Proof of Fermat Last Theorem based on successive presentations of pairs of odd numbers

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Abstract

A proof of Fermat Last Theorem (FLT) is proposed. FLT was formulated by Fermat in 1637, and proved by A. Wiles in 1995. Here, a simpler proof is considered. It is based mostly on new concepts. The proposed methods and ideas can be used for studying other problems in number theory. The initial equation *x^n + y^n = z^n* is considered not in natural, but in integer numbers. It is subdivided into four equations based on parity of terms and their powers. Although these four cases can be converted to only one equation (case 2), cases 1 and 4 are also studied separately. The first equation is presented as a binomial expansion of its terms. The second one is considered using presentations of pairs of odd numbers with a successively increasing factor of 2^*r*. The third equation is equivalent to the second one. The fourth equation uses presentation of pairs of odd numbers with a factor of four, and transformation to the second power. All four equations have no solution in integer numbers. Thus, the original FLT equation has no solution too.

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1. Introduction

One of the reasons that FLT still attracts people is that the known solution [1], in their view, is too complicated for the problem. Earlier, an approach for analysis of Diophantine equations, and FLT equation in particular, was proposed in [2]. Here, it is presented with some additions.

2. FLT sub-equations

Let us consider an equation.

$$x^a + y^a = z^a \qquad (1)$$

*Mail *constructive, respectful* comments to user "shes" padded with a number 13^2 on server yahoo.

The power a is a natural number $a \geq 3$. Unlike in the original FLT equation, here, x, y, z belong to the set of integer numbers **Z**. Combinations with zero values are not considered as solutions. We assume that variables x, y, z have no common divisor. Indeed, if they have such a divisor d, both parts of equation can be divided by d^a, so that the new variables $x_1 = x/d$, $y_1 = y/d$, $z_1 = z/d$ will have no common divisor. We will call such a solution, without a common divisor, a *primitive solution*. From the formulas above, it is clear that any non-primitive solution can be reduced to a primitive solution by dividing by the greatest common divisor. The reverse is also true, that is any non-primitive solution can be obtained from a primitive solution by multiplying the primitive solution by a certain number. So, it is suffice to consider primitive solutions only.

Values x, y, z in (1) cannot be all even. Indeed, if this is so, this means that the solution is not primitive. By dividing it by the greatest common divisor, it can be reduced to a primitive solution. Obviously, x, y, z cannot be all odd. So, the only possible combinations left are when x and y are both odd, then z is even, or when one of the variables, x or y, is even, and the other is odd. In this case, z is odd. Thus, equation (1) can be subdivided into the following cases, which cover all permissible permutations of equation's parameters.

1. $a = 2n$; $x = 2k+1$; $y = 2p+1$. Then, z is even, $z = 2m$.
2. $a = 2n+1$; $x = 2p+1$; $y = 2m$. Then, z is odd, $z = 2k+1$.
3. $a = 2n+1$; $x = 2k+1$; $y = 2p+1$. Then, z is even, $z = 2m$.
4. $a = 2n$; $x = 2p+1$; $y = 2m$. Then, z is odd, $z = 2k+1$.

It will be shown later that case 3 is equivalent to case 2. Regarding cases 1 and 4, as Dr. M. J. Leamer noted in his comment, there is a well known way to show that considering equation (1) is equivalent (in terms of existence of solution) to the case when exponent a is represented as a product of number four and (or) odd prime numbers. Indeed, we can assume that $a = fp$, where $f \geq 1$ is a natural number, p is a product of one or more prime factors, so that p is odd. (Certainly, prime factors of a can be distributed between f and p.) We can rewrite (1) as

$$(x^f)^p + (y^f)^p = (z^f)^p$$

Then, if there is no integer solution for the odd exponent p, then there is no solution for the exponent a (since the power of an integer number, such as x^f, is an integer). When a has no prime factors, $p = 1$, f is even. Since $a \geq 3$, $f \geq 4$. When f is divisible by four, we can use a known theorem that (1) has no solution for $a = 4$, representing the terms in (1) - say, the first one, as $(x^{f/4})^4$. When f is divisible by two, but not four, that would mean that $p \geq 3$ (since $a \geq 3$), and we again can convert (1) to an equation with an odd power.

Thus, the cases 1 and 4 with even powers can be converted to cases 3 and 2 accordingly. Since case 3 is equivalent to case 2, this means that all four cases, in fact, converge to case 2. So, it is suffice to only prove that there is no integer solution for case 2. However, although considering cases 1 and 4 is optional, we will do that too for methodological reasons.

3. Case 1

Let us assume that (1) has a solution for the following terms.

$$(2k+1)^{2n} + (2p+1)^{2n} = (2m)^{2n} \tag{2}$$

Binomial expansion of the left part of (2) is as follows.

$$\left[\sum_{i=0}^{2n-2} C_i^{2n}(2k)^{2n-i} + 2n(2k) + 1\right] + \left[\sum_{i=0}^{2n-2} C_i^{2n}(2p)^{2n-i} + 2n(2p) + 1\right] = (2m)^{2n} \tag{3}$$

Transforming (3), we obtain

$$\sum_{i=0}^{2n-2} C_i^{2n}\left[(2k)^{2n-i} + (2p)^{2n-i}\right] + 4n(k+p) + 2 = (2m)^{2n} \tag{4}$$

The lowest power of terms 2*k* and 2*p* in the sum is $2n-(2n-2)=2$. So, all summands have a factor of two in a degree of two or greater. The second term has a factor of four. Let us divide both parts of (4) by two. We obtain.

$$\sum_{i=0}^{2n-2} C_i^{2n}\left[k(2k)^{2n-i-1}+p(2p)^{2n-i-1}\right]+2n(k+p)+1=m(2m)^{2n-1} \tag{5}$$

The first two summands in the left part of (5) are even. So, the left part presents the sum of two even terms and the number one. Thus, the left part is odd.

Since we consider the values of $2n \geq 4$, the power $(2n-1) \geq 3$, so that the right part is even. So, (5) presents an equality of the odd and even integer numbers, which is impossible. Thus, the initial assumption that (2) has no solution in *integer* numbers, since the parity of the right and left parts of (5) does not depend on algebraic signs of variables.

That (2) has no solution, can be proved through modulo arithmetic too. Indeed, for the left part $(2k+1)^{2n}+(2p+1)^{2n} \equiv 2(\bmod 4)$. For the right part, $(2m)^{2n} \equiv 0(\bmod 4)$.

4. Cases 2 and 3

We will need several Lemmas for these cases.

4.1. Presentation of numbers in a binary form

Lemma 1: *Each non-negative integer number n can be presented in a form*

$$n=\sum_{i=0}^{r} 2^i K_i \tag{6}$$

where $K_i=\{0,1\}$.

Proof: Effectively, this Lemma states the fact that any number can be written in a binary presentation. Paper [2] provides more detail.

From Lemma 1, the following Corollary follows.

Corollary 1: Any negative integer number *n* can be presented as $n=\sum_{i=0}^{r} 2^i B_i$, where $B_i=\{-1,0\}$.

4.2. Presentation of equation (1) for cases 2 and 3

For the case 3, we have $a=2n+1$; $x=2k_1+1$; $y=2p_1+1$. Then, (1) transforms to

$$(2k_1+1)^{2n+1}+(2p_1+1)^{2n+1}=(2m)^{2n+1} \tag{7}$$

For the case 2, the power $a=2n+1$; $x=2p+1$; $y=2m$. Then, *z* is odd, $z=2k+1$.

$$(2p+1)^{2n+1}+(2m)^{2n+1}=(2k+1)^{2n+1}$$

It can be rewritten in a form

$$(2k+1)^{2n+1}-(2p+1)^{2n+1}=(2m)^{2n+1} \tag{8}$$

We can present *m* as $m=2^{\mu} m_1$, where $\mu \geq 0$, and m_1 is an odd number. Then, (8) transforms to

$$(2k+1)^N-(2p+1)^N=2^{N(\mu+1)} m_1{}^N \tag{9}$$

where $N=2n+1$.

Note that the value $r_t=N(\mu+1)$ is a threshold one. If we divide both parts of the equation by 2^r, then for $r<r_t$ the right part is even, for $r=r_t$ it is odd, and for $r>r_t$ it is rational.

In the following, we will use a presentation of pairs of odd numbers with a factor of 2^r, where $r \geq 1$, whose properties are considered below.

4.3. Presentation of pairs of odd numbers with a factor of 2^r

Let us consider an infinite set of pairs of odd integer numbers $\{(2k+1),(2p+1)\}$, where k and p are *integers*. The set $\{(2k+1)\}$ can be presented with a factor of four as two sets $\{(4t+1)\}$ and $\{(4t+3)\}$. Similarly, the set $\{(2p+1)\}$ can be presented as sets $\{(4s+1)\}$ and $\{(4s+3)\}$. Here, t and s are integers, $(-\infty < t < \infty)$, $(-\infty < s < \infty)$. Thus, the original set $\{(2k+1),(2p+1)\}$ can be presented through four possible permutations of sets with a presentation factor of four.

Table 1 shows four possible *pairs of terms*, expressed with a factor of four. Note that such a presentation produces a *complete set* of pairs of odd integer numbers, since we considered all possible combinations of parities of k and p. (The completeness will be proved later for a general case of presentation with a factor 2^r).

We can continue presentations of pairs of odd numbers using a successively increasing factor of 2^r. Initial pairs for the next presentation level with a factor of 2^3 are pairs in cells (2,1)-(2,4). Table 2 shows the presentation with a factor of 2^3 for two pairs from cells (2,3), (2,4) in Table 1. Note that index '3' corresponds to power r=3 in a presentation factor 2^r. Such correspondence of the index to the power of two in a presentation factor will be used throughout the paper.

Table 1. All possible pairs of terms, defining odd numbers, expressed with a factor of four.

	0	1	2	3	4
1	k p	$2t_2$ $2s_2+1$	$2t_2+1$ $2s_2$	$2t_2$ $2s_2$	$2t_2+1$ $2s_2+1$
2	$2k+1$ $2p+1$	**$4t_2+1$** **$4s_2+3$**	**$4t_2+3$** **$4s_2+1$**	$4t_2+1$ $4s_2+1$	$4t_2+3$ $4s_2+3$

Table 2. Pairs of terms, expressed with a factor of 2^3, corresponding to initial pairs $[4t_2+1, 4s_2+1]$, $[4t_2+3, 4s_2+3]$.

	0	1	2	3	4
1	t_2 s_2	$2t_3$ $2s_3+1$	$2t_3+1$ $2s_3$	$2t_3$ $2s_3$	$2t_3+1$ $2s_3+1$
2	$4t_2+1$ $4s_2+1$	**$8t_3+1$** **$8s_3+5$**	**$8t_3+5$** **$8s_3+1$**	$8t_3+1$ $8s_3+1$	$8t_3+5$ $8s_3+5$
3	$4t_2+3$ $4s_2+3$	**$8t_3+3$** **$8s_3+7$**	**$8t_3+7$** **$8s_3+3$**	$8t_3+3$ $8s_3+3$	$8t_3+7$ $8s_3+7$

4.3.1. The concept of the proof

Each pair of terms in Tables 1 and 2, and in subsequent presentations, defines an *infinite* set of pairs of odd numbers. However, the number of pairs of terms at each presentation level is *finite*. All pairs of terms at every single presentation level produce the *whole set of pairs of odd numbers*.

(In the following, unless it is stated otherwise or said explicitly, the term "pair" will mean a *pair of terms*, defined through parameters t and s (k and p for the first level), while the term "pairs of odd numbers" refers to an infinite set, produced by one or more pairs of terms.)

A subset of pairs of terms from one level can be transformed to a subset of pairs of terms at another presentation level. The subset of pairs of odd numbers, associated the first subset of pairs of terms, will remain the same in such a transition. Using such transformations, we can distribute the original set of pairs of terms across different presentation levels, and vice versa (that is to transform pairs of terms from different presentation levels back to one level). Accordingly, the pairs of odd numbers, associated with pairs of terms at different presentation levels, will be associated with one presentation level again. If such transformations are unique, that is they are one-to-one transformations in both directions, then the "reversed" subset of pairs of odd numbers will be the same as the original one. The properties of a subset of pairs of odd numbers, acquired at other presentation levels (say, that (1) has no solution on this subset), certainly remain with this subset at other presentation level (because these are just the same combinations of numbers).

It will be proved later in Lemma 3 that the infinite sets, defined by pairs of terms at the same level, indeed, are unique and do not intersect. At each presentation level, equation (8) has no solution for a certain fraction of pairs (of terms). Such "no solution" fractions accumulate through subsequent presentation levels, producing a greater and greater total fraction of pairs of terms, for which (8) has no solution. In the limit, this total "no solution" fraction becomes equal to one, which would mean that (8) has no solution for *all* possible pairs of odd numbers. Why one can make such an inference, associating the total fraction with a whole set of pairs of odd numbers, will be discussed in more detail at the end of Case 2, once we get acquainted with the proof specifics.

At this point, it is very important to understand that we deal with *deterministic* objects, which are *pairs of terms* and associated with them particular values of "no solution" and "uncertain" fractions. *Absolutely no notion of probability* is involved in this proof. Finding the aforementioned limit would mean that all concrete (but not probabilistic!) values of fractions, uniquely associated with the appropriate sets of pairs of terms, are summed up to the value of one in infinity. Why this is so important? The reason is that if one would resort to *probabilistic* approach, say using the notion of asymptotic density [3,4], it could be still possible to obtain the value of one in the limit for the total asymptotic density, even if for one pair - from infinity - (8) has a solution. The only way to cope with such an issue is to use *only* deterministic countable values, and finding the required total value (in our case, this will be the total "no solution" fraction) for *all* these deterministic values.

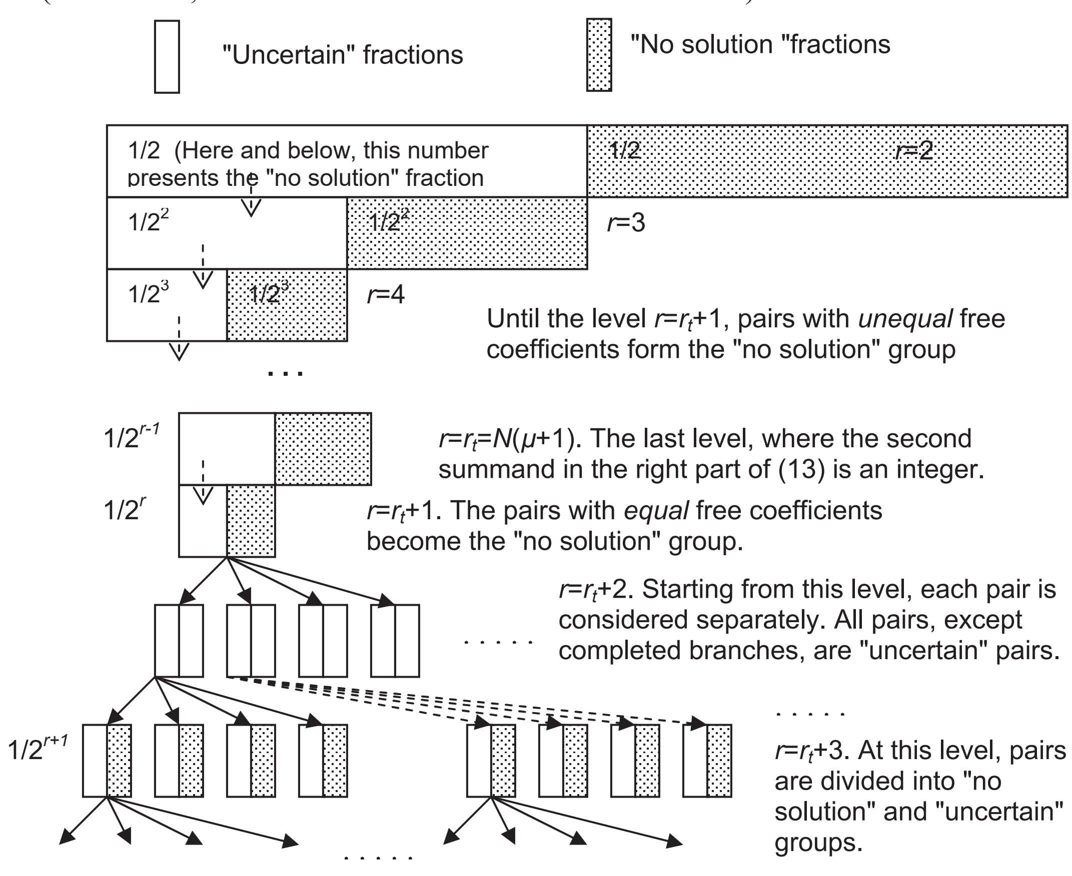

Fig. 1. Graphical presentation of how the "No solution" fraction accumulates through presentation levels, and the appropriate decrease of "Uncertain" fraction. The value of $r=r_t=N(\mu+1)$ is a threshold value, where transition begins from even right parts of equations to integer or rational ones.

We begin with the *whole set* of all possible pairs of odd integer numbers, represented by a pair of terms $[(2k+1),(2p+1)]$, *k* and *p* are integers. Equations, corresponding to a *half* of pairs of terms from this set, have no solution. Then, the half of pairs, for which equations have no solution, is set aside (the "no solution" fraction f_{2ns}). The remaining pairs compose an "uncertain" fraction, for which solution is uncertain. The "uncertain" fraction is equal to $f_{2u}=1-f_{2ns}=1/2$.

The following example illustrates the approach. (The actual algorithm is different, but the general idea is similar.) Equation (8) can be transformed as a difference of two numbers in odd powers.

$$2(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^i=(2m)^{2n+1}$$

Dividing both parts by two, one obtains

$$(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^i=m(2m)^{2n}$$

Here, the sum is odd as an odd quantity of odd numbers. If the factor $(k-p)$ is odd, then the left part is odd, while the right part is even (since $n > 0$). This means that there is no solution in this case. The value of $(k-p)$ is odd when one of the terms is odd and the other is even, which are the values of *k* and *p* in cells (1,1), (1,2) in Table 1, corresponding to pairs $[4t+1,4s+3]$ and $[4t+3,4s+1]$ with a presentation factor of 2^2. The change of algebraic signs of *k* and *p* does not change the parity of the left part. So, the result is valid for *integer* numbers *k* and *p*. When $(k-p)=0$, the left part is zero, while the right part is an integer. So, there is no solution in this case.

When $(k-p)$ is even, both parts of equation are even, and solution is uncertain. This corresponds to values of *k* and *p* in cells (1,3), (1,4) in Table 1, with corresponding pairs of terms $[4t+1,4s+1]$ and $[4t+3,4s+3]$. These "uncertain" pairs should be used as initial pairs for the next presentation level with a factor of 2^3 (Table 2).

At the presentation level with r=3, we again find that a half of pairs (the ones in bold in Table 2) correspond to a "no solution" fraction, which is found as $f_{3ns}=f_{2u}\times 1/2=1/4$. The fraction of remaining uncertain pairs is accordingly $f_{3u}=f_{2u}-f_{3ns}=1/2-1/4=1/4$. Therefore, two presentation levels produce the following total fraction of pairs of terms, for which (15) has no solution, $F_{3NS}=f_{2ns}+f_{3ns}=1/2+1/4=3/4$. The "uncertain" fraction $f_{3u}=1-3/4=1/4$, gives initial pairs for the next presentation level (with r=4), and so forth, until in infinity the "no solution" fraction accumulates to one. (The real situation with the "no solution" fractions is slightly more complicated, since such fractions can be greater than 1/2, when equations, corresponding to certain pairs of terms, have no solution for all pairs of terms, and such a branch is closed. However, the total "no solution" fraction is still equal to one in the limit.) Fig. 1 illustrates the concept in more detail.

4.3.2. Properties of presentations of pairs of odd number with a factor of 2^r

Lemma 2: *Successive presentations of odd numbers with a factor of* 2^r *cannot contain a free coefficient greater or equal to* 2^r.

Proof: Presentations of odd numbers with factors 2^2 and 2^3 satisfy this requirement. Let us assume that this is true for a presentation level *r*, that is the free coefficient *v* in a term $(2^r t_r+v)$ satisfies the condition $v<2^r$. At a presentation level $(r+1)$, this term is presented as $(2^{r+1}t_{r+1}+2^r+v)$ or $(2^{r+1}t_{r+1}+v)$. In the latter term, the condition is already fulfilled. In the first term, $2^r+v<2^r+2^r=2^{r+1}$, since $v<2^r$ is true for level *r* by assumption. So, assuming that the condition is fulfilled at the level r, we obtained that it is also fulfilled at the level $(r + 1)$. According to principle of mathematical induction, this means the validity of the assumption. This proves the Lemma.

The number of pairs of terms grows for successive *complete* presentations in a geometrical progression with a common ratio of *four*, since each initial pair produces four new pairs at the next presentation level. (Each new pair corresponds to one of the four possible parity combinations of input parameters, like t_2, s_2 in Table 2, whose parity is expressed through t_3, s_3.)

For the following, we need to prove that (a) such a presentation produces the whole set of pairs of odd numbers at each level; (b) the presentation is unique, that is two different pairs of odd numbers cannot produce the same pair of odd numbers at higher levels of presentation.

Lemma 3: *Successive presentations of pairs of odd numbers with a factor of* 2^r, $r \geq 2$, *produce the same set of pairs of odd numbers at each presentation level. Such presentations are unique, that is two different pairs of odd numbers from the previous levels cannot correspond to the same pair of odd numbers at higher presentation levels.*

Proof: The equivalency of sets of pairs of odd numbers at each presentation level r follows from the fact that each next presentation level $(r+1)$ is obtained prom the previous one through branching of each initial pair (from level r) into all four possible combinations of parities of parameters t_r and s_r, so that there are no any other possible combinations of parities. This means that any pair of terms from level r is fully represented at level $(r+1)$, although in the form of four pairs of terms. Indeed, the initial term $(2^r t_r + v)$ can be presented at level $(r+1)$ only in two forms (for even and odd values of t), that is as $2^r(2t_{r+1}) + v = 2^{r+1}t_{r+1} + v$, or $2^r(2t_{r+1} + 1) + v = 2^{r+1}t_{r+1} + 2^r + v$. Similarly, the term $(2^r s_r + w)$ can also be represented in the same two forms only. So, only four combinations of pairs of terms, containing both t and s parameters, are possible. These combinations are unique, because the combinations of free coefficients are unique, which are as follows: $[v,w]$, $[2^r + v, w]$, $[v, 2^r + w]$, $[2^r + v, 2^r + w]$. Consequently, no intersection of thus defined sets of pairs of odd numbers is possible.

The reverse is also true, that is four pairs of terms at presentation level $(r+1)$ converge to one initial pair of terms at lower level r. Indeed, two terms with parameter t converge to the same term $(2^r t_r + v)$.

$$2^{r+1}t_{r+1} + v = 2^r(2t_{r+1}) + v = 2^r t_r + v \qquad (10)$$

$$2^{r+1}t_{r+1} + 2^{r+1} + v = 2^r(2t_{r+1} + 1) + v = 2^r t_r + v \qquad (11)$$

where $t_r = \{2t_{r+1}, 2t_{r+1} + 1\}$.

Note that the same convergence to a single term can be obtained for a general case of presenting two terms at level $(r+1)$ using Lemma 1, and then transforming them to level r.

$$2^{r+1}t_{r+1} + \sum_{i=1}^{r} 2^i K_i + 1 = 2^r(2t_{r+1} + K_r) + \sum_{i=1}^{r-1} 2^i K_i + 1 = 2^r t_r + \sum_{i=1}^{r-1} 2^i K_i + 1 \qquad (12)$$

For a positive number, $K_r = \{0,1\}$, and we obtain $t_r = \{2t_{r+1}, 2t_{r+1} + 1\}$, that is the same set of integer numbers, on which t_r was defined originally. The same is true for negative numbers.

Similarly, one can convert two possible terms with parameter s at level $(r+1)$ to a single term with parameter s at level r. So, four pairs of terms at level $(r+1)$, indeed, converge to one pair of terms $[2^r t_r + v, 2^r s_r + w]$ at level r. Therefore, such transformations from level r to level $(r+1)$ and backward include all possible, while non-intersecting, pairs of terms. This means that presentations of pairs of odd numbers at these two levels are equivalent, that is for each pair of odd numbers at level r there is one and only one pair of odd numbers at level (r+1).

4.4. Datasets of integer numbers with a factor of four, symmetrical relative to zero

Lemma 4: *The dataset* $\boldsymbol{Z_1}=\{4s+1\}$*, defined on the set of integer numbers* $(-\infty<s<\infty)$ *is symmetrical to the dataset* $\boldsymbol{Z_3}=\{(4s_1+3)\}$*,* $(-\infty<s_1<\infty)$ *relative to zero, meaning that for each number w in* $\boldsymbol{Z_1}$ *there is one and only one number (-w) in the dataset* $\boldsymbol{Z_3}$ *, and vice versa (meaning the swap of datasets).*

Proof: Let us consider $s_1=-(s+1)$. Then, we can write the following for $\mathbf{Z_3}$.
$(4s_1+3)=(4(-s-1)+3)=(-4s-1)=-(4s+1)$
Assuming $s_1=-s-1$, we obtain for $\mathbf{Z_1}$.
$(4s_1+1)=(4(-s-1)+1)=-(4s+3)$
or $(4s+3)=-(4s_1+1)$
Since the above transformations are one-to-one, it means one-to-one relationship between any number in one dataset and its algebraic opposite in another dataset. Note that values of s and s_1 have the same ranges of definition, so that they are interchangeable in the above expressions. This proves the Lemma.

Fig. 2 illustrates the algebraically opposite numbers in two datasets.

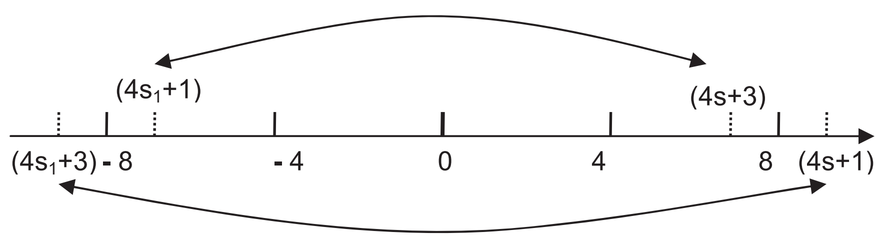

Fig. 2. Symmetrical subsets of odd integer numbers, expressed with a factor of four.

The symmetry of obtained sets can be illustrated by congruencies. Indeed, $4s+1\equiv 1(\mathrm{mod}\,4)$, while the congruency for the matching value $(-4s_1-3)\equiv -3(\mathrm{mod}\,4)$ transforms to $(-4s_1+1)\equiv 1(\mathrm{mod}\,4)$, so that both values are congruent to number one.

The following corollary follows from Lemma 4.

Corollary 2: *Dataset* $\mathbf{Z_1}=\{4s+1\}$, $(-\infty<s<\infty)$ *can be substituted by dataset* -$\mathbf{Z_3}=\{-(4s_1+3)\}$, $(-\infty<s_1<\infty)$*, and vice versa.*

4.5. Properties of equations, corresponding to pairs of odd numbers with a factor of 2^r

This section introduces an equation, to which all equations, corresponding to pairs of odd numbers, can be transformed, and explores its properties.

Lemma 5: *Let us consider an equation*

$$(2^r t_r+v)^N-(2^r s_r+w)^N=2^{N(\mu+1)}m_1^N \tag{13}$$

where t_r *and* s_r *are integers; N=2n+1;* m_1 *is odd; v, w are positive odd (possibly equal) numbers, obtained through successive presentations of pairs of odd numbers. Then, for any* $r\geq 3$*, such equations can be transformed to the following form*

$(t_r - s_r)A_r = A_r c/2^r + m_1^N / 2^{r-r_t}$ (14)

where $A_r = \sum_{i=0}^{N-1}(2^r t_r + v)^{N-1-i}(2^r s_r + w)^i$ *is an odd integer; c is an integer;* $r_t = N(\mu+1)$.

Proof: Equation (13) is equation (8), rewritten for a presentation with a factor of 2^r.

$$[2^r(t_r - s_r) + (v-w)]\sum_{i=0}^{N-1}(2^r t_r + v)^{N-1-i}(2^r s_r + w)^i = 2^{N(\mu+1)} m_1^N \quad (15)$$

The sum in (15) is odd, because it presents the sum of odd quantity of odd values. Let us denote it

$$A_r = \sum_{i=0}^{N-1}(2^r t_r + v)^{N-1-i}(2^r s_r + w)^i$$

Since *v* and *w* are odd, their difference is even. Also, in successive presentation of odd numbers, according to Lemma 3, $v < 2^r$, $w < 2^r$. Since both values are positive, their absolute difference is also less than 2^r. According to Lemma 1 and Corollary 1, (*v* - *w*) can be presented as a sum of powers of two with coefficients, having the same algebraic sign. Since $|v-w| < 2^r$, such a sum cannot contain a summand with a power greater than 2^{r-1}, when all coefficients K_i have the same algebraic sign.

$$[2^r(t_r - s_r) + \sum_{i=1}^{r-1}2^i K_i]A_r = 2^{N(\mu+1)} m_1^N \quad (16)$$

Then, (16) can be rewritten as follows.

$$2^r(t_r - s_r)A_r = -\left(\sum_{i=1}^{r-1}2^i K_i\right)A_r + 2^{N(\mu+1)} m_1^N \quad (17)$$

Let us denote $c = -\sum_{i=1}^{r-1}2^i K_i$. Since $c = w - v$, when $w = v$ (that is free coefficients are equal), $c = 0$. When $w \neq v$, the value of $c \neq 0$. Dividing both parts of (17) by 2^r, and taking into account that $r_t = N(\mu+1)$, we obtain

$$(t_r - s_r)A_r = A_r c/2^r + m_1^N / 2^{r-r_t} \quad (18)$$

This proves the Lemma.

Lemma 6: *If* $c \neq 0$ *in (18), then* $A_r c/2^r$ *is a rational number.*

Proof: It was indicated in Lemma 5 that when free coefficients *w* and *v* are unequal, $c \neq 0$. According to Lemma 1, we can always use a presentation $\sum_{i=1}^{r-1}2^i K_i$ with the range of values $K_i = \{0,1\}$, $1 \le i \le r-1$, when $c > 0$, and $K_i = \{-1,0\}$ when $c < 0$. Then

$$|c| = \left|\sum_{i=1}^{r-1}2^i K_i\right| \le \sum_{i=1}^{r-1}2^i = 2(2^{r-1}-1)/(2-1) = 2^r - 2 \quad (19)$$

(Here, we substituted the sum of a geometrical progression with a common ratio of two and the first term of two.) Accordingly

$$|A_r c/2^r| \le |A_r|(1 - 1/2^{r-1}) \quad (20)$$

Dividing inequality (20) by a positive number $|A_r|$, one obtains

$$|c/2^r| \le (1 - 1/2^{r-1}) \quad (21)$$

Thus, $c/2^r$ is a rational number. The term A_r is an odd number, which, consequently, contains no dividers of two. In turn, this means that $A_r c/2^r$ is a rational number. This proves the Lemma.

Lemma 7: *Equation (14) has no solution for pairs with unequal free coefficients when* $r \le N(\mu+1)$*, while solution is uncertain for pairs with equal free coefficients.*

Proof: For $r \le N(\mu+1) = r_t$, the term $2^{r_t - r} m_1^N$ in (14) is an integer. According to Lemma 6, the summand $A_r c / 2^r$ is rational for pairs with unequal free coefficients. So, the right part of (14) is rational. On the other hand, the left part is an integer when $(t_r - s_r) \ne 0$. This means that (14) has no solution in this case. When $(t_r - s_r) = 0$, (14) presents equality of zero (in the left part), and of a rational number, which is impossible too. So, (14) has no solution for pairs with unequal free coefficients.

When free coefficients are equal, $c = 0$, and (14) transforms to

$$(t_r - s_r) A_r = 2^{r_t - r} m_1^N \qquad (22)$$

For $r < r_t$, the right part is even, for $r = r_t$ it is odd. The left part can be odd, or even, or zero. So, the solution of this equation is uncertain. Consequently, the pairs, whose terms have equal free coefficients, should be used as initial pairs of terms for the next presentation level.

This proves the Lemma.

Now, we should establish relationships between the sizes of groups, corresponding to pairs of terms with equal and unequal free coefficients, and the parity of the term $(t_r - s_r)$ in (14).

Lemma 8: *When initial pairs of terms, obtained from the r-level of presentation, have equal free coefficients, the number of pairs of terms with equal and unequal free coefficients at the next presentation level (r+1) is the same and is equal to 1/2 of the whole set of pairs at level (r+1). The group of pairs with equal free coefficients correspond to even values of* $(t_r - s_r)$*, while pairs with unequal free coefficients correspond to odd* $(t_r - s_r)$*, so that it is equivalent subdividing the pairs based on parity of* $(t_r - s_r)$*, or on the basis of equal and unequal free coefficients.*

Proof: It follows from Table 1 that for $r_2 = 2$ the quantities of pairs of terms with equal and unequal free coefficients are equal. Consequently, each group constitutes a half of all pairs of terms. Odd values of $(t_{r_2} - s_{r_2})$ correspond to pairs at level $r = 3$ with unequal free coefficients. Accordingly, even values of $(t_{r_2} - s_{r_2})$ correspond to pairs of terms with equal free coefficients. Let us assume that the same is true for an initial pair of terms with equal free coefficients at the greater level r, $r \ge 2$. The presentation for all possible parity combinations of t_r and s_r at level (r+1) is shown in Table 3 for one generic pair with equal free coefficients.

It follows from Table 3 that the number of pairs with equal and unequal free coefficients is the same, and is equal to 1/2 of quantity of all pairs. Unequal free coefficients correspond to odd values of $(t_r - s_r)$, while even values $(t_r - s_r)$ correspond to pairs with equal free coefficients. So, we obtained the same results as for $r = 2$. Since the rest of initial pairs of terms have the same form (in all of them free coefficients are equal), depending on the parity of $(t_r - s_r)$, they also produce a half of pairs with equal free coefficients, and a half with unequal ones. According to principle of mathematical induction, this means that the found properties are valid for any presentation level $r \ge 2$. This proves the Lemma.

Table 3. Presentation with a factor 2^r for a pair with equal free coefficients.

	0	1	2	3	4
1	t_r	$2t_{r+1}$	$2t_{r+1}+1$	$2t_{r+1}$	$2t_{r+1}+1$

	s_r	$2s_{r+1}+1$	$2s_{r+1}$	$2s_{r+1}$	$2s_{r+1}+1$
2	$2^r t_r+v_i$ $2^r s_r+v_i$	$2^{r+1}t_{r+1}+v_i$ $2^{r+1}s_{r+1}+2^r+v_i$	$2^{r+1}t_{r+1}+2^r+v_i$ $2^{r+1}s_{r+1}+v_i$	$2^{r+1}t_{r+1}+v_i$ $2^{r+1}s_{r+1}+v_i$	$2^{r+1}t_{r+1}+2^r+v_i$ $2^{r+1}s_{r+1}+2^r+v_i$

Corollary 3: *Consider successive presentations of pairs of odd numbers with pairs of terms having a factor of* 2^r*, which use initial pairs of terms with equal free coefficients from the previous level, beginning with one pair of terms. Then, the number of initial pairs of terms at level r is equal to* 2^{r-1}.

Proof: For a factor of two, we have one pair of terms; for a factor of 2^2 there are two pairs of terms with equal free coefficients (Table 1); for a factor of 2^3 there are 2^2 such pairs (Table 2), and so forth. The total number of pairs of terms increases by four times for the next presentation level (since each initial pair produces four new pairs, one per parity combination of t_r, s_r). From this amount, a half of pairs correspond to pairs with equal free coefficients, according to Lemma 8. The value of 2^{r-1} reflects on the fact that at each presentation level the number of pairs with equal free coefficients doubles. This proves the Corollary.

Corollary 4: *For* $r \le r_t = N(\mu+1)$*, the fraction of pairs of terms, for which equation (8) has no solution for a presentation level r, is equal to*

$$f_r = (1/2)^{r-1} \qquad (23)$$

Proof: It was shown in Lemma 7 that in this case (13) has no solution for pairs with unequal free coefficients, while, according to Lemma 8, these pairs constitute half of all pairs of terms at a given presentation level. Thus, (23) is true for $r = 2$. Let us assume that Lemma is valid for the value of $r > 2$. According to Lemma 7, for $r \le r_t$, the corresponding equations have no solution for pairs of terms with unequal free coefficients, so that initial pairs for the next level are always pairs with equal free coefficients. Then, the fraction f_{ru} of pairs, for which solution is uncertain, is the same, as the fraction of "no solution" pairs, that is $f_{ru} = (1/2)^{r-1}$. This fraction contains initial pairs for the presentation level (*r*+1). At this level, all pairs are again divided into two equal groups of "no solution" and "uncertain" pairs, so that the "no solution" fraction is

$$f_{r+1} = f_{ru} \times (1/2) = (1/2)^{r-1}/2 = (1/2)^r,$$

which is formula (23) for the level (*r*+1). According to principle of mathematical induction, this means validity of (23). This proves the Corollary.

Lemma 9: *At each next presentation level (r+1), the number of pairs, corresponding to odd and even values of* $(t_r - s_r)$*, are equal.*

Proof: Suppose we have p_{r+1} initial pairs at a presentation level (*r*+1). Each initial pair of terms produces four pairs at level (*r*+1), one pair per each possible parity combination of terms t_r, s_r, listed in the first row of Table 3. These parity combinations do not depend, whether the initial pairs have equal or unequal free terms, and also do not depend on the value of *r* compared to r_t. Two of these parity combinations (in cells (1,1), (1,2) in Table 3) produce odd values of $(t_r - s_r)$, namely when t_r, s_r are equal to $[2t_{r+1}, 2s_{r+1}+1]$, $[2t_{r+1}+1, 2s_{r+1}]$. Two other combinations, in cells (1,3), (1,4), produce even values of $(t_r - s_r)$ for pairs $[2t_{r+1}, 2s_{r+1}]$, $[2t_{r+1}+1, 2s_{r+1}+1]$. So, the number of pairs, for which $(t_r - s_r)$ is odd is equal to $2p_{r+1}$. The number of pairs, for which $(t_r - s_r)$ is even, is also $2p_{r+1}$. So, quantities of pairs of terms, corresponding to odd and even values of $(t_r - s_r)$, are equal. This proves the Lemma.

Note: At the presentation level (r+1), odd values $(t_r - s_r)$ cannot be zero, given the presentation of t_r and s_r through t_{r+1} and s_{r+1} in Table 3. Even values of $(t_r - s_r)$ can be zero. However, from the perspective of solution, such a zero term can be transformed to a non-zero even term (such a transition is addressed by Lemma 10).

4.6. Finding fraction of "no solution" pairs for presentation levels with $r \le r_t = N(\mu+1)$

We found so far that for $r \le r_t = N(\mu+1)$ the following is true:

(a) Initial pairs of terms with equal free coefficients, taken from level r, produce equal number of pairs of terms with equal and unequal free coefficients at a presentation level (r+1), Lemma 8;
(b) Corresponding to pairs of terms equations have no solution for pairs with unequal free coefficients, while solution is uncertain for pairs with equal free coefficients, Lemma 7;
(c) Each presentation level adds a "no solution" fraction of pairs of terms equal to $f_r = (1/2)^{r-1}$.

So, each previous level supplies to the next presentation level "uncertain" pairs of terms, which constitutes half of all pairs of the previous level. These initial pairs have equal free coefficients. This allows finding a "no solution" fraction of pairs of terms from successive presentations with a factor of 2^r. Since each level adds 1/2 of pairs to a "no solution" fraction, the total such fraction F_r is equal to a sum of geometrical progression with a common ratio $q = 1/2$, and the first term $f_2 = 1/2$ (the "no solution" fraction at level r=2). Fig. 1 illustrates this consideration.

So, we can write

$$F_r = \sum_{i=2}^{r} f_i = f_2 \sum_{i=2}^{r} q^{i-2} = f_2 (1 - q^{r-1})/(1-q) \qquad (24)$$

For example, for r=5, $F_r = 15/16$. Note that if such a progression is valid to infinity, the total fraction in the limit would be

$$\lim_{r\to\infty} F_r = f_2/(1-q) = (1/2)/(1/2) = 1 \qquad (25)$$

(Here, the limit is understood as an ordinary Cauchy's limit.) In other words, equation (8) would not have a solution for *all* possible pairs of terms. However, in order to obtain such a result, one needs to confirm that such a progression is true for $r > r_t = N(\mu+1)$.

4.7. Transcending the threshold level $r = r_t$

Presentation level $(r_t + 1)$

Table 4 shows pairs of terms for level $(r_t + 1)$. The number of initial pairs is defined by Corollary 3, and is equal to 2^{r_t} for this level. For pairs with *equal* free coefficients (columns 3 and 4 in Table 4), (14) transform to

$$(t_{r_t+1} - s_{r_t+1}) A_{r_t+1,ij} = m_1^N / 2 \qquad (26)$$

The right part of (26) is rational (m_1 is an odd number). The left part is an integer. So, (26) has no solution for pairs with equal free coefficients (and, consequently, for even $(t_{r_t} - s_{r_t})$, according to Lemmas 8 and 9). When $(t_{r_t} - s_{r_t}) = 0$, the left part is zero, while the right part is rational. So, (26) has no solution too. This group of pairs of terms constitutes 1/2 of all pairs (Lemma 9), so that the common ratio remains equal to 1/2, and formula (24) stays valid.

Table 4. Pairs presented with a factor of 2^{r_t+1}. It is assumed that $r=r_t$.

	0	1	2	3	4
	t_r s_r	$2t_{r+1}$ $2s_{r+1}+1$	$2t_{r+1}+1$ $2s_{r+1}$	$2t_{r+1}$ $2s_{r+1}$	$2t_{r+1}+1$ $2s_{r+1}+1$
1	$2^r t_r + v_{r1}$ $2^r s_r + v_{r1}$	$2^{r+1} t_{r+1} + v_{r1}$ $2^{r+1} s_{r+1} + 2^r + v_{r1}$	$2^{r+1} t_{r+1} + 2^r + v_{r1}$ $2^{r+1} s_{r+1} + v_{r1}$	$2^{r+1} t_{r+1} + v_{r1}$ $2^{r+1} s_{r+1} + v_{r1}$	$2^{r+1} t_{r+1} + 2^r + v_{r1}$ $2^{r+1} s_{r+1} + 2^r + v_{r1}$
2	$2^r t_r + v_{r2}$ $2^r s_r + v_{r2}$	$2^{r+1} t_{r+1} + v_{r2}$ $2^{r+1} s_{r+1} + 2^r + v_{r2}$	$2^{r+1} t_{r+1} + 2^r + v_{r2}$ $2^{r+1} s_{r+1} + v_{r2}$	$2^{r+1} t_{r+1} + v_{r2}$ $2^{r+1} s_{r+1} + v_{r2}$	$2^{r+1} t_{r+1} + 2^r + v_{r2}$ $2^{r+1} s_{r+1} + 2^r + v_{r2}$
...					
2^r	$2^r t_r + v_{rR}$ $2^r s_r + v_{rR}$	$2^{r+1} t_{r+1} + v_{rR}$ $2^{r+1} s_{r+1} + 2^r + v_{rR}$	$2^{r+1} t_{r+1} + 2^r + v_{rR}$ $2^{r+1} s_{r+1} + v_{rR}$	$2^{r+1} t_{r+1} + v_{rR}$ $2^{r+1} s_{r+1} + v_{rR}$	$2^{r+1} t_{r+1} + 2^r + v_{rR}$ $2^{r+1} s_{r+1} + 2^r + v_{rR}$

For pairs with unequal free coefficients (and consequently odd $(t_{r_t} - s_{r_t})$, Lemma 9), (14) transforms to

$$(t_{r_t+1} - s_{r_t+1}) A_{r_t+1,j} = A_{r_t+1,j} c / 2^{r_t} + m_1^N / 2 \tag{27}$$

The right part can be rational, an integer or zero. Since the sums $A_{r+1,j}$ are all odd, parity of the left part in (27) is defined by the term $(t_{r_t+1} - s_{r_t+1})$, which can be odd, even or zero. So, solution of (27) for odd $(t_{r_t} - s_{r_t})$ is uncertain, and such pairs should be used as initial pairs of terms for the next presentation level $(r_t + 2)$. As it was mentioned (a note after Lemma 9), for odd $(t_{r_t} - s_{r_t})$, the term $(t_{r_t+1} - s_{r_t+1}) \neq 0$.

Recall that before the level $(r_t + 1)$, the pairs with *unequal* free coefficients had no solution, while (26) has no solution for *even* $(t_{r_t+1} - s_{r_t+1})$, corresponding to pairs with *equal* free coefficients. In this regard, the level $(r_t + 1)$ reverses the groups of pairs. The "uncertain" group of pairs is now composed of pairs with *unequal* free coefficients (and accordingly with *odd* $(t_{r_t} - s_{r_t})$). These pairs of terms (in columns 1 and 2 in Table 4) should be used as initial pairs at the next presentation level $(r_t + 2)$.

Transition in the presentation level $(r_t + 2)$

Level $(r_t + 1)$ supplied initial pairs of terms with unequal free coefficients. This means that we do not have anymore distinct groups with equal and unequal free coefficients at level $(r_t + 2)$, as before, since the initial pairs with unequal free coefficients produce mostly pairs with unequal free coefficients, with occasional inclusion of pairs with equal ones. Previously, we have seen that the parity of parameter $(t_r - s_r)$ defined the absence or uncertainty of solution. However, beginning from level $(r_t + 2)$, this parameter lost association with groups of pairs of terms with equal and unequal free coefficients. This is due to the fact that the right part of equation (27) can be an integer, a rational number, or zero *per pair basis*, and so we should consider the use of parameter $(t_r - s_r)$ this way. We will still have a half of "no solution" and a half of "uncertain" pairs of terms, but only for a block of four pairs, corresponding to each initial pair. This is the assembly of such "uncertain" pairs from each block, which goes to the next level. Table 5 shows pairs of terms for level $(r_t + 2)$.

Table 5. Pairs of terms with a factor of 2^{r_t+2}, obtained from initial pairs in Table 4, for which $(t_r - s_r)$ is odd. First two rows correspond to cells (1,1), (1,2) in Table 4. It is assumed that $r = r_t$.

	0	1	2
	t_{r+1} s_{r+1}	$2t_{r+2}$ $2s_{r+2}+1$	$2t_{r+2}+1$ $2s_{r+2}$
1	$2^{r+1}t_{r+1} + v_{r1}$ $2^{r+1}s_{r+1} + 2^r + v_{r1}$	$2^{r+2}t_{r+2} + v_{r1}$ $2^{r+2}s_{r+2} + 2^{r+1} + 2^r + v_{r1}$	$2^{r+2}t_{r+2} + 2^{r+1} + v_{r1}$ $2^{r+2}s_{r+2} + 2^r + v_{r1}$
2	$2^{r+1}t_r + 2^r + v_{r1}$ $2^{r+1}s_{r+1} + v_{r1}$	$2^{r+2}t_{r+2} + 2^r + v_{r1}$ $2^{r+2}s_{r+2} + 2^r + v_{r1}$	$2^{r+2}t_{r+2} + 2^{r+1} + 2^r + v_{r1}$ $2^{r+2}s_{r+2} + v_{r1}$
. . .	. . .	. . .	. . .
2^{r+1}	. . .	. . .	. . .

Table 5 continued

3	4
$2t_{r+2}$ $2s_{r+2}$	$2t_{r+2}+1$ $2s_{r+2}+1$
$2^{r+2}t_{r+2} + v_{r1}$ $2^{r+2}s_{r+2} + 2^r + v_{r1}$	$2^{r+2}t_{r+2} + 2^{r+1} + v_{r1}$ $2^{r+2}s_{r+2} + 2^{r+1} + 2^r + v_{r1}$
$2^{r+2}t_{r+2} + 2^r + v_{r1}$ $2^{r+2}s_{r+2} + v_{r1}$	$2^{r+2}t_{r+2} + 2^{r+1} + 2^r + v_{r1}$ $2^{r+2}s_{r+2} + 2^{r+1} + v_{r1}$
. . .	. . .
. . .	. . .

When r=(r_t+2), (14) transforms to

$$(t_{r_t+2} - s_{r_t+2})A_{r+2,ij} = A_{r_t+2,ij}c/2^{r_t+2} + m_1^N/4 \qquad (28)$$

where index '*ij*' denotes the cell number. The right part of (28) can be rational, an integer, or zero. When the left part is an integer (the case, when it's zero, will be considered later), (28) has no solution for any $(t_{r_t+2} - s_{r_t+2})$ for the rational or zero right part, and, consequently, this branch is completed. (Compared to continuing branches, the completed branch delivers *double* fraction of pairs, for which (8) has no solution, since in this case two equal "no solution" and "uncertain" fractions compose one "no solution" fraction.) If the right part is an integer, (28) has no solution when $(t_{r_t+2} - s_{r_t+2})$ has the opposite parity, and the solution is uncertain for another parity of $(t_{r_t+2} - s_{r_t+2})$. The number of combinations of parameters t_{r_t+2} and s_{r_t+2}, corresponding to each parity, is equal to two from four in this case, and so we still have equal division between the "no solution" and "uncertain" pairs of terms. However, at this level, we have no distinction between the odd and even values of t_{r_t+2} and s_{r_t+2} in the same way, as before, when there was an association with equal and unequal free coefficients. Such distinction can be done *only* at the next presentation level (r_t+3). All pairs at level r_t+2 correspond to "uncertain" equations, except for the cases when the pair's branch is completed.

The case of $(t_{r_t+2} - s_{r_t+2}) = 0$ is also an "uncertain" one, since there is a possibility that two terms in the right part are equal in absolute values and have the opposite algebraic signs.

Note that values $A_{r_t+2,ij}$ are different, so that the right parts of corresponding equations, transformed to a form (14), may have dissimilar parities (as well as may be rational or zeros) for different pairs. (The right part can be an integer, provided $c \neq 0$ in (14), otherwise the right part is equal to $m_1^N / 2^{r-r_t}$, which is always rational for $r > r_t$, so that such a branch is completed.) This is why, starting from this level, one should consider each pair of terms *separately* (Fig. 1). (In fact, it is possible to show that at level $(r_t + 2)$, when $c \neq 0$, integer right parts of these equations have the same parity. However, this is not necessarily true for the next levels, so we use the same generic approach for this level and above.)

With regard to accumulation of a total "no solution" fraction, we have the same common ratio of 1/2, although it is obtained differently - not per group, as previously, but per pair, and then such "per pair" fractions are summed up, in order to obtain the total "no solution" fraction. We will consider this assembling process in detail later.

So, we found that the corresponding equations for pairs of terms in both groups (meaning groups of pairs, having either even or odd values of $(t_{r_t+2} - s_{r_t+2})$) converge to equations, which have no solution for one parity of $(t_{r_t+2} - s_{r_t+2})$, and accordingly for one half of pairs of terms (according to Lemma 9), while solution is uncertain for the other parity, corresponding to the second half of pairs. So, the common ratio for a geometric progression, defining fractions of "no solution" pairs, will remain equal to 1/2. However, because we can specify particular pairs, corresponding to odd or even $(t_{r_t+2} - s_{r_t+2})$, at the next level only, this common ratio accordingly should be assigned to a presentation level, where such a specification actually happens; in this case, this is the next level $(r_t + 3)$. At level $r_t + 2$, all equations, corresponding to initial pairs, have the same form (14), and consequently, the same "uncertain" status. All pairs (except for completed ones) are "uncertain" pairs.

Presentation level $(r_t + 3)$

We will need the following Lemma to address zero values of $(t_r - s_r) = 0$ in equation (14). Note that $(t_r - s_r) = 0$ only when both parameters are equal (and, of course, have the same parity), including when both are equal to zero. When $(t_r - s_r)$ is odd (parameters have different parity), $(t_r - s_r) \neq 0$.

Lemma 10: *Equation (14), that is* $(t_r - s_r)A_r = A_r c / 2^r + m_1^N / 2^{r-r_t}$*, is equivalent to equation* $(t_{1r} - s_{1r})A_r = A_r c / 2^r + m_1^N / 2^{r-r_t}$ *in terms of parities of both parts, with the substitutions* $t_r = t_{1r} - 2a$ *and* $s_r = s_{1r} - 2b$*, where a and b are integers. If the second equation has no solution based on parity or rationality considerations, then the first equation also has no solution, and vice versa.*

Proof: According to the notion of presentation of odd numbers with a factor of 2^r, the terms t_r and s_r are integers, having ranges of definition $(-\infty < t_r < \infty)$ and $(-\infty < s_r < \infty)$. The only property, which is of importance with regard to such a presentation, is that these parameters should be defined on the whole set of integer numbers, in order to include *all* possible numbers, corresponding to a particular presentation; for instance, the term $(2^r t_r + v_r)$ should produce the whole set of the appropriate "stroboscopic" numbers in the range $(-\infty, \infty)$, located at the distance 2^r from each other. As long as this condition is fulfilled, that is such a set can be reproduced, we can make an equivalent substitution for parameters t_r, s_r. For instance, the substitution $t_r = t_{1r} - 2a$ is an equivalent one. Indeed, it preserves the range of definition $(-\infty < t_{1r} < \infty)$, and accordingly produces all numbers, which parameter t_r produces (only with a shift of $(-2a \times 2^r)$ for the same values of t_r and t_{1r}). However, this shift makes no difference with regard to the range of produced numbers, since our range $(-\infty,\infty)$ is infinite in both directions. On the other hand, when $(t_r - s_r) = 0$, we have $(t_{1r} - s_r) \neq 0$, and vice versa. So, for $(t_r - s_r) = 0$, such a substitution produces an equation with a non-zero left part.

Substituting $t_r = t_{1r} - 2a$ into (14), one obtains the equation

$$(t_{1r} - s_r)A_r = 2aA_r + A_r c/2^r + m_1^N / 2^{r-r_t} \quad (29)$$

When $(t_r - s_r) = 0$, we have $(t_{1r} - s_r) = 2a \neq 0$. Also, the appearance of the even term $2aA_r$ does not change the parity of the right part, nor the substitution $t_r = t_{1r} - 2a$ changes the parity of the left part (if it is not zero; if it is zero, the substitution still provides an even increment). Thus, with regard to parities, (14) and (29), indeed, are equivalent equations.

If the equivalent equation (29) has no solution, then the original equation (14) has no solution too. The proof is as follows. Let us assume that (29) has no solution, while (14) has a solution, so that

$$(t_r - s_r)A_r = A_r c/2^r + m_1^N / 2^{r-r_t}$$

Adding $2aA_r$ to the left and right parts of this equation, one obtains an equivalent equation, which also should have a solution.

$$(t_r + 2a - s_r)A_r = 2aA_r + A_r c/2^r + m_1^N / 2^{r-r_t}$$

According to the substitution, $t_r = t_{1r} - 2a$, so that $t_r + 2a = t_{1r}$, and the obtained equation transforms to (29), which should also have a solution. However, according to our assumption, it has no solution. The obtained contradiction means that the assumption that (14) has a solution is invalid, and, in fact, it has no solution.

Similarly, we can assume that (29) has a solution, while (14) does not, and show that then (14) should have a solution, which would contradict to the initial assumption.

Although we proved the equivalency of equations with regard to their solution properties in a general case, we need such equivalency only for the case when the left part of equivalent equations is zero (because $(t_r - s_r) = 0$ or $(t_{1r} - s_r) = 0$). The proposed substitution then makes the left part of the equivalent equation a non-zero value, and the inference about the absence of solution or its uncertainty can be made based on parities of the left and right parts. Certainly, one can do an analogous substitution for s_r, or both parameters. This proves the Lemma.

Table 6 shows an example of pairs of terms for the presentation level $(r_t + 3)$. Four initial pairs are from cells (1,1)-(1,4) in Table 5. If (28) has no solution for even $(t_{r_t+2} - s_{r_t+2})$, then these are pairs (1,3), (1,4) in Table 6, which satisfy this condition. Accordingly, pairs (1,1) and (1,2), for which $(t_{r_t+2} - s_{r_t+2})$ is odd, are "uncertain" pairs, which should be used as initial pairs for the next, $(r_t + 4)$, level. If, on the contrary, (28) has no solution for odd $(t_{r_t+2} - s_{r_t+2})$, then (1,1) and (1,2) are the "no solution" pairs, while (1,3), (1,4) become "uncertain" pairs, which should be used as initial pairs of terms for the next level. This way, all new pairs, four per each initial pair, are divided into two halves as before, so that the common ratio of geometrical progression remains equal to 1/2. The case $(t_{r_t+2} - s_{r_t+2}) = 0$ is addressed by Lemma 10 through equivalent equations.

In the same way, as we considered one pair above, we should consider the rest of initial pairs in Table 6 and find out, which two pairs should be used as initial pairs for the next level. Then, the same procedure should be repeated for each initial pair at level $(r_t + 3)$.

Then, the cycle is repeated for the next two levels $(r_t + 4)$ and $(r_t + 5)$, and so forth, to infinity, since there are no anymore threshold values of *r*, at which the right part could change the parity (if it's an integer), and the corresponding equations their form and properties. The following Lemma generalizes the discovered order.

Table 6. Pairs with a factor of 2^{r_t+3}. Initial pairs are (1,1)-(1,4) from Table 5. Here, $r = r_t$.

	0	1	2
	t_{r+2} s_{r+2}	$2t_{r+3}$ $2s_{r+3}+1$	$2t_{r+3}+1$ $2s_{r+3}$
1	$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$
2	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$
3	$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+v_{r2}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r2}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$
4	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+1}+v_{r2}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r2}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{+1r1}$

Table 6 continued

3	4
$2t_{r+3}$ $2s_{r+3}$	$2t_{r+3}+1$ $2s_{r+3}+1$
$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$

Lemma 11: *From the presentation level (r_t+2), the "no solution" fraction is accumulated across two sequential levels, and then the pattern repeats for each two successive levels, to infinity. Some branches can be completed at level (r_t+2), but otherwise this level provides no explicit division into the "no solution" and "uncertain" groups, as it was the case for the previous levels. Except for the pairs, corresponding to completed branches, the pairs become initial "uncertain" pairs of terms for the next presentation level. At level (r_t+3), all new pairs are divided into the "no solution" and "uncertain" groups (according to odd or even parity of $(t_r - s_r)$ in equation (14)). The "uncertain" pairs become initial pairs for the next presentation level, and the two-level cycle repeats to infinity.*

Proof: Previously, we have seen that the Lemma is true for the paired levels (r_t+2) and (r_t+3). Let us assume that Lemma is true for the previous (r_t+d-1) level, which then supplies initial "uncertain" pairs of terms for the level (r_t+d). We need to prove that Lemma is true for the next two levels (r_t+d) and (r_t+d+1). Initial pairs may have equal and unequal free coefficients.

Let us consider an equation for a pair with free coefficients *v* and *w*.

$$(2^{r_t+d}t_{r_t+d}+v)^N-(2^{r_t+d}s_{r_t+d}+w)^N=2^{N(\mu+1)}m_1^N \quad (30)$$

where $d\geq 2$.

According to Lemma 5, it can be transformed to an equation

$$(t_{r_t+d}-s_{r_t+d})A_{r_t+d}=A_{r_t+d}c/2^{r_t+d}+m_1^N/2^d \quad (31)$$

where $A_{r_t+d}=\sum_{i=0}^{N-1}(2^{r_t+d}t_{r_t+d}+v)^{N-1-i}(2^{r_t+d}s_{r_t+d}+w)^i$, $N=2n+1$.

The right part of (31) can be an integer, rational or zero. The left part is an integer (if $(t_{r_t+d}-s_{r_t+d})=0$, the left part can be transformed to an integer, using Lemma 10). When the right part is rational, (31) has no solution for any t_{r_t+d} and s_{r_t+d}, and the branch is completed. If the right part is even or odd, (31) has no solution when $(t_{r_t+d}-s_{r_t+d})$ has the opposite parity. Solution is uncertain for the other parity of $(t_{r_t+d}-s_{r_t+d})$, since both parts of (31) have the same parity in this case. However, at this level, we cannot specify particular parity of $(t_{r_t+d}-s_{r_t+d})$, which should be done at the next presentation level (r_t+d+1). When $c=0$, (31) has no solution, since the right part is a rational number, while the left part is an integer or zero, and so the branch is completed.

Even if the branch is completed for some pair, we still can assume that it is "uncertain", and use it as an initial pair at the next presentation level. There, the new pairs, corresponding to this initial pair, are then divided into the "no solution" and "uncertain" groups. The fraction of the former goes to the total "no solution" fraction, while the latter is used as initial pairs for the next level, besides other uncertain pairs. (Such an arrangement, without completed branches, is more convenient for calculation of the total "no solution" fraction.)

Table 7. New pairs of terms for the initial pair $[2^{r_t+d}t_{r_t+d}+v,\ 2^{r_t+d}s_{r_t+d}+w]$ at the presentation level (r_t+d+1) with a factor of 2^{r_t+d+1}.

	0	1	2
0	t_{r_t+d} s_{r_t+d}	$2t_{r_t+d+1}$ $2s_{r_t+d+1}+1$	$2t_{r_t+d+1}+1$ $2s_{r_t+d+1}$
1	$2^{r_t+d}t_{r_t+d}+v$ $2^{r_t+d}s_{r_t+d}+w$	$2^{r_t+d+1}t_{r_t+d+1}+v$ $2^{r_t+d+1}s_{r_t+d+1}+2^{r_t+d}+w$	$2^{r_t+d+1}t_{r_t+d+1}+2^{r_t+d}+v$ $2^{r_t+d+1}s_{r_t+d+1}+w$

Table 7 continued

3	4
$2t_{r+3}$ $2s_{r+3}$	$2t_{r+3}+1$ $2s_{r+3}+1$
$2^{r_t+d+1}t_{r_t+d+1}+v$ $2^{r_t+d+1}s_{r_t+d+1}+w$	$2^{r_t+d+1}t_{r_t+d+1}+2^{r_t+d}+v$ $2^{r_t+d+1}s_{r_t+d+1}+2^{r_t+d}+w$

Table 7 shows new pairs for the next presentation level for the initial pair from (30). At this level, we can choose the needed parities of pair's terms t_{r_t+d}, s_{r_t+d}, expressed through t_{r_t+d+1}, s_{r_t+d+1}, in order for (31) to have no solution. For instance, if (31) has no solution for even $(t_{r_t+d}-s_{r_t+d})$, then the "no solution" pairs are (1,3), (1,4). Accordingly, solution is uncertain for pairs (1,1), (1,2), since both parts of (31) have the same parity in this case. Consequently, these pairs should be used as initial "uncertain" pairs for the next presentation level.

We can see from Table 7 that when a pair of an actually completed branch is used as an "uncertain" pair for the next level, it produces no new pairs with some specific features, which could prevent their corresponding equations to be transformed into a form (31). We still obtain pairs of terms, satisfying conditions of Lemma 5, to which the same equation (14) is applicable. For instance, when $v = w$, then $c = 0$ in (31), and so the branch is completed. However, if we use it as an initial pair for the next presentation level $(r_t + d + 1)$, then we are free to choose new pairs, corresponding to either even or odd values of $(t_{r_t+d} - s_{r_t+d})$, since the corresponding equations have no solution for both scenarios. Then, the pairs with the opposite parity $(t_{r_t+d} - s_{r_t+d})$ will proceed to the next level as uncertain initial pairs. As before, such a division produces two equal groups of pairs of terms, and so the common ratio of the geometrical progression remains equal to 1/2.

So, with the assumption that Lemma is true for the previous level, we confirmed the same pattern earlier discovered for the coupled levels $[(r_t + 2), (r_t + 3)]$. According to principle of mathematical induction, this means that Lemma is true for any $d \geq 2$. This proves the Lemma.

In this Lemma, we also studied the useful property, considering completed branches as non-completed ones. This property is formulated below as a Corollary.

Corollary 5: *Pairs of terms, corresponding to completed branches, can be considered as regular "uncertain" pairs, which can be passed to the next level as initial pairs, so that such a branch is actually assigned a non-completed status.*

Lemma 12: *At presentation levels above* $(r_t + 1)$*, and in the absence of completed branches, the number of pairs of terms in "no solution" and "uncertain" groups are equal, when such a division takes place.*

Proof: According to Lemma 11 and Corollary 5, all pairs, both regular ones, with "no solution" and "uncertain" components, and the pairs, which could be completed, but continue to participate in the next levels as non-completed pairs, can be presented in a form of Table 7. The solution properties of equations, corresponding to pairs in Table 7, are defined by equation (14), or more particular, by equations in a form (31), whose solution properties depend on the term $(t_{r_t+d} - s_{r_t+d})$. (Unless the right part is rational, in which case equation has no solution for all parities, and the branch is completed. However, according to Corollary 5, we can still consider such a pair as a regular non-completed pair.)

The division of four pairs into two equal "no solution" and "uncertain" groups is based solely on the parity of $(t_{r_t+d} - s_{r_t+d})$, as Lemma 11 showed, with one parity corresponding to a "no solution" group, and with the opposite parity corresponding to "uncertain" group. The number of pairs, corresponding to one parity, is therefore equal to 2π, where π is the number of initial pairs, number two is the number of parity combinations of t_{r_t+d}, s_{r_t+d}, producing the same parity of $(t_{r_t+d} - s_{r_t+d})$, see Table 7. For the opposite parity of $(t_{r_t+d} - s_{r_t+d})$, the number of produced pairs is also 2π. Thus, the number of pairs in "no solution" and "uncertain" groups is the same. This proves the Lemma.

4.8. Calculating the total "no solution" fraction

Using Corollary 5, we consider all levels as if they have no completed branches. Then, according to Lemmas 8 and 9, until the level $(r_t + 2)$, all levels have two equal groups of pair combinations. One corresponds to a "no solution" fraction, and the other to "uncertain" fraction, so that the common ratio q = 1/2. Substituting these values into (24), one obtains

$$F_{r_t+1} = f_2(1-q^{r-1})/(1-q) = 1/2(1-(1/2)^{r_t+1-1})/(1/2) = 1-(1/2)^{r_t} \qquad (32)$$

The "no solution" fraction for the level (r_t+1) is defined by (23) as follows (the last term of a geometrical progression), taking into account that $f_2 = 1/2$.

$$f_{r_t+1} = f_2 q^{r_t+1-2} = (1/2)^{r_t} \qquad (33)$$

Since in the absence of completed branches the "no solution" and "uncertain" fractions are equal, according to Lemma 9, the "uncertain" fraction of pairs, which is passed to the level (r_t+2), is the same as the "no solution" fraction (33). This "uncertain" fraction, according to Lemma 12, is equally divided into "no solution" and "uncertain" fractions at each second level, beginning from level (r_t+3), so that the first term of the geometrical progression, representing the "no solution" fraction of two following coupled levels, is

$$f_{r_t+3} = f_{r_t+1} \times (1/2) \qquad (34)$$

Then, each next two levels add a half of the previous "uncertain" fraction", which is equal to "no solution" fraction. Let $D = \{2L, 2L+1\}$, $L = 1,2...$ This way, (r_t+D) defines the levels' numbers for $r \ge (r_t+2)$. Levels, at which pairs are divided into two groups, are levels (r_t+3), (r_t+5), …, (r_t+2L+1), so that the total "no solution" fraction, obtained by summation of "no solution" fractions of all levels above the (r_t+1) level, is equal to

$$F_{r_t+2,D} = (1/2)^{r_t}[1/2+(1/2)^2+(1/2)^3+...(1/2)^L] = (1/2)^{r_t}\sum_{i=1}^{L}(1/2)^i = (1/2)^{r_t}(1-(1/2)^L) \qquad (35)$$

when $D = 2L+1$, and

$$F_{r_t+2,D} = (1/2)^{r_t}[1/2+(1/2)^2+(1/2)^3+...(1/2)^{L-1}] = (1/2)^{r_t}\sum_{i=1}^{L-1}(1/2)^i = (1/2)^{r_t}(1-(1/2)^{L-1})$$

when $D = 2L$. (36)

In the last case, the division into the "no solution" and "uncertain" groups did not happen yet at the first level of coupled levels, since it occurs at the second level of the couple, as it was earlier discussed. This is why the power is (L - 1), but not L.

The total "no solution" fraction, accordingly, is defined as $F_{r_t+1+D} = F_{r_t+1} + F_{r_t+2,D}$. For $D = 2L+1$, we have

$$F_{r_t+1+D} = F_{r_t+1} + F_{r_t+2,D} = 1-(1/2)^{r_t} + (1/2)^{r_t} - (1/2)^{r_t+L} = 1-(1/2)^{r_t+L} \qquad (37)$$

It follows form (37) that in the limit

$$\lim_{L->\infty} F_{r_t+1+D} = \lim_{L->\infty}(1-(1/2)^{r_t+L}) = 1 \qquad (38)$$

The same is true for (36). So, when we consider all branches as non-completed, in the limit, the "no solution" fraction is equal to one. Of course, it may look awkward, considering completed branches as non-completed, but, as Lemma 11 and Corollary 5 showed, this is a legitimate procedure.

Accounting for completed branches. Let us assume that level r has k completed branches, to which the "no solution" fraction f_{rk} corresponds. Let us assume that these branches were not completed, and consider the pairs of terms, corresponding to these branches, as regular ones, with "no solution" and "uncertain" components, to infinity. In other words, we assume that there are no more completed branches in the following presentations of these k pairs, to infinity. (In real situation, if there are such pairs, we can also consider them as non-completed pairs, according to Corollary 5.) In this scenario, the fraction f_{rk} would be divided equally (Lemma 12) between the "no solution" and "uncertain" fractions on each subsequent level (or on the second level in coupled levels beyond the value of $r=(r_t+1)$). So, the total "no solution" fraction, accumulated at level *L*, is as follows.

$$F_{r+L}=f_{rk}\sum_{i=1}^{L}(1/2)^i=f_{rk}[1-(1/2)^L] \qquad (39)$$

In the limit, (39) transforms to

$$\lim_{L->\infty}F_{r+D}=\lim_{L->\infty}f_{rk}[1-(1/2)^L]=f_{rk} \qquad (40)$$

So, in the limit, we obtained in (40) exactly the same "no solution" fraction, which was taken by *k* completed branches. Since, according to (38), in the scenarios with non-completed branches the total "no solution" fraction is equal to one, the result (40) means that accounting for completed branches, in the limit, produces the same "no solution" fraction of one.

The correspondence between the total "no solution" fraction of one and the whole set of pairs of odd integer numbers

Now, we can revisit the subsection 4.3.1. It stated the principle difference between considering *deterministic* objects, the pairs of terms for different presentation levels and associated with them values of "no solution" fractions, and summing up *probabilistic* values, as asymptotic densities can be interpreted. The question is, does the total fraction of one, found as the limit of sum of "no solution" fractions, mean that the *whole* set of pairs of odd numbers has no solution?

Suppose we could prove that (8) has no solution for all pairs of terms, corresponding to one presentation level, say for the initial pair of terms $[(2k+1),(2p+1)]$, producing the whole set of pairs of odd numbers. The "no solution" fraction is equal to one for this one level. Would it mean that (8) has no solution for all possible pairs of odd numbers? The answer is "yes", because all pairs of terms at one presentation level generate *all* possible pairs of odd numbers. There is no question about it, since this is what "not having a solution" for a regular equation mean. Now, suppose we could prove that (8) has no solution for the first half of all pairs of terms at presentation level *r*, and for a half of pairs of terms at another presentation level *R*, corresponding to "uncertain" pairs from level *r*. Would it mean that (8) has no solution for all possible pairs of odd integer numbers? The answer is also "yes", because that half of pairs of terms from level *R* can be uniquely transformed to level *r*, using formulas (10) - (12), thus recreating the second half of pairs at level *r*, for which the solution was previously uncertain. However, now, these are also the "no solution" pairs of terms (which was proved at level *R*). (With regard to uniqueness of such a transformation, recall the result of Lemma 3 that each pair of terms produces a unique subset of pairs of odd integer numbers, which do not intersect with subsets produced by other pairs of terms from the same level.) This way, we again obtain that (8) has no solution for *all* pairs of terms *at level r*, and consequently for the whole set of pairs of odd numbers.

When we assemble the "no solution" fractions from all presentation levels, we essentially do the same. The original pair of terms $[(2k+1),(2p+1)]$ just was uniquely distributed across all presentation levels. All "no solution" pairs of terms at different levels can be transformed back to this first level, preserving uniqueness of the subsets of pairs of odd numbers each such pair of terms represents. The only specific is that in the last case we deal with infinite set of pairs of terms, but that makes no difference. We still deal with *deterministic* countable objects. This, accordingly, makes the obtained result (the total fraction of one) also *deterministic* value. It is due to *this* determinism, that it is legitimate making the following inference: Because the "no solution" fractions accumulate to one, (8) has no solution for the *whole* set of pairs of odd numbers.

However, a similar inference cannot be made, if one resorts to *probabilistic* notions, such as asymptotic densities, and obtains that the sum of appropriate asymptotic densities tends to one in the limit (assuming that the asymptotic density of the whole set of pairs of odd numbers is one). The reason is that in this case the mere definition of asymptotic density still allows to have finite number of solutions from infinity. Maybe new developments on the basis of asymptotic densities could address such an issue, introducing new varieties of densities. However, at present, we should use deterministic objects, which have no such problems.

4.9. Cases 2 and 3 as equivalent equations

For the case 3, we have $a = 2n+1$; $x = 2k_1 + 1$; $y = 2p_1 + 1$. Then, (1) transforms to (7).

$$(2k_1 + 1)^{2n+1} + (2p_1 + 1)^{2n+1} = (2m)^{2n+1} \qquad (41)$$

Using an approach, similar to one for equation (8), it is possible to prove that it has no solution. The shorter way could be to show the equivalency of (8) and (41) in terms of solution properties. Then, since (8), as it was found, has no solution, that would mean that (41) has no solution too.

The notion of equivalent equations. It means that for each set of input variables for one equation there is one and only one matching set of corresponding input variables for the other equation, such that the terms in both equations are the same. For instance, with regard to equations (8) and (41), defined on the set of integer numbers, their equivalency would mean that for any combination of terms $(2k+1)$, $(2p+1)$, $2m$ in (8) there is only one combination of terms $(2k_1+1)$, $(2p_1+1)$, $2m_1$ in (41), such, that $(2k+1) = (2k_1+1)$, $(2p+1) = -(2p_1+1)$, $m = m_1$, so that with such a substitution equation (8) becomes equation (41). Similarly, the substitution $(2k_1+1) = (2k+1)$, $(2p_1+1) = -(2p+1)$, $m_1 = m$ in (41) produces equation (8). It was proved that (8) has no solution in integer numbers, so that it has no solution for any combination of these terms. However, on the set of all possible pairs of odd numbers, on which both equations are defined, these are equivalent equations, as it will be shown. Then, since (8) has no solution, (41) will have no solution too.

Lemma 13: *Equation (8) is equivalent to equation (41) on the set of integer numbers. If one of these equations has no solution in integer numbers, then the other equation also has no solution.*

Proof: Since the odd power does not change the algebraic sign, we can rewrite (8) as follows.

$$(2k+1)^{2n+1} + (-2p-1)^{2n+1} = (2m)^{2n+1} \qquad (42)$$

k, p and k_1, p_1 in (8), (41) are integers defined on the range $(-\infty, +\infty)$. So, we can do a substitution $p = -p_1 - 1$.

$$(2k_1 + 1)^{2n+1} + (2p_1 + 1)^{2n+1} = (2m)^{2n+1} \qquad (43)$$

where $k=k_1$. In this transformation, the range of parameters and equations' terms remains the same, that is $(-\infty < p < \infty)$, $(-\infty < p_1 < \infty)$, and so $(-\infty < (2p+1) < \infty)$ $(-\infty < (2p_1+1) < \infty)$. Thus, equation (42), which is (8), became equation (43). The substitution $p = -p_1 - 1$ is an equivalent one, because (i) it does not change the range of the substituted parameter, neither it changes the ranges of the terms, defined by these parameters; (ii) this is a one-to-one substitution.

Similarly, we can obtain equation (8) from (41), using substitution $p_1 = -p - 1$ in (41).

$$(2k_1 + 1)^{2n+1} + (2(-p-1)+1)^{2n+1} = (2m)^{2n+1} \qquad (44)$$

This transforms into equation (8).

$$(2k+1)^{2n+1} - (2p+1)^{2n+1} = (2m)^{2n+1} \qquad (45)$$

where $k_1 = k$.

Thus, (8) and (41), indeed, are equivalent equations.

Now, we should prove that if one of these equations has no solution, then the other equation also has no solution. For that, let us assume that equation (41) has no solution, while the equivalent equation (8) has a solution for the parameters (k_0, p_0, m_0), that is

$$(2k_0 + 1)^{2n+1} - (2p_0 + 1)^{2n+1} = (2m_0)^{2n+1} \qquad (46)$$

Doing an equivalent substitution $p_0 = -p_1 - 1$, one obtains

$$(2k_0+1)^{2n+1}+(2p_1+1)^{2n+1}=(2m_0)^{2n+1} \quad (47)$$

Equation (47) (which is the original equation (41)), accordingly, has a solution for the parameters (k_0, p_1, m_0). However, this contradicts to the assumption that (41) has no solution. So, the equivalent equation (8) also has no solution. Similarly, we can assume that (8) has no solution, while (41) has a solution, and find through similar contradiction that (41) has no solution.

This completes the proof of Lemma.

It follows from Lemma 13 that it is suffice to prove that only one of the equations, (8) or (41), has no solution, in order to prove that both equations have no solution. Previously, we found that (8) has no solution in integer numbers. So, according to Lemma 13, (41), which presents case 3 for equation (1), also has no solution.

5. Case 4

In this case, $a=2n$; $x=2p+1$; $y=2m$, $z=2k+1$. Equation (1) can be presented in two forms.

$$(2k+1)^{2n}-(2p+1)^{2n}=(2m)^{2n} \quad (48)$$

$$(2p+1)^{2n}+(2m)^{2n}=(2k+1)^{2n} \quad (49)$$

Because of the even power $a=2n$, we may consider (48) and (49) as defined on the set of integer numbers.

Let us consider (49). It can be rewritten as follows.

$$[(2p+1)^n]^2+[(2m)^n]^2=[(2k+1)^n]^2 \quad (50)$$

We will use Theorem 1 (p. 38) from Chapter 2 in [5]. The Theorem says the following: *All the primitive solutions of the equation* $x^2+y^2=z^2$ *for which y is even number are given by the formulae* $x=M^2-N^2$, $y=2MN$, $z=M^2+N^2$, *where M, N are taken to be pairs of relatively prime numbers, one of them even and the other odd and M greater than N.*

The Theorem was obtained with the assumption that $\{x,y,z\}$ are natural numbers. When $\{x,y,z\}$ are integers, following the logic of the original Theorem, one finds that *M* and *N* can be integers, and the condition "*M greater than N*" may not be fulfilled. In addition, we should also consider the second set $\{x,z\}$, when simultaneously $x=N^2-M^2$; $z=-M^2-N^2$ (for details, see Appendix A).

All solutions of (50) are defined as follows.

$$(2p+1)^n=(M^2-N^2)L;\ (2m)^n=2MNL;\ (2k+1)^n=(M^2+N^2)L \quad (51)$$

Here, *M* and *N* are pairs of relatively prime integer numbers, one of them even and the other is odd [5]. Substituting (51) into (50), we can see that by dividing both parts by L^2, it can be reduced to an equation, whose terms have no common divisor. So, if a solution of such an equation exists, it can be reduced to a primitive solution, and vice versa - any non-primitive solution can be obtained from a primitive solution. Thus, it is suffice to consider only primitive solutions. The same considerations are applicable to the second set $\{x,z\}$.

For the primitive solution, using the first and the third formulas from (51), we can write.

$$(2k+1)^n=(M^2+N^2) \quad (52)$$

$$(2p+1)^n=(M^2-N^2) \quad (53)$$

Equations (52) and (53) are independent. Indeed, there is no way to obtain one from another by transformations. (Formally, the independence can be proved considering the matrix rank of these equations in a linear representation).

Below, we assume that *M* and *N* are interchangeably equal to $(2c+1)$ and $(2d)$.

5.1. The case of odd n

Lemma 14: Equation

$$(2p+1)^{2n}+(2m)^{2n}=(2k+1)^{2n} \qquad (54)$$

has no solution in integer numbers, when n is odd.

Proof: Let $n=2q+1$. We will consider scenario 1 first, when $M=(2c+1)$, $N=(2d)$

$$(2k+1)^n=(M^2+N^2) \qquad (55)$$

Applying binomial expansion to the left part, one obtains.

$$\sum_{i=0}^{2q-1} C_i^{2q+1}(2k)^{2q+1-i}+(2q+1)(2k)+1=4c^2+4c+1+4d^2 \qquad (56)$$

It transforms into the following (after the division of both parts by two)

$$\sum_{i=0}^{2q-1} C_i^{2q+1}k(2k)^{2q-i}+(2q+1)k=2(c^2+c+d^2) \qquad (57)$$

The right part is even. The left part is odd when k is an *odd integer*, (k=2t+1, $-\infty<k<\infty$). In this case, (57) has no integer solution. (Note that the difference in parities of the left and right parts *does not depend* on the algebraic sign of k.)

Similarly, we consider equation

$$(2p+1)^n=(M^2-N^2) \qquad (58)$$

$$\sum_{i=0}^{2q-1} C_i^{2q+1}(2p)^{2q+1-i}+(2q+1)(2p)+1=4c^2+4c+1-4d^2 \qquad (59)$$

Transforming this equation and dividing both parts by two, one obtains

$$\sum_{i=0}^{2q-1} C_i^{2q+1}p(2p)^{2q-i}+(2q+1)p=2(c^2+c-d^2) \qquad (60)$$

This equation has no integer solution for *odd integer p* (*p=2s*+1, $-\infty<p<\infty$), since the left part is odd, and the right part is even in this case. Note that this difference in parities of the left and right parts in (60) does not depend on the algebraic sign of p, as well as of the sign of the right part.

Table 8 presents values of parameters when (54), and consequently (49), have no solution (the first scenario, row 1). Odd values of k=2t+1 and p=2s+1 correspond to $(2k+1)=(4t+3)$ and $(2p+1)=(4s+3)$, where t and s are integers.

Table 8. Values of parameters, when (49) has no solution because (52) or (53) have no solution.

Scenario	*M*	*N*	*k*	*p*	*2k+1*	*2p+1*
1	*2c+1*	*2d*	*2t+1*	*2s+1*	*4t+3*	*4s+3*
2	*2d*	*2c+1*	*2t+1*	*2s*	*4t+3*	*4s+1*

Since equations (55) and (58) are independent, the obtained values of $(4t+3)$ and $(4s+3)$ can be paired in equation (57) with any odd number. (When expressed with a factor of four, these are the numbers $(4s+1)$ and $(4s+3)$ for $(4t+3)$, and $(4t+1)$ and $(4t+3)$ for $(4s+3)$). Three found pairs, for which (49) has no solution, are shown in Table 9 in bold. The missing pair is [$(4t+1)$, $(4s+1)$].

Table 9. Found pairs of odd numbers (bold) for scenario 1, for which there is no integer solution.

	0	1	2	3	4
a	*k* *p*	***2t*** ***2s+1***	***2t+1*** ***2s***	*2t* *2s*	***2t+1*** ***2s+1***
b	*2k+1* *2p+1*	***4t+1*** ***4s+3***	***4t+3*** ***4s+1***	*4t+1* *4s+1*	***4t+3*** ***4s+3***

Let us consider scenario 2, when M is even and N is odd (row 2 in Table 8). Equation (55) has no solution for the *odd integer k* in this case (it is obvious that swapping M and N in (55) does not influence the previous result). Equation (58) becomes as follows.

$$\sum_{i=0}^{2q-1} C_i^{2q+1}(2p)^{2q+1-i} + (2q+1)(2p) + 1 = 4d^2 - 4c^2 - 4c - 1 \quad (61)$$

Transforming this equation and dividing both parts by two, one obtains

$$\sum_{i=0}^{2q-1} C_i^{2q+1} p(2p)^{2q-i} + (2q+1)p + 1 = 2(d^2 - c^2 - c) \quad (62)$$

This equation has no integer solution for *even integer p* $(-\infty < p < \infty)$ and for any algebraic sign of the right part. The obtained values of odd integer k and even integer p correspond to numbers $(4t+3)$ and $(4s+1)$, where t and s are integers. Since these numbers are obtained independently, each can be combined in pair with any odd number. The resulting combinations, shown in Table 10, are in bold. This time, we obtained all possible combinations of odd numbers, expressed with a factor of four. In other words, all possible combinations of pairs of odd numbers.

Table 10. Found pairs of odd numbers for scenario 2 (from Table 8), when (49) has no solution.

	0	1	2	3	4
a	*k* *p*	***2t*** ***2s+1***	***2t+1*** ***2s***	***2t*** ***2s***	***2t+1*** ***2s+1***
b	*2k+1* *2p+1*	***4t+1*** ***4s+3***	***4t+3*** ***4s+1***	***4t+1*** ***4s+1***	***4t+3*** ***4s+3***

So, we need to prove that equation (49) has no solution for the pair of odd numbers $[(4t+1),(4s+1)]$ from scenario 1. Let us substitute the pair $[(4t+3),(4s+3)]$ from Table 9, for which, as it was found, (49) has no solution, into this equation. One obtains

$$(4t+3)^{2n} + (2m)^{2n} = (4s+3)^{2n} \quad (63)$$

Equation (63) is defined on the set of integer numbers. So, we can use Corollary 2 and do equivalent substitutions of $(4t+3)$ by $(-(4t_1+1))$, and $(4s+3)$ by $(-(4s_1+1))$, where t_1 and s_1 are integers, thus obtaining an equivalent equation (equivalent in that regard that it produces all possible combinations of its terms the same as in (63)).

$$(-(4t_1+1))^{2n} + (2m)^{2n} = (-(4s_1+1))^{2n} \quad (64)$$

which, due to even power, can be rewritten as

$$(4t_1+1)^{2n} + (2m)^{2n} = (4s_1+1)^{2n} \quad (65)$$

Since (63) has no solution, and (65) is equivalent to (63), this means that (65) also has no solution. This proves that (49) has no solution for the pair $[(4t+1),(4s+1)]$. Now, we found that (49) has no solution for all possible pairs of odd numbers with a factor of four in Table 9, so that (49) has no solution for both scenarios.

The second set, when $x = N^2 - M^2$; $z = -M^2 - N^2$, is considered very similarly (Appendix A). This proves the Lemma.

5.2. Even n

Lemma 15: Equation $(2p+1)^{2n} + (2m)^{2n} = (2k+1)^{2n}$ has no solution in integer numbers when n is even.

Proof: Let $n = 2q$. Then, the above equation (which is equation (49)) can be presented as follows.

$$[(2k+1)^q]^4 - [(2p+1)^q]^4 = [(2m)^{2q}]^2 \quad (66)$$

According to Corollary 1 (p. 52) from Chapter 2 in [5], equation (66) has no solutions in natural numbers (because of the even power, this also means that (66) has no solution in integer numbers). The corollary is read as follows: *There are no natural numbers a, b, c such that* $a^4 - b^4 = c^2$.

Since $(2k+1)^q$, $(2p+1)^q$ and $(2m)^{2q}$ cannot be natural numbers, $(2k+1)$, $(2p+1)$ and $(2m)$ cannot be natural numbers too. Indeed, if one assumes that these are natural numbers, then, raised to appropriate powers, such numbers have to be natural numbers too, which contradicts to the aforementioned Corollary.

So, equations (48), (49) have no solution for even *n*.

The same result can be obtained using the property that there is no Pythagorean triangle, whose sides are squares. Indeed, we can rewrite (66) as follows.

$$[(2p+1)^q]^4 + [(2m)^q]^4 = [(2k+1)^q]^4 \quad (67)$$

Corollary 2 on p. 53, Chapter 2 in [5], says: *There are no natural numbers x, y, z satisfying the equation* $x^4 + y^4 = z^4$. This means that (66) and (67), and consequently (48) and (49) for even *n*, have no solution in natural numbers. However, because of the even power, the result is valid for integer numbers too. This proves the Lemma.

Thus, we proved that (48), (49) have no solution in integer numbers for odd and even *n*, that is for the case 4.

6. Conclusion

We found that in each of four cases, corresponding to equation (1), the appropriate equations have no solution in integer numbers. This means that (1) has no solution in integer numbers.

Introduced concepts and approaches can be applied to other problems of number theory.

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8. Appendix A

It was shown in [5] (p. 36) that if the set $\{x,y,z\}$ is a solution of equation $x^2 + y^2 = z^2$, then one number is even (say *y*), and two others are odd. The equation can be rewritten as

$$y^2 = (z+x)(z-z) \quad (1a)$$

Both factors in (1a) are even (*x* and *z* are odd). So, one can write

$$(z+x) = 2a; \; (z-x) = 2b \quad (2a)$$

where *a* and *b* are integers. Consequently.

$$z = a+b; \; x = a-b \quad (3a)$$

It was proved in [5] that *a* and *b* are relatively prime, (*a*,*b*)=1. In virtue of (2a) and (1a), for *y*=2*c*

$$c^2 = ab \quad (4a)$$

Since (*a*,*b*)=1, it follows from Theorem 8 in [5] (p. 13) that *a* and *b* are squares, that is $a = M^2$, $b = N^2$, where *M* and *N* are integers. Hence

$$x = M^2 - N^2; \; y = 2MN; \; z = M^2 + N^2 \quad (5a)$$

However, (4a) is also fulfilled if *simultaneously* $a = -M^2$, $b = -N^2$. For this the second set $\{x,y\}$

$$x = N^2 - M^2; \; z = -M^2 - N^2 \quad (6a)$$

Possibility of negative value of *y* in (5a) is irrelevant, since we use y^2 only. However, the values of *x* and *z*, defined by (6a), should be accounted for, in addition to their values in (5a).

Values (6a) can be studied very similarly to values (5a) in section 5.1. In notations of section 5.1, for scenario 1, we assume $M=(2c+1)$, $N=(2d)$.

$$(2k+1)^n = -(M^2+N^2) \tag{7a}$$

Applying binomial expansion to the left part and dividing by two, one obtains.

$$\sum_{i=0}^{2q-1} C_i^{2q+1} k(2k)^{2q-i} + (2q+1)k + 1 = -2(c^2+c+d^2) \tag{8a}$$

The right part is even. The left part is odd when k is an *even integer*, ($k=2t$, $-\infty < k,t < \infty$). In this case (8a), and consequently (7a), have no integer solution.

Similarly, we consider the equation

$$(2p+1)^n = (N^2 - M^2) \tag{9a}$$

Transforming this equation and dividing both parts by two, one obtains

$$\sum_{i=0}^{2q-1} C_i^{2q+1} p(2p)^{2q-i} + (2q+1)p + 1 = 2(d^2 - c^2 - c) \tag{10a}$$

This equation has no integer solution for *even integer p* ($p=2s$, $-\infty < p,s < \infty$), since the left part is odd, and the right part is even in this case.

Table 1a presents values of parameters, when the original equation (54) has no solution, for the first scenario. Even values of $k=2t$ and $p=2s$ correspond to $(2k+1)=(4t+1)$ and $(2p+1)=(4s+1)$, where $-\infty < t,s < \infty$.

Table 1a. Values of parameters, when (54) has no solution because (7a) or (9a) have no solution.

Scenario	*M*	*N*	*k*	*p*	*2k+1*	*2p+1*
1	*2c+1*	*2d*	*2t*	*2s*	*4t+1*	*4s+1*
2	*2d*	*2c+1*	*2t+1*	*2s*	*4t+3*	*4s+1*

Then, we should draw a table similar to Table 9. The only difference will be that the missing pair is $[(4t+3),(4s+3)]$.

For scenario 2, when M is even and N is odd (row 2 in Table 1a), equation (54) has no solution for *even integer k* (swapping M and N in (54) does not influence the previous result), $(-\infty < k < \infty)$. Equation (9a) transforms as follows.

$$\sum_{i=0}^{2q-1} C_i^{2q+1} p(2p)^{2q-i} + (2q+1)p = 2(c^2 + c - d^2) \tag{11a}$$

It has no integer solution for *odd integer* $p=2s+1$ $(-\infty < p,s < \infty)$, and for any algebraic sign in the right part. Even integer k and odd integer p correspond to numbers $(4t+1)$ and $(4s+3)$, where t and s are integers. These values are obtained independently. So, each can be combined in pair with any odd number, thus covering all possible pairs of odd numbers expressed with a factor of four (similar to Table 10).

That equation (54), and consequently (49), have no solution for the pair of odd numbers $[(4t+3),(4s+3)]$ (the missing pair in scenario 1), is proved similarly to the missing pair $[(4t+1),(4s+1)]$ in Table 9 (equations (63-65). So, (49) has no solution for all possible pairs of odd numbers with a factor of four for both scenarios, when x and z are defined by (6a).

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