that there are no more completed branches in the following presentations of these *k* PPTs, to infinity. (In real situation, if there are such PPTs, we can also consider them as non-completed ones, according to Corollary 5.) In this scenario, the fraction f_{rk} would be divided equally (Lemma 11) between the "no solution" and "uncertain" fractions on each subsequent level (or on the second level in coupled levels beyond the value of $r=(r_t+1)$). So, the total "no solution" fraction, accumulated at level *L*, is as follows.

$$F_{r+L} = f_{rk}\sum_{i=1}^{L}(1/2)^i = f_{rk}[1-(1/2)^L] \qquad (39)$$

In the limit, (39) transforms to

$$\lim_{L->\infty} F_{r+D} = \lim_{L->\infty} f_{rk}[1-(1/2)^L] = f_{rk} \qquad (40)$$

So, in the limit, we obtained in (40) exactly the same "no solution" fraction, which was taken by *k* completed branches. Since, according to (38), in the scenarios with non-completed branches the total "no solution" fraction is equal to one, the result (40) means that accounting for completed branches, in the limit, produces the same "no solution" fraction of one.

4.9. Belonging of any pair of odd numbers to some "no solution" PPT

This issue is considered using two approaches. The first, which is Approach 1, is in Appendix. In the author's view, it is correct, but it can be considered as too subtle, and some people may not accept it.

Dr. A. Y. Shestopaloff suggested, and A. A. Tantsur supported the idea to present a more conventional proof, showing that any pair of odd numbers will eventually belong to some "no solution" PPT, when the presentation level *r* increases. This is Approach 2, presented below.

Let assume that a certain pair of odd numbers (*g*,*h*) is a solution of (8). Then, at each presentation level, it can be written in the form $(2^r t_{0r}+v,\ 2^r s_{0r}+w)$, presented by some PPT $[(2^r t_r+v), (2^r s_r+w)]$. In order to be a solution of (8), this pair of odd numbers should be able to pass through all successive presentation levels, to infinity, "within" (meaning, defined by) "uncertain" PPTs. Fig. 2 shows such a transition schematically. Since until level $r=r_t$ "uncertain" PPTs have to have *equal* free coefficients, our pair of numbers should be defined by such PPTs. If the transition to the next presentation level produces unequal free coefficients, then our pair is represented by a "no solution" PPT (Lemma 6 and Fig. 2). For instance, the pair of numbers $(2^3\times36+5,\ 2^3\times7+5)$ is presented at the next level as $(2^4\times18+5,\ 2^4\times3+2^3+5)$, so that the PPT $[2^4 t_4+5,\ 2^4 s_4+13]$, which defines this pair, has unequal free coefficients. If such a transformation happens before the threshold value r_t, then this is a "no solution" PPT, and consequently our pair (*g*,*h*), contrary to the assumption, is not a solution of (8). However, if $r_t=3$, then the presentation $r=4$ corresponds to r_t+1, for which "uncertain" PPTs are the ones with *unequal* free coefficients, which means that our pair preserves the "uncertain" status, and thus the possibility still to be a solution of (8).

The inequality of free coefficients in PPT at the level r_t+1 could come only as a result of appearance of an additional summand 2^r in the presentation of one number, while the presentation form for the second number should remain the same. Indeed, the transition from, say, $(2^r s_r+v)$ to the next level r_t+1 can be done only in two ways: $(2^{r+1}s_{r+1}+v)$ or as $(2^{r+1}s_{r+1}+2^r+v)$. Thus, the pair of numbers that manages to pass the threshold level and remain uncertain, should have unequal free coefficients and be represented by PPTs of the form $[(2^{r+1}t_{r+1}+v),(2^{r+1}s_{r+1}+2^r+v)]$ or $[(2^{r+1}t_{r+1}+2^r+v),(2^{r+1}s_{r+1}+v)]$ (here, $r=r_t$). There are no other possibilities. The same value of v in both terms is due to the need for our pair to pass the level $r=r_t$ in uncertain status, for which both terms have to have equal free coefficients.

Now, we can formulate the following Lemma.

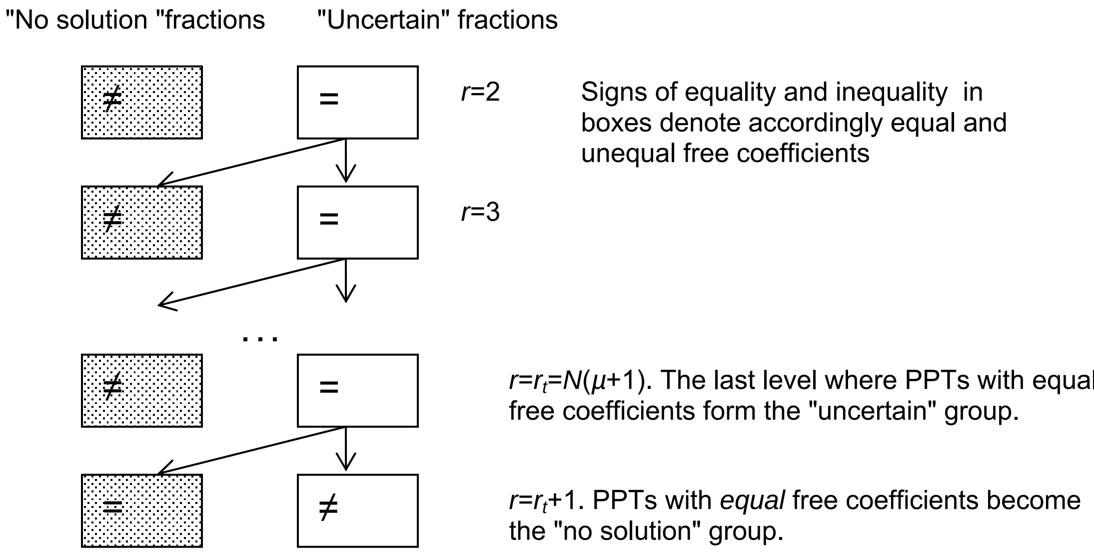

Fig. 2. Transition of pairs of odd numbers through presentation levels. Before the level $r=r_t+1$, PPTs with equal free coefficients are composing the "uncertain" group. At level $r=r_t+1$, PPTs with equal free coefficients acquire the "no solution" status.

Lemma 12: *With increase of presentation level, any pair of odd numbers will eventually belong to a "no solution" PPT, and this branch will be a completed one.*

Proof: Let us consider equation (8) transformed to (9), which accordingly can be transformed to (14). We assume that the pair (g,h) is a solution of these equations for the fixed power of N. According to Lemmas 4 and 5, coefficient c in equation (14) is defined as $c=-\sum_{j=1}^{r-1}2^jK_j$ for the presentation level r. Here, K_j may have different algebraic signs. According to Lemma 1, we can represent c also as $c=-\sum_{i=r_t}^{r-1}2^iK_i$, where $K_i=\{0,1\}$ for $c > 0$, and $K_i=\{-1,0\}$ for $c < 0$, $1\le i\le r-1$. However, in our case, for equations corresponding to PPTs at levels $r\ge r_t+1$, the smallest value of index i is $i=r_t$ (see preliminary discussion), so that $c=-\sum_{i=r_t}^{r-1}2^iK_i=-2^{r_t}\sum_{i=r_t}^{r-1}2^{i-r_t}K_i$. Substituting this value into the equation (14), one obtains.

$$(t_r-s_r)A_r=(-A_r\sum_{i=r_t}^{r-1}2^{i-r_t}K_i)/2^{r-r_t}+m_1^N/2^{r-r_t} \qquad (41)$$

According to Lemma 4, $A_r=\sum_{i=0}^{N-1}(2^rt_r+v)^{N-1-i}(2^rs_r+w)^i$. However, in our case, (2^rt_r+v) and (2^rs_r+w) are numbers of our pair, which remain the same at each presentation level, and so the value of A_r does not depend on *r*. Let denote it as $A = A_r$. Since the numbers of our pair are fixed values, the right part of the original equation (9), which is $2^{N(\mu+1)} m_1^N$, is also a fixed value, and so is the value of m_1^N.

We want to show that the range $[R_{\min}, R_{\max}]$ of possible values of the right part of equation (41) is finite for all $r \to \infty$, so that (a) the number of integers N_R in this interval $[R_{\min}, R_{\max}]$ is finite. Suppose we will manage to prove (b) the monotonic change of the right part when *r* increases. Then, that would mean that the right part of (41) cannot be equal to the same integer for different values of *r* more than one time (assuming the right part has integer values only). Since $r \to \infty$, the number *r* of values of the right part of (41) also goes to infinity (since each value of *r* corresponds to one value of the right part), while the number of integers N_R within the interval $[R_{\min}, R_{\max}]$ is finite. Then, such a value of r_b always exists that for all $r > r_b$ we have $r > N_R$. Then, there will be not enough integers in this interval to correspond to each value of the right part, and so some of them have to be *rational*. The left part of (41) is an integer, and so the equation has no solution in this case. Thus, the pair (*g*,*h*), supposed to be a solution, will always eventually belong to a "no solution" branch. That would mean that, contrary to the assumption, this pair is not a solution. Obviously, the same will be true for any pair of odd numbers belonging to the same presentation branch, since only values of *A* and m_1^N will be different for different numbers. Note that such a branch will be entirely a "no solution" one, and consequently will be closed.

The above considerations remain valid, if the right part of (41) as a function of *r* has finite number of extremums (see Fig. 3), since in this case the number of integer values, the right part of (41) can take, also will be finite. This consideration should be added to condition (b).

Since $|K_i| \le 1$, we can write

$$\left|\sum_{i=r_t}^{r-1} 2^{i-r_t} K_i\right| \le \sum_{i=r_t}^{r-1} 2^{i-r_t}$$

Calculating the sum on the right as a sum of geometrical progression with a common ratio of two, we find

$$\sum_{i=r_t}^{r-1} 2^{i-r_t} = 2^{r-1-r_t} \times 2 - 1 = 2^{r-r_t} - 1$$

So that

$$-(2^{r-r_t} - 1) \le \sum_{i=r_t}^{r-1} 2^{i-r_t} K_i \le 2^{r-r_t} - 1 \qquad (42)$$

Substituting the upper limit into the right part of (41), one obtains.

$$-A(2^{r-r_t} - 1)/2^{r-r_t} + m_1^N/2^{r-r_t} = -A + (m_1^N + A)/2^{r-r_t} \qquad (43)$$

For the lower value

$$-A(-2^{r-r_t} + 1)/2^{r-r_t} + m_1^N/2^{r-r_t} = A + (m_1^N - A)/2^{r-r_t} \qquad (44)$$

Let us denote the right part of (41) as a function $\rho(r)$. Then, based on (43) and (44), we have

$$|\rho(r)| \le \max\{|A + (m_1^N - A)/2^{r-r_t}|, |A + (m_1^N - A)/2^{r-r_t}\}|$$

When $r \to \infty$, $(m_1^N - A)/2^{r-r_t} \to 0$, so that the limits of (43) and (44) are accordingly $(-A)$ and *A*. Since we consider $r \ge r_t + 1$, the other possible boundaries are $[-A + (m_1^N + A)/2]$ and $[A + (m_1^N - A)/2]$, that is also finite. Thus, the boundaries of the interval for all possible values of the right part of (41) are

$$R_{\min} = \min\{A, -A, [-A + (m_1^N + A)/2], [A + (m_1^N - A)/2]\}$$
$$R_{\max} = \max\{A, -A, [-A + (m_1^N + A)/2], [A + (m_1^N - A)/2]\}$$

So, the interval $[R_{\min}, R_{\max}]$ is finite.

Since the right part of (43) and (44) are monotonic functions of *r*, for the maximum and minimum values $\sum_{i=r_t}^{r-1} 2^{i-r_t} K_i = 2^{r-r_t} - 1$ and $\sum_{i=r_t}^{r-1} 2^{i-r_t} K_i = -2^{r-r_t} + 1$ both conditions (a) and (b) are fulfilled. So, in this case, the pair (*g*,*h*), indeed, will eventually belong to a "no solution" PPT.

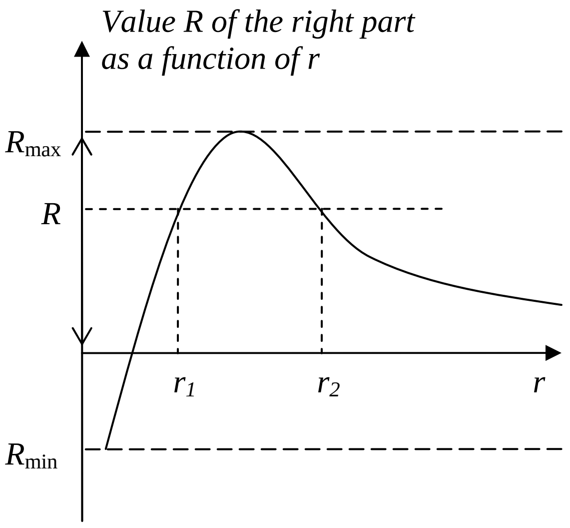

Fig. 3. A possible graph of the value of the right part as a function of *r*. To one value of *R* from the interval $[R_{\min}, R_{\max}]$, only two values of *r* can correspond (r_1 and r_2 on the figure).

So, we found that the right part of (41) is restricted for all $r > r_t$. However, for a general case, we still have to prove that it cannot have an infinite number of integer values in the interval $[R_{\min}, R_{\max}]$ when K_i can be zero, for which we need to find the first derivate of the right part $\rho(r)$. The upper limit of the sum $\sum_{i=r_t}^{r-1} 2^{i-r_t} K_i$ is variable. However, we can substitute the upper limit by infinity, assuming $K_i = 0$ for all $i > r-1$, so that the sum transforms to $\sum_{i=r_t}^{\infty} 2^{i-r_t} K_i$. Then, we can find the first derivative as follows.

$$\frac{\partial \rho(r)}{\partial r} = \frac{\partial[(-A\sum_{i=r_t}^{\infty} 2^{i-r_t} K_i)/2^{r-r_t} + m_1^N / 2^{r-r_t}]}{\partial r} = A\sum_{i=r_t}^{\infty} 2^{i-r} K_i \ln 2 - m_1^N 2^{r_t - r} \ln 2 \qquad (45)$$

Since for all $i > r-1$ we have $K_i = 0$, it can be rewritten as

$$\frac{\partial \rho(r)}{\partial r} = \ln 2 (A\sum_{i=r_t}^{r-1} 2^{i-r} K_i \ln 2 - m_1^N 2^{r_t - r}) \qquad (46)$$

Because K_i are either all non-negative, or all are non-positive, all summands in the sum have the same algebraic sign. So, only one change of algebraic sign for the first derivative of $\rho(r)$ in (46) is possible for $r > r_t$. The number of solutions of equations, composed of the sum of exponential functions, is defined by the same Descartes' Rule of Signs, as for polynomial equations (the proof can be found in [6] and elsewhere). Thus, (46) can be equal to zero one time maximum, for all possible values of $r > r_t$. This means that on the left and on the right of possible extremum the function $\rho(r)$ changes *monotonically*. So, the same integer number *R* from the finite interval $[R_{\min}, R_{\max}]$ can correspond to only two values of *r* (Fig. 3). Although the number of possible integer values for the right part of (41) can double, compared to scenarios (43) and (44), it is still finite. So, in this case, too, such a value of r_b always exists that for all $r > r_b$ we will have $r > N_R$ (where N_R, recall, is the number of integers in the finite interval $[R_{\min}, R_{\max}]$, within which all possible values of $\rho(r)$ are located). Thus, eventually, the right part of (41) will necessarily become rational. Since the left part of (41) is an integer (if both parts of (41) are zero, the right part can be converted to an integer, according to Lemma 9), this means that (41), contrary to the assumption, has no solution for the pair of odd numbers (*g*,*h*), and, in fact, this pair belongs to an entirely "no solution" branch, which is consequently closed.

A note should be made with regard to legitimacy of finding a derivative for a discrete right part of (41) as if it is a continuous function. We can think of it as a derivative in the vicinity of integer value of *r*, until it increases to enter the realm of value of (*r*+1), in which case (45) and (46) represent a correct operation.

This completes the proof of Lemma.

So, we proved that when the presentation level *r* increases, any pair of odd numbers eventually will belong to a "no solution" branch. Thus, equation (14), and consequently equation (8), have no solution on the whole set of pairs of odd integer numbers.

4.10. Cases 2 and 3 as equivalent equations

For the case 3, we have $a = 2n+1$; $x = 2k_1 + 1$; $y = 2p_1 + 1$. Then, (1) transforms to (7).

$$(2k_1+1)^{2n+1} + (2p_1+1)^{2n+1} = (2m)^{2n+1} \qquad (47)$$

We will show the equivalency of (8) and (47) in terms of solution properties. Then, since (8), as it was found, has no solution, that would mean that (47) has no solution too.

The notion of equivalent equations. It means that for each set of input variables for one equation there is one and only one matching set of corresponding input variables for the other equation, such that the terms in both equations are the same. For instance, with regard to equations (8) and (47), defined on the set of integer numbers, their equivalency would mean that for any combination of terms $(2k+1)$, $(2p+1)$, $2m$ in (8) there is only one combination of terms $(2k_1+1)$, $(2p_1+1)$, $2m_1$ in (47), such, that $(2k+1) = (2k_1+1)$, $(2p+1) = -(2p_1+1)$, $m = m_1$, so that with such a substitution equation (8) becomes equation (47). Similarly, the substitution $(2k_1+1) = (2k+1)$, $(2p_1+1) = -(2p+1)$, $m_1 = m$ in (47) produces equation (8). It was proved that (8) has no solution in integer numbers, so that it has no solution for any combination of these terms. However, on the set of all possible pairs of odd numbers, on which both equations are defined, these are equivalent equations, as it will be shown. Then, since (8) has no solution, (47) has no solution too.

Lemma 13: *Equation (8) is equivalent to equation (47) on the set of integer numbers. If one of these equations has no solution in integer numbers, then the other equation also has no solution.*

Proof: Since the odd power does not change the algebraic sign, we can rewrite (8) as follows.

$$(2k+1)^{2n+1} + (-2p-1)^{2n+1} = (2m)^{2n+1} \qquad (48)$$

k, p and *k₁, p₁* in (8), (47) are integers defined on the range $(-\infty, +\infty)$. So, we can do a substitution $p = -p_1 - 1$.

$$(2k_1+1)^{2n+1} + (2p_1+1)^{2n+1} = (2m)^{2n+1} \qquad (49)$$

where $k = k_1$. In this transformation, the range of parameters and equations' terms remains the same, that is $(-\infty < p < \infty)$, $(-\infty < p_1 < \infty)$, and so $(-\infty < (2p+1) < \infty)$ $(-\infty < (2p_1+1) < \infty)$. Thus, equation (48), which is (8), became equation (49). The substitution $p = -p_1 - 1$ is an equivalent one, because (i) it does not change the range of the substituted parameter, neither it changes the ranges of the terms, defined by these parameters; (ii) this is a one-to-one substitution.

Similarly, we can obtain equation (8) from (47), using substitution $p_1 = -p - 1$ in (47).

$$(2k_1+1)^{2n+1} + (2(-p-1)+1)^{2n+1} = (2m)^{2n+1} \quad (50)$$

This transforms into equation (8).

$$(2k+1)^{2n+1} - (2p+1)^{2n+1} = (2m)^{2n+1} \quad (51)$$

where $k_1 = k$.

Thus, (8) and (47), indeed, are equivalent equations.

Now, we should prove that if one of these equations has no solution, then the other equation also has no solution. For that, let us assume that equation (47) has no solution, while the equivalent equation (8) has a solution for the parameters (k_0, p_0, m_0), that is

$$(2k_0+1)^{2n+1} - (2p_0+1)^{2n+1} = (2m_0)^{2n+1} \quad (52)$$

Doing an equivalent substitution $p_0 = -p_1 - 1$, one obtains

$$(2k_0+1)^{2n+1} + (2p_1+1)^{2n+1} = (2m_0)^{2n+1} \quad (53)$$

Equation (53) (which is the original equation (47)), accordingly, has a solution for the parameters (k_0, p_1, m_0). However, this contradicts to the assumption that (47) has no solution. So, the equivalent equation (8) also has no solution. Similarly, we can assume that (8) has no solution, while (47) has a solution, and find through similar contradiction that (47) has no solution.

This completes the proof of Lemma.

It follows from Lemma 13 that it is suffice to prove that only one of the equations, (8) or (47), has no solution, in order to prove that both equations have no solution. Previously, we found that (8) has no solution in integer numbers. So, according to Lemma 13, (47), which presents case 3 for equation (1), also has no solution.

5. Conclusion

We found that cases 1, 3 and 4 can be converged to case 2. We proved that the corresponding to this case equation (8) has no solution in integer numbers. This means that (1) has no solution in integer numbers, which proves FLT.

Apparently, introduced concepts and approaches can be applied to other problems of number theory.

6. Acknowledgements

The author thanks all Commenters and Reviewers, whose comments helped to improve and clarify the paper. Dr. M. J. Leamer pinpointed that cases 1 and 4 can be reduced to case 2, which was a significant advancement making cases 1 and 4 optional for the proof (their independent proofs are in [2]). Dr. A. Y. Shestopaloff advised, and A. A. Tantsur supported the idea, to add a conventional proof that any pair of odd numbers eventually will belong to a "no solution" PPT. The author indebted to A. A. Tantsur for multiple reviews, discussions and thoughtful suggestions, which much contributed to elimination of errors and improvement of the paper.

Appendix

Approach 1 for association the limit of "no solution" fractions of one with the whole set of pairs of odd integers

Here, we consider, if the limit of one of the sum of "no solution" fractions in (25) corresponds to the whole set of pairs of odd integers? First of all, let us emphasize that in this paper we consider *only deterministic* values. It is only when one uses a *stochastic* approach, we can have convergence, while the finite number of elements do not posses the properties the convergence is proving.

According to Lemma 3, the found limit of the sum represents a union of (a) non-intersecting PPTs, which (b) have no duplicate elements, (c) together represent a one-to-one transformation between adjacent presentation levels, in both directions. Consequently, the limit of the sum is equivalent to initial PPT. The last one defines the whole set of pairs of odd integers. Therefore, the limit of one of the sum also defines the whole set of pairs of odd integers

The completeness of the set of PPTs, defined by the limit of the sum (completeness means equivalence to the initial set $[(2k+1),(2p+1)]$), means that *for any pair of odd numbers there is a "no solution" PPT in this set this pair is represented by*. Since all PPTs in the sum define non-intersecting sets, this means that such a pair is unique, and cannot be defined by any other PPT at the same level of presentation. Thus, all possible pairs of odd numbers are defined by some "no solution" PPT, so that (8), representing case 2 of the general FLT equation, has no solution on the whole set of all possible pairs of odd numbers.

The objection to such an approach is linked to a rather philosophical level, how one understands a limit. Some see a limit (and we are talking, of course, about *deterministic* limit) in the frame of a famous Zeno's paradox, when Achilles tries to catch up a tortoise, but will never do. Our situation, rephrased through the paradox, should sound as follows. Suppose we have a pair of odd numbers within the "undefined" PPT. When it is divided into the "no solution" and "undefined" PPTs, our number stays in the "undefined" fraction. Regardless of how many presentation levels we ascend, our pair will always be within the "undefined" fraction of PPT. So, in the same way as Zeno concluded that Achilles would never catch up the tortoise, we should conclude that there will always be an "uncertain" fraction containing our number. Consequently, the result that the total fraction goes to one is not the proof that (8) has no integer solutions.

Another approach, which is in better agreement with a common sense, is to see the limit of infinite number of summands in the same way as the sums with finite number of terms. It is the *actual value of the limit*, which defines the obtained domain of pairs of odd numbers, but not the successive iterative values, approaching to the limit. In our case, there are no any special conditions requiring us to treat the limit differently, than the sum with finite number of summands. Since the limit produces the whole set of PPTs, defining the whole set of pairs of odd integer numbers, this means that (8) has no solution for *all* possible pairs of odd numbers, but not for the sets, corresponding successive iterations to the limit.

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