Proof of Fermat Last Theorem based on successive presentations of pairs of odd numbers

Yuri K. Shestopaloff *

A proof of Fermat Last Theorem (FLT) is proposed. FLT was formulated by Fermat in 1637, and proved by A. Wiles in 1995. Here, a simpler approach is studied. The initial equation *x^n + y^n = z^n* is considered not in natural, but in integer numbers. It is subdivided into four equations based on parity of terms and their powers. Then, each such equation is studied separately. One equation is considered using presentations of pairs of odd numbers with a successively increasing factor of 2^*r*. The other equation is equivalent to the first one with regard to absence of solution. The third equation is considered using binomial expansion of its terms. The fourth equation uses presentation of pairs of odd numbers with a factor of four, and transformation to the second power. All four equations have no solution in integer numbers. Thus, the original FLT equation has no solution too.

2010 Mathematics Subject Classification (MSC) 11D41 (Primary)
Keywords: Diophantine equations; integer numbers; binomial expansion; parity

Table of contents

*Mail: to user "shes" padded with a number 13^2 on server yahoo.ca

(Version 4). http://doi.org/10.5281/zenodo.3786524.

1. Introduction

One of the reasons that FLT still attracts people is that the known solution [1] is too complicated. Earlier, a general approach for analysis of Diophantine equations, and FLT equation in particular, was proposed in [2]. Here, it is presented with some additions.

2. FLT sub-equations

Let us consider an equation.

$$x^a + y^a = z^a \tag{1}$$

The power *a* is a natural number $a \geq 3$. Unlike in the original FLT equation, here, *x*, *y*, *z* belong to the set of integer numbers **Z**. We assume that variables *x*, *y*, *z* have no common divisor. Indeed, if they have such a divisor *d*, both parts of equation can be divided by d^a, so that the new variables $x_1 = x/d$, $y_1 = y/d$, $z_1 = z/d$ will have no common divisor. We will call such a solution, without a common divisor, a *primitive solution*. From the formulas above, it is clear that any non-primitive solution can be reduced to a primitive solution by dividing by the greatest common divisor. The reverse is also true, that is any non-primitive solution can be obtained from a primitive solution by multiplying the primitive solution by a certain number. So, it is suffice to consider primitive solutions only.

Values *x*, *y*, *z* in (1) cannot be all even. Indeed, if this is so, this means that the solution is not primitive. By dividing it by the greatest common divisor, it can be reduced to a primitive solution. Obviously, *x*, *y*, *z* cannot be all odd. So, the only possible combinations left are when *x* and *y* are both odd, then *z* is even, or when one of the variables, *x* or *y*, is even, and the other is odd. In this case, *z* is odd. Thus, equation (1) can be subdivided into the following cases, which cover all permissible permutations of equation's parameters.

1. $a = 2n$; $x = 2k+1$; $y = 2p+1$. Then, *z* is even, $z = 2m$.
2. $a = 2n+1$; $x = 2p+1$; $y = 2m$. Then, *z* is odd, $z = 2k+1$.
3. $a = 2n+1$; $x = 2k+1$; $y = 2p+1$. Then, *z* is even, $z = 2m$.
4. $a = 2n$; $x = 2p+1$; $y = 2m$. Then, *z* is odd, $z = 2k+1$.

3. Case 1

Let us assume that (1) has a solution for the following terms.

$$(2k+1)^{2n} + (2p+1)^{2n} = (2m)^{2n} \tag{2}$$

Binomial expansion of the left part of (2) is as follows.

$$\left[\sum_{i=0}^{2n-2} C_i^{2n} (2k)^{2n-i} + 2n(2k) + 1\right] + \left[\sum_{i=0}^{2n-2} C_i^{2n} (2p)^{2n-i} + 2n(2p) + 1\right] = (2m)^{2n} \tag{3}$$

Transforming (3), we obtain

$$\sum_{i=0}^{2n-2} C_i^{2n} \left[(2k)^{2n-i} + (2p)^{2n-i}\right] + 4n(k+p) + 2 = (2m)^{2n} \tag{4}$$

The lowest power of terms 2*k* and 2*p* in the sum is $2n-(2n-2)=2$. In other words, all summands in the sum are even, having a factor of two in a degree of two or greater. The second term has a factor of four. Let us divide both parts of (4) by two. We obtain.

$$\sum_{i=0}^{2n-2} C_i^{2n} \left[k(2k)^{2n-i-1} + p(2p)^{2n-i-1}\right] + 2n(k+p) + 1 = m(2m)^{2n-1} \tag{5}$$

The first two summands in the left part of (5) are even. So, the left part presents the sum of two even terms and the number one. Thus, the left part is odd.

Since we consider the values of $2n \geq 4$, the power $(2n-1) \geq 3$, so that the right part is even. So, (5) presents an equality of the odd and even integer numbers, which is impossible. Thus, the initial assumption that (2) has a solution is invalid. So, it has no solution in *integer* numbers, since the parity of the right and left parts of (5) does not depend on algebraic signs of variables.

4. Cases 2 and 3

4.1. Sums with merging terms

We will need several Lemmas for these cases.

Lemma 1: *Each non-negative integer number n can be presented in a form*

$$n = \sum_{i=0}^{r} 2^i K_i \qquad (6)$$

where $K_i = \{0,1\}$.

Proof: We will use method of mathematical induction. Lemma is true for numbers $3 = 2^1 + 1$; $4 = 2^2 + 0$; $5 = 2^2 + 1$. Let us assume that (6) is valid for number *n*. Then, the next number $n+1$ is

$$n+1 = \sum_{i=0}^{r} 2^i K_i + 1 = (K_0 + 1) + \sum_{i=1}^{r} 2^i K_i \qquad (7)$$

If $K_0 = 0$, then (7) converges to the form (6) as follows.

$$n+1 = \sum_{i=0}^{r} 2^i K_{1i} \qquad (8)$$

where $K_{10} = 1$ for $i = 0$, $K_{1i} = K_i$ for $i \geq 1$.

If $K_0 = 1$, we may have potentially a "falling domino" effect, when the previous term merges with the next one, thus transforming this next term into a term, whose power of two is greater by one.

$$n+1 = (K_0 + 1) + \sum_{i=1}^{r} 2^i K_i = (1+1) + \sum_{i=1}^{r} 2^i K_i = 2^1 + \sum_{i=1}^{r} 2^i K_i \qquad (9)$$

So, $K_{10} = 0$.

In (9), we should merge the summand 2^1 with the term $2^1 K_1$, which becomes $2^1(1+K_1)$. For the values of $K_1 = \{0,1\}$, this new term can be equal to accordingly $\{2, 2^2\}$. For the first value, $K_{11} = 1$. The second value of 2^2 should be merged with the term $2^2 K_2$, while K_{11} becomes zero, since in this case we don't have non-zero terms with 2^1. The merge with the term $2^2 K_2$ results in a term $2^2(1+K_2)$. For the values of $K_2 = \{0,1\}$, it accordingly produces the values of $K_{12} = \{1,0\}$, Zero value occurs because when $K_2 = 1$, we have $2^2(1+K_2) = 2^3$, and the term should be merged with the term $2^3 K_3$.

Formally, one can assume that such a merge happened at an arbitrary step with the term $2^s K_s$. The resultant merged term is $2^s(1+K_s)$. Then, for $K_s = \{0,1\}$ we have $K_{1s} = \{1,0\}$. For $K_s = 1$, the new term should be merged with the term $2^{s+1} K_{s+1}$, thus making $K_{1s} = 0$.

So, the form (8) is preserved in a general case as well. Thus, the assumption that the procedure is repeated at an arbitrary step led to the same values of K_{1s} for the corresponding values of K_s, as it was obtained for particular powers, and also led to a merge at the next step when $K_s = 1$.

The procedure is repeated until the term with a zero value of K_i is met, or a new non-zero term $2^{r+1}K_{r+1}$ is created at the very end, by merging the last term $2^r K_r$ in the sum, and the value of 2^r obtained from the merge of previous terms. In all instances, the form (8) is preserved. So, the sum (8) eventually becomes

$$n+1=\sum_{i=0}^{r+1} 2^i K_{1i} \tag{10}$$

where $K_{1,r+1}=\{0,1\}$.

So, the presentation (6) is valid for number (n+1). According to principle of mathematical induction, this means the validity of (6) for all numbers. This proves the Lemma.

Corollary 1: *Any negative integer number n can be presented in a form*

$$n=\sum_{i=0}^{r} 2^i K_i \tag{11}$$

where $K_i=\{-1,0\}$.

Proof: The proof follows from Lemma 1. For $n < 0$, we can write

$$n=-|n|=-\sum_{i=0}^{r} 2^i K_i=\sum_{i=0}^{r} 2^i K_{1i} \tag{12}$$

where $K_{1i}=-K_i$. According to Lemma 1, $K_i=\{0,1\}$. So, $K_{1i}=\{-1,0\}$. This proves the Corollary.

Corollary 2: *Any integer number n can be presented in a form*

$$n=\sum_{i=0}^{r} 2^i K_i \tag{13}$$

where $K_i=\{-1,0,1\}$.

Proof: Suppose $n > 0$, and in presentation (13) $K_{i_0}=1$, $0\le i_0\le r$, so that the appropriate term is $2^{i_0}K_{i_0}$. We can add and subtract the value of 2^{i_0}, such that the term becomes $[(2^{i_0}K_{i_0}+2^{i_0})-2^{i_0}]$. Then, we can assume that $K_{i_0}=-1$, while the summand $(2^{i_0}K_{i_0}+2^{i_0})=2\times 2^{i_0}$ should be added to the term $2^{i_0+1}K_{i_0+1}$, in the same way, as it was done in Lemma 1.

Similarly, when $n < 0$, one can obtain a positive coefficient $K_{i_0}=1$. This proves the Lemma.

4.2. Presentation of equation (1) for cases 2 and 3

For the case 3, we have $a=2n+1$; $x=2k_1+1$; $y=2p_1+1$. Then, (1) transforms to

$$(2k_1+1)^{2n+1}+(2p_1+1)^{2n+1}=(2m)^{2n+1} \tag{14}$$

For the case 2, the power $a=2n+1$; $x=2p+1$; $y=2m$. Then, z is odd, $z=2k+1$.

$$(2p+1)^{2n+1}+(2m)^{2n+1}=(2k+1)^{2n+1}$$

It can be rewritten in a form

$$(2k+1)^{2n+1}-(2p+1)^{2n+1}=(2m)^{2n+1} \tag{15}$$

We can present m as $m=2^{\mu}m_1$, where $\mu\ge 0$, and m_1 is an odd number. Then, (15) transforms to

$$(2k+1)^N-(2p+1)^N=2^{N(\mu+1)}m_1{}^N \tag{16}$$

where $N=2n+1$.

Note that the value $r_t = N(\mu+1)$ is a threshold one. If we divide both parts of the equation by 2^r, then for $r < r_t$ the right part is even, for $r = r_t$ it is odd, and for $r > r_t$ it is rational.

The left part of (16) can be presented in a form (13). It can be rewritten as follows.

$$2^{i_0} K_{i_0} + \sum_{i=i_0+1}^{r} 2^i K_i = 2^{r_t} m_1^N \tag{17}$$

Here, i_0 is a minimum value of the index, for which $K_i \neq 0$.

The following Lemma gives an example of using the presentation from Lemma 1.

Lemma 2: *Equation (16), and consequently (15), has no solution in integer numbers for* $N \geq 3$ *when* $i_0 \neq r_t$, where $N = 2n+1$.

Proof: Dividing both parts of (17) by 2^{i_0}, one obtains

$$K_{i_0} + \sum_{i=i_0+1}^{r} 2^{i-i_0} K_i = 2^{r_t - i_0} m_1^N \tag{18}$$

Since $i > i_0$, all terms in the sum in the left part are either zeros or even, so that the sum is even, while $K_{i_0} = \{-1,1\}$ is odd. So, the left part is odd. Excluding $i_0 \neq r_t$, two scenarios are possible.

(a) $r_t - i_0 > 0$. In this case, the right part is even, while the left part is odd. So, (18), and consequently (16), has no solution.

(b) $r_t - i_0 < 0$. Since m_1 is odd, m_1^N is odd too. Thus, the right part $m_1^N / 2^{i_0 - r_t}$ is rational, while the left part is an odd integer. So, (18) and the original equation (16) have no solution in integer numbers. This proves the Lemma.

In the following, we will use a presentation of pairs of odd numbers with a factor of 2^r, where $r \geq 1$, whose properties are considered below.

4.3. Presentation of pairs of odd numbers with a factor of 2^r

Let us consider an infinite set of pairs of odd integer numbers $\{(2k+1), (2p+1)\}$, where *k* and *p* are *integers*. Each set $\{(2k+1)\}$ and $\{(2p+1)\}$ can be accordingly presented with a factor of four as sets $\{(4t+1), (4t+3)\}$ and $\{(4s+1), (4s+3)\}$, where t and s are integers, $(-\infty < t < \infty)$, $(-\infty < s < \infty)$. Table 1 shows four possible pairs of odd numbers, expressed with a factor of four, composed from these terms. Note that such a presentation produces a *complete set* of pairs of odd integer numbers, since we considered all possible combinations of parities of *k* and *p*. (The completeness will be proved later for a general case of presentation with a factor 2^r).

We can continue presentations of pairs of odd numbers using a successively increasing factor of 2^r. Initial pairs for the next presentation level with a factor of 2^3 are pairs in cells (2,1)-(2,4). Table 2 shows the presentation with a factor of 2^3 for two pairs from cells (2,3), (2,4) in Table 1. Note that index '3' corresponds to power *r*=3 in a presentation factor 2^r. Such correspondence of the index to the power of two in a presentation factor will be used throughout the paper.

Table 1. All possible pairs of odd numbers, expressed with a factor of four.

	0	1	2	3	4
1	k p	$2t_2$ $2s_2+1$	$2t_2+1$ $2s_2$	$2t_2$ $2s_2$	$2t_2+1$ $2s_2+1$
2	$2k+1$ $2p+1$	$\mathbf{4t_2+1}$ $\mathbf{4s_2+3}$	$\mathbf{4t_2+3}$ $\mathbf{4s_2+1}$	$4t_2+1$ $4s_2+1$	$4t_2+3$ $4s_2+3$

Table 2. Pairs of odd numbers, expressed with a factor of 2^3, corresponding to initial pairs $[4t_2+1, 4s_2+1]$, $[4t_2+3, 4s_2+3]$.

	0	1	2	3	4
1	t_2 s_2	$2t_3$ $2s_3+1$	$2t_3+1$ $2s_3$	$2t_3$ $2s_3$	$2t_3+1$ $2s_3+1$
2	$4t_2+1$ $4s_2+1$	$\mathbf{8t_3+1}$ $\mathbf{8s_3+5}$	$\mathbf{8t_3+5}$ $\mathbf{8s_3+1}$	$8t_3+1$ $8s_3+1$	$8t_3+5$ $8s_3+5$
3	$4t_2+3$ $4s_2+3$	$\mathbf{8t_3+3}$ $\mathbf{8s_3+7}$	$\mathbf{8t_3+7}$ $\mathbf{8s_3+3}$	$8t_3+3$ $8s_3+3$	$8t_3+7$ $8s_3+7$

4.3.1. The concept of the proof

Each pair of terms in Tables 1 and 2, and in subsequent presentations, defines an infinite set of pairs of odd numbers. All such pairs of terms at each presentation level produce the whole set of pairs of odd numbers. The infinite sets, defined by pairs of terms, are unique and do not intersect (this will be proved later). At each presentation level, equation (15), corresponding to certain pairs of terms, has no solution. Such "no solution" pairs accumulate through subsequent presentation levels, producing a greater and greater total fraction of pairs of terms, for which (15) has no solution. In the limit, this total "no solution" fraction becomes equal to one, which would mean that (15) has no solution for all possible pairs of odd numbers.

At the first presentation level, with a factor of 2^2, we begin with the *whole set* of all possible pairs of odd integer numbers, represented as a pair of terms $\{(2k+1), (2p+1)\}$, k and p are integers. Equations, corresponding to a *half* of pairs from this set, have no solution. Then, the half of pairs, for which equations have no solution, is set aside (the "no solution" fraction f_{2ns}). The remaining pairs compose an "uncertain" fraction, for which solution is uncertain. The "uncertain" fraction is equal to $f_{2u} = 1 - f_{2ns} = 1/2$. The following example illustrates the approach. (The actual algorithm is different, but the general idea is similar.)

Equation (15) can be transformed as a difference of two numbers in odd powers.

$$2(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^{i} = (2m)^{2n+1}$$

Dividing both parts by two, one obtains

$$(k-p)\sum_{i=0}^{2n}(2k+1)^{2n-i}(2p+1)^{i} = m(2m)^{2n}$$

Here, the sum is odd as an odd quantity of odd numbers. If the factor $(k-p)$ is odd, then the left part is odd, while the right part is even (since $n > 0$). This means that there is no solution in this case. The value of $(k-p)$ is odd when one of the terms is odd and the other is even, which are the values of k and p in cells (1,1), (1,2) in Table 1, corresponding to pairs $[4t+1, 4s+3]$ and $[4t+3, 4s+1]$.

The change of algebraic signs of k and p does not change the parity of the left part. So, the result is valid for *integer* numbers k and p. When $(k-p)=0$, the left part is zero, while the right part is an integer, so that there is no solution in this case.

When $(k-p)$ is even, both parts of equation are even, and solution is uncertain. This corresponds to values of k and p in cells (1,3), (1,4) in Table 1, with corresponding pairs of terms $[4t+1, 4s+1]$ and $[4t+3, 4s+3]$. These "uncertain" pairs should be used as initial pairs for the next presentation level with a factor of 2^3 (Table 2).

At the presentation level with r=3, we again find that a half of pairs (the ones in bold in Table 2) correspond to a "no solution" fraction, which is found as $f_{3ns}=f_{2u}\times 1/2=1/4$. The fraction of remaining uncertain pairs is accordingly $f_{3u}=f_{2u}-f_{3ns}=1/2-1/4=1/4$. Therefore, two presentation levels produce the following total fraction of pairs, for which (15) has no solution, $F_{3NS}=f_{2ns}+f_{3ns}=1/2+1/4=3/4$. The "uncertain" fraction $f_{3u}=1-3/4=1/4$, gives initial pairs for the next presentation level (with r=4), and so forth, until in infinity the "no solution" fraction accumulates to one. (The real situation with the "no solution" fractions is slightly more complicated, since such fractions can be greater than 1/2, when equations, corresponding to certain pairs of terms, have no solution for all pairs, and such a branch is closed. However, the total "no solution" fraction is still equal to one in the limit.)

Above, we used the term "pair" in two connotations - as a pair of terms, such as $[4t_2+1, 4s_2+1]$, which defines an infinite set, and as a pair of particular odd numbers within a set. In the following, for brevity, the term "pair" will mean a pair of terms, generating a set, unless it is clear from context that particular pairs of odd numbers are meant. In less obvious situations, the elements of a set, particular pairs, will be called "pairs of odd numbers".

4.3.2. Properties of presentations of pairs of odd number with a factor of 2^r

Lemma 3: *Successive presentations of odd numbers with a factor of* 2^r *cannot contain a free coefficient greater or equal to* 2^r.

Proof: Presentations of odd numbers with factors 2^2 and 2^3 satisfy this requirement. Let us assume that this is true for a presentation level r, that is the free coefficient v in a term $(2^r t_r + v)$ satisfies the condition $v<2^r$. At a presentation level (r + 1), this term is presented as $(2^{r+1}t_{r+1}+2^r+v)$ or $(2^{r+1}t_{r+1}+v)$. In the latter term, the condition is already fulfilled. In the first term, $2^r+v<2^r+2^r=2^{r+1}$, since $v<2^r$ is true for level r by assumption. So, assuming that the condition is fulfilled at the level r, we obtained that it is also fulfilled at the level (r + 1). According to principle of mathematical induction, this means the validity of the assumption. This proves the Lemma.

The number of pairs grows for successive *complete* presentations in a geometrical progression with a common ratio of *four*, since each initial pair produces four new pairs at the next presentation level. (Each new pair corresponds to one of the four possible parity combinations of input parameters, like t_2, s_2 in Table 2, whose parity is expressed through t_3, s_3.)

For the following, we need to prove that (a) such a presentation produces the whole set of pairs of odd numbers at each level; (b) the presentation is unique, that is two different pairs of odd numbers cannot produce the same pair of odd numbers at higher levels of presentation.

Lemma 4: *Successive presentations of pairs of odd numbers with a factor of* 2^r, $r \geq 2$, *produce the same set of pairs of odd numbers at each presentation level, both for subsets of such pairs, and for the whole set of pairs of odd integer numbers. Such presentations are unique, that is two different pairs of odd numbers from the previous levels cannot correspond to the same pair at higher presentation levels.*

Proof: The equivalency of sets of pairs of odd numbers at each presentation level r follows from the fact that each next presentation level (r+1) is obtained prom the previous one through branching of each initial pair (from level r) into all four possible combinations of parities of parameters t_r and s_r, so that there are no any other possible combinations of parities. This means that any pair from level r is fully represented at level (r+1), although in the form of four pairs. Indeed, the initial term $(2^r t_r + v)$ can be presented at level (r+1) only in two forms (for even and odd values of t), that is as $2^r(2t_{r+1}) + v = 2^{r+1}t_{r+1} + v$, or $2^r(2t_{r+1} + 1) + v = 2^{r+1}t_{r+1} + 2^r + v$. Similarly, the term $(2^r s_r + w)$ can also be represented in the same two forms only. So, only four combinations of pairs, containing both t and s parameters, are possible. These combinations are unique, because the combinations of free coefficients are unique, which are as follows: [v,w], $[2^r + v, w]$, [v, $2^r + w$], $[2^r + v, 2^r + w]$. Consequently, no intersection of thus defined sets of pairs of odd numbers is possible.

The reverse is also true, that is four pairs at presentation level (r+1) converge to one initial pair at lower level r. Indeed, two terms with parameter t converge to the same term $(2^r t_r + v)$.

$$2^{r+1}t_{r+1} + v = 2^r(2t_{r+1}) + v = 2^r t_r + v$$

$$2^{r+1}t_{r+1} + 2^{r+1} + v = 2^r(2t_{r+1} + 1) + v = 2^r t_r + v$$

where $t_r = \{2t_{r+1}, 2t_{r+1} + 1\}$.

Note that the same convergence to a single term can be obtained for a general case of presenting two terms at level (r+1) using Lemma 1, and then transforming them to level r.

$$2^{r+1}t_{r+1} + \sum_{i=1}^{r} 2^i K_i + 1 = 2^r(2t_{r+1} + K_r) + \sum_{i=1}^{r-1} 2^i K_i + 1 = 2^r t_r + \sum_{i=1}^{r-1} 2^i K_i + 1 \qquad (19)$$

Since $K_r = \{0,1\}$, we obtain the same set $t_r = \{2t_{r+1}, 2t_{r+1} + 1\}$.

Similarly, one can convert two possible terms with parameter s at level (r+1) to a single term with parameter s at level r. So, four pairs at level (r+1), indeed, converge to one pair $[2^r t_r + v, 2^r s_r + w]$ at level r. Therefore, such transformations from level r to level (r+1) and backward include all possible, while non-intersecting, pairs of terms. This means that presentations of pairs of odd numbers at these two levels are equivalent, that is for each pair of odd numbers at level r there is only one pair of odd numbers at level (r+1), and vice versa.

Another approach involves arrangements with repetitions and their extension to infinite sets. For that, we need to use the notion of *density* for finite, and *asymptotic density* for infinite, countable sets [3,4]. In the last case, the asymptotic density should be defined as the limit of the ratio $A(n)/n$, that is of the count of certain elements in a given set of n elements, when $n \to \infty$, if such a limit exists.

Let us consider a set of 2n odd numbers at level r. The number of arrangements of two (the number of pairs) with repetitions is

$$P_2^{2n} = (2n)^2 = 4n^2 \qquad (20)$$

At the next level of presentation (r+1), the density of the set of odd numbers for each term is twice as less, that is n (since the presentation factor increases twice, and so the distance between the numbers in the set increases twice as well). For instance, the presentation (4t+1) misses each next number compared to the presentation (2k+1). Indeed, on a finite interval [a,b], the quantity Q_r of numbers $(2^r + v)$ is equal to $Q_r = [(b-a)/2^r]$. The quantity of numbers $(2^{r+1} + v_1)$ is $Q_{r+1} = [(b-a)/2^{r+1}]$. Here, the square brackets denote an integer part. The density $\alpha = Q_{r+1}/Q_r$ can be equal only to 1/2, or $(h+1)/(2h+1)$, where $Q_r = 2h+1$. For instance, in the interval [4,14], the number of multiples of four is three, the number of even numbers is six, and so the ratio is 1/2. On the interval [4,16], the ratio is 4/7. In the limit, when $h \to \infty$, and the left *a* and right *b* boundaries of the interval go to infinities, that is $a \to -\infty$, $b \to \infty$, the asymptotic density $\lim_{h\to\infty}(h+1)/(2h+1) = 1/2$. It is in this sense of asymptotic density that we can compare infinite countable sets of integer numbers. (The notion of cardinality is not applicable in this case.) So, the density of sets, defined by each pair at the next presentation level, is twice as less compared to the density of the set at the previous presentation level. Thus, for a finite set 2*n*, the size of each new set is equal to *n*. Accordingly, for each new pair out of four (such as pairs in cells (2,1)-(2,4) in Tables 1 and 2), the number of arrangements of two with repetitions is $P_2^n = n^2$, and the total number of arrangements for all four new pairs is $4P_2^n = 4n^2$, that is the same as for the original set in (20). The obtained equality is valid for any *n*, so that it remains true for subsets of odd numbers when $n \to \infty$. So, both presentations, at levels *r* and (*r*+1), produce the same (in size) set of pairs of odd integer numbers.

The uniqueness of presentations with factors 2^r can be proved as follows. Let us assume that two pairs from a presentation level r_0 produced the same pair of odd numbers in the next presentation level (r_0+1). That would mean that the number of arrangements of two for one of the set of odd numbers at the level (r_0+1) will be $P_2^n = n^2 - 1$, and the total number of arrangements of two for this level will be $(4n^2 - 1) \neq 4n^2$. The same inequality remains true for any *n* when $n \to \infty$. Thus, we obtained that the number of arrangements in two different presentation levels is different, which contradicts to the earlier proved statement that these numbers are the same. This means that our assumption is invalid, and the presentations at each level contain unique elements.

So, the transformations from the lower presentation levels to higher levels, and from a higher level downward, are unique, one-to-one, transformations. This proves the Lemma.

Corollary 3: The asymptotic densities of four subsets of pairs at level (*r*+1) are the same.

Proof: It was found in Lemma 4 that the number of arrangements of two with repetitions for each of four subsets at the next presentation is $P_2^n = n^2$, for any large *n* when $n \to \infty$. Thus, the ratio of numbers of elements in these subsets is one for any large *n*, and so it remains one when $n \to \infty$. This proves the Corollary.

It follows from Lemma 4 and Corollary 3 that (a) for any pair of odd numbers in one presentation level there is one and only one corresponding pair in another presentation level, which means that the sets of pairs of odd numbers at two different levels are the same (provided all pairs from the previous presentation level become initial pairs for the next one); (b) asymptotic densities of sets, corresponding to four pairs at level (*r*+1), obtained from one pair at level *r*, are the same; (c) these four sets are non-intersecting. Since these four sets do not intersect, and together define the same set of pairs of odd numbers, as at the previous level, this means that the asymptotic density of a union of four sets is the same as the density *D* of the set, produced by the initial pair. Since all four sets have the same density, it means that the asymptotic density of each of them is equal to (1/4)*D*. In other words, these four sets are equivalent in terms of densities. This opens another venue of considering accumulation of densities instead of fractions of pairs. However, we use the last one.

Note that such successive presentations can be used for any number, not only for the odd ones.

4.4. Datasets of integer numbers with a factor of four, symmetrical relative to zero

Lemma 5: *The dataset* $\mathbf{Z_1}$ *=*{ $4s+1$ }, *defined on the set of integer numbers* $(-\infty < s < \infty)$ *is symmetrical to the dataset* $\mathbf{Z_3}$ *=*{ $(4s_1+3)$ }, $(-\infty < s_1 < \infty)$ *relative to zero, meaning that for each number w in* $\mathbf{Z_1}$ *there is one and only one number (-w) in the dataset* $\mathbf{Z_3}$ *, and vice versa (meaning the swap of datasets).*

Proof: Let us consider $s_1 = -(s+1)$. Then, we can write the following for $\mathbf{Z_3}$.
$(4s_1+3) = (4(-s-1)+3) = (-4s-1) = -(4s+1)$
Assuming $s_1 = -s-1$, we obtain for $\mathbf{Z_1}$.
$(4s_1+1) = (4(-s-1)+1) = -(4s+3)$
or $(4s+3) = -(4s_1+1)$
Since the above transformations are one-to-one, it means one-to-one relationship between any number in one dataset and its algebraic opposite in another dataset. Note that values of s and s_1 have the same ranges of definition, so that they are interchangeable in the above expressions. This proves the Lemma.

Fig. 1 illustrates the algebraically opposite numbers in two datasets.

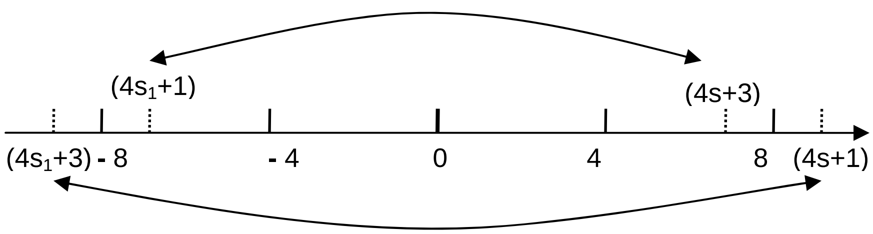

Fig. 1. Symmetrical subsets of odd integer numbers, expressed with a factor of four.

The symmetry of obtained sets can be illustrated by congruencies. Indeed, $4s+1 \equiv 1(\bmod 4)$, while the congruency for the matching value $(-4s_1-3) \equiv -3(\bmod 4)$ transforms to $(-4s_1+1) \equiv 1(\bmod 4)$, so that both values are congruent to number one.

The following corollary follows from Lemma 5.

Corollary 4: *Dataset* $\mathbf{Z_1}$ ={ $4s+1$ }, $(-\infty < s < \infty)$ *can be substituted by dataset* **-**$\mathbf{Z_3}$ ={ $-(4s_1+3)$ }, $(-\infty < s_1 < \infty)$, *and vice versa.*

4.5. Properties of equations, corresponding to pairs of odd numbers with a factor of 2^r

This section introduces an equation, to which all equations, corresponding to pairs of odd numbers, can be transformed, and explores its properties.

Lemma 6: *Let us consider an equation*

$$(2^r t_r + v)^N - (2^r s_r + w)^N = 2^{N(\mu+1)} m_1^N \qquad (21)$$

where t_r and s_r are integers; N=2n+1; m_1 is odd; v, w are positive odd (possibly equal) numbers, obtained through successive presentations of pairs of odd numbers. Then, for any $r \geq 3$, such equations can be transformed to the following form

$$(t_r - s_r)A_r = A_r c / 2^r + m_1^N / 2^{r-r_t} \quad (22)$$

where $A_r = \sum_{i=0}^{N-1}(2^r t_r + v)^{N-1-i}(2^r s_r + w)^i$ *is an odd integer; c is an integer;* $r_t = N(\mu + 1)$.

Proof: Equation (21) presents equation (15), rewritten for a presentation with a factor of 2^r.

$$[2^r(t_r - s_r) + (v - w)]\sum_{i=0}^{N-1}(2^r t_r + v)^{N-1-i}(2^r s_r + w)^i = 2^{N(\mu+1)} m_1^N \quad (23)$$

The sum in (23) is odd, because it presents the sum of odd number of odd values. Let us denote it

$$A_r = \sum_{i=0}^{N-1}(2^r t_r + v)^{N-1-i}(2^r s_r + w)^i$$

Since *v* and *w* are odd, their difference is even. Also, in successive presentation of odd numbers, according to Lemma 3, $v < 2^r$, $w < 2^r$. Since both values are positive, their absolute difference is also less than 2^r. According to Lemma 1, (*v* - *w*) can be presented as a sum of powers of two with coefficients, having the same algebraic sign. Since $|v - w| < 2^r$, such a sum cannot contain a summand with a power greater than 2^{r-1}, when all coefficients K_i have the same algebraic sign.

$$[2^r(t_r - s_r) + \sum_{i=1}^{r-1} 2^i K_i]A_r = 2^{N(\mu+1)} m_1^N \quad (24)$$

Then, (24) can be rewritten as follows.

$$2^r(t_r - s_r)A_r = -\left(\sum_{i=1}^{r-1} 2^i K_i\right)A_r + 2^{N(\mu+1)} m_1^N \quad (25)$$

Let us denote $c = -\sum_{i=1}^{r-1} 2^i K_i$. Since $c = w - v$, when $w = v$ (that is free coefficients are equal), $c = 0$. When $w \neq v$, the value of $c \neq 0$. Dividing both parts of (25) by 2^r, and taking into account that $r_t = N(\mu + 1)$, we obtain

$$(t_r - s_r)A_r = A_r c / 2^r + m_1^N / 2^{r-r_t} \quad (26)$$

This proves the Lemma.

Lemma 7: *If $c \neq 0$ in (26), then $A_r c / 2^r$ is a rational number.*

Proof: It was indicated in Lemma 6 that when free coefficients *w* and *v* are unequal, $c \neq 0$. According to Lemma 1, we can always use a presentation $\sum_{i=1}^{r-1} 2^i K_i$ with the range of values $K_i = \{0,1\}$, $1 \leq i \leq r - 1$, when $c > 0$, and $K_i = \{-1,0\}$ when $c < 0$. Then

$$|c| = \left|\sum_{i=1}^{r-1} 2^i K_i\right| \leq \sum_{i=1}^{r-1} 2^i = 2(2^{r-1} - 1)/(2 - 1) = 2^r - 2 \quad (27)$$

(Here, we substituted the sum of a geometrical progression with a common ratio of two and the first term of two.) Accordingly

$$|A_r c / 2^r| \leq |A_r|(1 - 1/2^{r-1}) \quad (28)$$

Dividing inequality (28) by a positive number $|A_r|$, one obtains

$$|c/2^r| \le (1-1/2^{r-1}) \qquad (29)$$

Thus, $c/2^r$ is a rational number. The term A_r is an odd number, which, consequently, contains no dividers of two. In turn, this means that $A_r c/2^r$ is a rational number. This proves the Lemma.

Lemma 8: *Equation (22) has no solution for pairs with unequal free coefficients when* $r \le N(\mu+1)$*, while solution is uncertain for pairs with equal free coefficients.*

Proof: For $r \le N(\mu+1) = r_t$, the term $2^{r_t - r} m_1^N$ in (22) is an integer. According to Lemma 7, the summand $A_r c/2^r$ is rational for pairs with unequal free coefficients. So, the right part of (22) is rational. On the other hand, the left part is an integer when $(t_r - s_r) \ne 0$. This means that (22) has no solution in this case. When $(t_r - s_r) = 0$, (22) presents equality of zero (in the left part), and of a rational number, which is impossible too. So, (22) has no solution for pairs with unequal free coefficients.

When free coefficients are equal, $c = 0$, and (22) transforms to

$$(t_r - s_r)A_r = 2^{r_t - r} m_1^N \qquad (30)$$

For $r < r_t$, the right part is even, for $r = r_t$ it is odd. The left part can be odd, or even, or zero. So, the solution of this equation is uncertain. Consequently, the pairs, whose terms have equal free coefficients, should be used as initial pairs for the next presentation level.

This proves the Lemma.

Now, we should establish relationships between the sizes of groups, corresponding to pairs with equal and unequal free coefficients, and the parity of the term $(t_r - s_r)$ in (22).

Lemma 9: *When initial pairs, obtained from the r-level of presentation, have equal free coefficients, the number of pairs with equal and unequal free coefficients at the next presentation level (r+1) is the same and is equal to 1/2 of the whole set of pairs at level (r+1). The group of pairs with equal free coefficients correspond to even values of* $(t_r - s_r)$*, while pairs with unequal free coefficients correspond to odd* $(t_r - s_r)$*, so that it is equivalent subdividing the pairs based on parity of* $(t_r - s_r)$*, or on the basis of equal and unequal free coefficients.*

Proof: It follows from Table 1 that for $r_2 = 2$, the quantities of pairs with equal and unequal free coefficients are equal. Consequently, each group constitutes a half of all pairs. Odd values of $(t_{r_2} - s_{r_2})$ correspond to pairs at level $r = 3$ with unequal free coefficients. Accordingly, even values of $(t_{r_2} - s_{r_2})$ correspond to pairs with equal free coefficients. Let us assume that the same is true for an initial pair with equal free coefficients at the greater level r, $r \ge 2$. The presentation for all possible parity combinations of t_r and s_r at level (r+1) is shown in Table 3 for one generic pair with equal free coefficients.

It follows from Table 3 that the number of pairs with equal and unequal free coefficients is the same, and is equal to 1/2 of quantity of all pairs. Unequal free coefficients correspond to odd values of $(t_r - s_r)$, while even values $(t_r - s_r)$ correspond to pairs with equal free coefficients. So, we obtained the same results as for $r = 2$. Since the rest of initial pairs have the same form (in all of them free coefficients are equal), depending on the parity of $(t_r - s_r)$, they also produce a half of pairs with equal free coefficients, and a half with unequal ones. According to principle of mathematical induction, this means that the found properties are valid for any presentation level $r \geq 2$. This proves the Lemma.

Table 3. Presentation with a factor 2^r for a pair with equal free coefficients.

	0	1	2	3	4
1	t_r s_r	$2t_{r+1}$ $2s_{r+1}+1$	$2t_{r+1}+1$ $2s_{r+1}$	$2t_{r+1}$ $2s_{r+1}$	$2t_{r+1}+1$ $2s_{r+1}+1$
2	$2^r t_r+v_i$ $2^r s_r+v_i$	$2^{r+1}t_{r+1}+v_i$ $2^{r+1}s_{r+1}+2^r+v_i$	$2^{r+1}t_{r+1}+2^r+v_i$ $2^{r+1}s_{r+1}+v_i$	$2^{r+1}t_{r+1}+v_i$ $2^{r+1}s_{r+1}+v_i$	$2^{r+1}t_{r+1}+2^r+v_i$ $2^{r+1}s_{r+1}+2^r+v_i$

Corollary 5: *Consider successive presentations of pairs of odd numbers with a factor of* 2^r, *which use initial pairs with equal free coefficients from the previous level, beginning with one pair. Then, the number of initial pairs at level r is equal to* 2^{r-1}.

Proof: For a factor of two, we have one pair; for a factor of 2^2 there are two pairs with equal free coefficients (Table 1); for a factor of 2^3 there are 2^2 such pairs (Table 2), and so forth. The total number of pairs increases by four times for the next presentation level (since each initial pair produces four new pairs, one per parity combination of t_r, s_r). From this amount, a half of pairs correspond to pairs with equal free coefficients, according to Lemma 9. The value of 2^{r-1} reflects on the fact that at each presentation level the number of pairs with equal free coefficients doubles. This proves the Corollary.

Corollary 6: *For* $r \leq r_t = N(\mu+1)$, *the fraction of pairs of odd numbers, for which equation (15) has no solution for a presentation level r, is equal to*

$$f_r = (1/2)^{r-1} \tag{31}$$

Proof: It was shown in Lemma 8 that (22) has no solution for pairs with unequal free coefficients, while, according to Lemma 9, these pairs constitute half of all pairs at a given presentation level. Thus, (31) is true for $r = 2$. Let us assume that Lemma is valid for the value of $r > 2$. According to Lemma 8, for $r \leq r_t$, the corresponding equations have no solution for pairs with unequal free coefficients, so that initial pairs for the next level are always pairs with equal free coefficients. Then, the fraction f_{ru} of pairs, for which solution is uncertain, is the same, as the fraction of "no solution" pairs, that is $f_{ru} = (1/2)^{r-1}$. This fraction contains initial pairs for the presentation level (r+1). At this level, all pairs are again divided into two equal groups of "no solution" and "uncertain" pairs, so that the "no solution" fraction is

$$f_{r+1} = f_{ru} \times (1/2) = (1/2)^{r-1}/2 = (1/2)^r,$$

which is formula (31) for the level (r+1). According to principle of mathematical induction, this means validity of (31). This proves the Corollary.

Lemma 10: *At each next presentation level (r+1), the number of pairs, corresponding to odd and even values of* $(t_r - s_r)$, *are equal.*

Proof: Suppose we have p_{r+1} initial pairs at a presentation level (*r*+1). Each initial pair produces four equivalent (in terms of asymptotic densities, Lemma 4) pairs at level (*r*+1), one pair per each possible parity combination of terms t_r, s_r, listed in the first row of Table 3. These parity combinations do not depend, whether the initial pairs have equal or unequal free terms, and also do not depend on the value of *r* compared to r_t. Two of these parity combinations (in cells (1,1), (1,2) in Table 3) produce odd values of $(t_r - s_r)$, namely when t_r, s_r are equal to $[2t_{r+1}, 2s_{r+1}+1]$, $[2t_{r+1}+1, 2s_{r+1}]$. Two other combinations, in cells (1,3), (1,4), produce even values of $(t_r - s_r)$ for pairs $[2t_{r+1}, 2s_{r+1}]$, $[2t_{r+1}+1, 2s_{r+1}+1]$. So, the number of pairs, for which $(t_r - s_r)$ is odd is equal to $2p_{r+1}$. The number of pairs, for which $(t_r - s_r)$ is even, is also $2p_{r+1}$. So, quantities of pairs, corresponding to odd and even values of $(t_r - s_r)$, are equal. This proves the Lemma.

Note: At the presentation level (*r*+1), odd values $(t_r - s_r)$ cannot be zero, given the presentation of t_r and s_r through t_{r+1} and s_{r+1} in Table 3. Even values of $(t_r - s_r)$ can be zero. However, from the perspective of solution, such a zero term can be transformed to a non-zero even term (such a transition is addressed by Lemma 11).

4.6. Finding fraction of "no solution" pairs for presentation levels with $r \le r_t = N(\mu+1)$

We found so far that for $r \le r_t = N(\mu+1)$ the following is true:

(a) Initial pairs with equal free coefficients, taken from level *r*, produce equal number of pairs with equal and unequal free coefficients at a presentation level (*r*+1), Lemma 9;
(b) Corresponding to pairs equations have no solution for pairs with unequal free coefficients, while solution is uncertain for pairs with equal free coefficients, Lemma 8;
(c) Each presentation level adds a "no solution" fraction of pairs equal to $f_r = (1/2)^{r-1}$.

So, each previous level supplies to the next presentation level "uncertain" pairs, which constitutes half of all pairs of the previous level. These initial pairs have equal free coefficients. This allows finding a "no solution" fraction of pairs from successive presentations with a factor of 2^r. Since each level adds 1/2 of pairs to a "no solution" fraction, the total such fraction F_r is equal to a sum of geometrical progression with a common ratio $q = 1/2$, and the first term $f_2 = 1/2$ (the "no solution" fraction at level *r*=2). Fig. 2 illustrates this consideration.

So, we can write

$$F_r = \sum_{i=2}^{r} f_i = f_2 \sum_{i=2}^{r} q^{i-2} = f_2 (1 - q^{r-1})/(1-q) \qquad (32)$$

For example, for *r*=5, $F_r = 15/16$. Note that if such a progression is valid to infinity, the total fraction in the limit would be

$$\lim_{r\to\infty} F_r = f_2/(1-q) = (1/2)/(1/2) = 1 \qquad (33)$$

(Here, the limit is understood as an ordinary Cauchy's limit.) In other words, equation (15) would not have a solution for all possible pairs of odd integer numbers. However, in order to obtain the result, one needs to confirm that such a progression is true for $r > r_t = N(\mu+1)$.

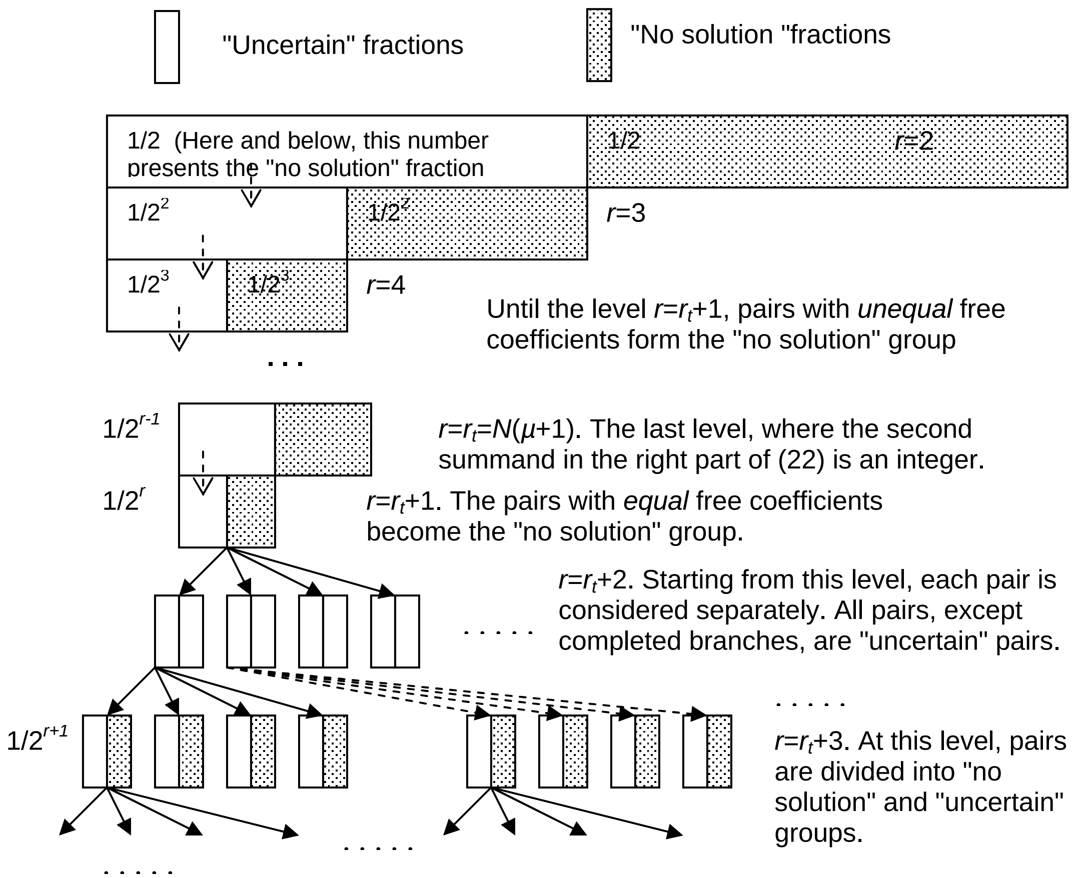

Fig. 2. Graphical presentation of how the "No solution" fraction accumulates through presentation levels, and the appropriate decrease of "Uncertain" fraction. The value of $r=r_t=N(\mu+1)$ is a threshold value, where transition begins from even right parts of equations to integer or rational ones.

4.7. Transcending the threshold level $r=r_t$

Presentation level (r_t+1)

Table 4 shows pairs of odd numbers for level (r_t+1). The number of initial pairs is defined by Corollary 5, and is equal to 2^{r_t} for this level. For pairs with *equal* free coefficients (columns 3 and 4 in Table 4), (22) transform to

$$(t_{r_t+1}-s_{r_t+1})A_{r_t+1,ij}=m_1^N/2 \qquad (34)$$

The right part of (34) is rational (m_1 is an odd number). The left part is an integer. So, (34) has no solution for pairs with equal free coefficients (and, consequently, for even $(t_{r_t}-s_{r_t})$, according to Lemmas 9 and 10). When $(t_{r_t}-s_{r_t})=0$, the left part is zero, while the right part is rational. So, (34) has no solution too. This group of pairs constitutes 1/2 of all pairs (Lemma 10), so that the common ratio remains equal to 1/2, and formula (32) stays valid.

Table 4. Pairs presented with a factor of 2^{r_t+1}. It is assumed that $r=r_t$.

	0	1	2	3	4
	t_r s_r	$2t_{r+1}$ $2s_{r+1}+1$	$2t_{r+1}+1$ $2s_{r+1}$	$2t_{r+1}$ $2s_{r+1}$	$2t_{r+1}+1$ $2s_{r+1}+1$

1	$2^r t_r + v_{r1}$ $2^r s_r + v_{r1}$	$2^{r+1} t_{r+1} + v_{r1}$ $2^{r+1} s_{r+1} + 2^r + v_{r1}$	$2^{r+1} t_{r+1} + 2^r + v_{r1}$ $2^{r+1} s_{r+1} + v_{r1}$	$2^{r+1} t_{r+1} + v_{r1}$ $2^{r+1} s_{r+1} + v_{r1}$	$2^{r+1} t_{r+1} + 2^r + v_{r1}$ $2^{r+1} s_{r+1} + 2^r + v_{r1}$
2	$2^r t_r + v_{r2}$ $2^r s_r + v_{r2}$	$2^{r+1} t_{r+1} + v_{r2}$ $2^{r+1} s_{r+1} + 2^r + v_{r2}$	$2^{r+1} t_{r+1} + 2^r + v_{r2}$ $2^{r+1} s_{r+1} + v_{r2}$	$2^{r+1} t_{r+1} + v_{r2}$ $2^{r+1} s_{r+1} + v_{r2}$	$2^{r+1} t_{r+1} + 2^r + v_{r2}$ $2^{r+1} s_{r+1} + 2^r + v_{r2}$
…					
2^r	$2^r t_r + v_{rR}$ $2^r s_r + v_{rR}$	$2^{r+1} t_{r+1} + v_{rR}$ $2^{r+1} s_{r+1} + 2^r + v_{rR}$	$2^{r+1} t_{r+1} + 2^r + v_{rR}$ $2^{r+1} s_{r+1} + v_{rR}$	$2^{r+1} t_{r+1} + v_{rR}$ $2^{r+1} s_{r+1} + v_{rR}$	$2^{r+1} t_{r+1} + 2^r + v_{rR}$ $2^{r+1} s_{r+1} + 2^r + v_{rR}$

For pairs with unequal free coefficients (and consequently odd $(t_{r_t} - s_{r_t})$, Lemma 10), (22) transforms to

$$(t_{r_t+1} - s_{r_t+1})A_{r_t+1,j} = A_{r_t+1,j}c/2^{r_t} + m_1^N/2 \qquad (35)$$

The right part can be rational, an integer or zero. Since the sums $A_{r+1,j}$ are all odd, parity of the left part in (35) is defined by the term $(t_{r_t+1} - s_{r_t+1})$, which can be odd, even or zero. So, solution of (35) for odd $(t_{r_t} - s_{r_t})$ is uncertain, and such pairs should be used as initial pairs for the next presentation level $(r_t + 2)$. As it was mentioned (a note after Lemma 10), for odd $(t_{r_t} - s_{r_t})$, the term $(t_{r_t+1} - s_{r_t+1}) \neq 0$.

Recall that before the level $(r_t + 1)$, the pairs with *unequal* free coefficients had no solution, while (34) has no solution for *even* $(t_{r_t+1} - s_{r_t+1})$, corresponding to pairs with *equal* free coefficients. In this regard, the level $(r_t + 1)$ reverses the groups of pairs. The "uncertain" group of pairs is now composed of pairs with *unequal* free coefficients (and accordingly with *odd* $(t_{r_t} - s_{r_t})$). These pairs (in columns 1 and 2 in Table 4) should be used as initial pairs at the next presentation level $(r_t + 2)$.

Transition in the presentation level $(r_t + 2)$

Level $(r_t + 1)$ supplied initial pairs with unequal free coefficients. This means that we do not have anymore distinct groups with equal and unequal free coefficients at level $(r_t + 2)$, as before, since the initial pairs with unequal free coefficients produce mostly pairs with unequal free coefficients, with occasional inclusion of pairs with equal ones. Previously, we have seen that the parity of parameter $(t_r - s_r)$ defined the absence or uncertainty of solution. However, beginning from level $(r_t + 2)$, this parameter lost association with groups of pairs with equal and unequal free coefficients. This is due to the fact that the right part of equation (35) can be an integer, a rational number, or zero *per pair basis*, and so we should consider the use of parameter $(t_r - s_r)$ this way. We will still have a half of "no solution" and a half of "uncertain" pairs, but only for a block of four pairs, corresponding to each initial pair. This is the assembly of such "uncertain" pairs from each block, which goes to the next level. Table 5 shows pairs for level $(r_t + 2)$.

Table 5. Pairs presented with a factor of 2^{r_t+2}, obtained from initial pairs in Table 4, for which $(t_r - s_r)$ is odd. First two rows correspond to cells (1,1), (1,2) in Table 4. It is assumed that $r = r_t$.

	0	1	2
	t_{r+1}	$2t_{r+2}$	$2t_{r+2}+1$

	s_{r+1}	$2s_{r+2}+1$	$2s_{r+2}$
1	$2^{r+1}t_{r+1}+v_{r1}$ $2^{r+1}s_{r+1}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$
2	$2^{r+1}t_r+2^r+v_{r1}$ $2^{r+1}s_{r+1}+v_{r1}$	$2^{r+2}t_{r+2}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+v_{r1}$
. . .	. . .	. . .	. . .
2^{r+1}	. . .	. . .	. . .

Table 5 continued

3	4
$2t_{r+2}$ $2s_{r+2}$	$2t_{r+2}+1$ $2s_{r+2}+1$
$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$
$2^{r+2}t_{r+2}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+v_{r1}$	$2^{r+2}t_{r+2}+2^{r+1}+2^r+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+v_{r1}$
. . .	. . .
. . .	. . .

When $r=(r_t+2)$, (22) transforms to

$$(t_{r_t+2}-s_{r_t+2})A_{r+2,ij}=A_{r_t+2,ij}c/2^{r_t+2}+m_1^N/4 \qquad (36)$$

where index '*ij*' denotes the cell number. The right part of (36) can be rational, an integer, or zero. When the left part is an integer (the case, when it's zero, will be considered later), (36) has no solution for any $(t_{r_t+2}-s_{r_t+2})$ for the rational or zero right part, and, consequently, this branch is completed. (Compared to continuing branches, the completed branch delivers *double* fraction of pairs, for which (15) has no solution, since in this case two equal "no solution" and "uncertain" fractions compose one "no solution" fraction.) If the right part is an integer, (36) has no solution when $(t_{r_t+2}-s_{r_t+2})$ has the opposite parity, and the solution is uncertain for another parity of $(t_{r_t+2}-s_{r_t+2})$. The number of combinations of parameters t_{r_t+2} and s_{r_t+2}, corresponding to each parity, is equal to two from four in this case, and so we still have equal division between the "no solution" and "uncertain" pairs. However, at this level, we have no distinction between the odd and even values of t_{r_t+2} and s_{r_t+2} in the same way, as before, when there was an association with equal and unequal free coefficients. Such distinction can be done *only* at the next presentation level (r_t+3). All pairs at level r_t+2 correspond to "uncertain" equations, except for the cases when the pair branch is completed.

The case of $(t_{r_t+2}-s_{r_t+2})=0$ is also an "uncertain" one, since there is a possibility that two terms in the right part are equal in absolute values and have the opposite algebraic signs.

Note that values $A_{r_t+2,ij}$ are different, so that the right parts of corresponding equations, transformed to a form (22), may have dissimilar parities (as well as may be rational or zeros) for different pairs. (The right part can be an integer, provided $c\neq 0$ in (22), otherwise the right part is equal to $m_1^N/2^{r-r_t}$, which is always rational for $r>r_t$, so that such a branch is completed.) Therefore, starting from this level, one should consider each pair *separately* (Fig. 2). (In fact, it is possible to show that at level (r_t+2), when $c \neq 0$, integer right parts of these equations have the same parity. However, this is not necessarily true for the next levels, so we use the same generic approach for this level and above.)

With regard to accumulation of a total "no solution" fraction, we have the same common ratio of 1/2, although it is obtained differently - not per group, as previously, but per pair, and then such "per pair" fractions are summed up, in order to obtain the total "no solution" fraction. We will consider this assembling process in detail later.

So, we found that the corresponding equations for pairs in both groups (meaning groups of pairs, having either even or odd values of $(t_{r_t+2} - s_{r_t+2})$) converge to equations, which have no solution for one parity of $(t_{r_t+2} - s_{r_t+2})$, and accordingly for one half of pairs (according to Lemma 10), while solution is uncertain for the other parity, corresponding to the second half of pairs. So, the common ratio for a geometric progression, defining fractions of "no solution" pairs, will remain equal to 1/2. However, because we can specify particular pairs, corresponding to odd or even $(t_{r_t+2} - s_{r_t+2})$, at the next level only, this common ratio accordingly should be assigned to a presentation level, where such a specification actually happens; in this case, this is the next level (r_t+3). At level r_t+2, all equations, corresponding to initial pairs, have the same form (22), and consequently, the same "uncertain" status. All pairs (except for completed ones) are "uncertain" pairs.

Presentation level (r_t+3)

We will need the following Lemma to address zero values of $(t_r - s_r) = 0$ in equation (22). Note that $(t_r - s_r) = 0$ only when both parameters are equal (and, of course, have the same parity), including when both are equal to zero. When $(t_r - s_r)$ is odd (parameters have different parity), $(t_r - s_r) \neq 0$.

Lemma 11: *Equation (22), that is* $(t_r - s_r)A_r = A_r c/2^r + m_1^N/2^{r-r_t}$, *is equivalent to equation* $(t_{1r} - s_{1r})A_r = A_r c/2^r + m_1^N/2^{r-r_t}$ *in terms of parities of both parts, with the substitutions* $t_r = t_{1r} - 2a$ *and* $s_r = s_{1r} - 2b$, *where a and b are integers. If the second equation has no solution based on parity or rationality considerations, then the first equation also has no solution, and vice versa.*

Proof: According to the notion of presentation of odd numbers with a factor of 2^r, the terms t_r and s_r are integers, having ranges of definition $(-\infty < t_r < \infty)$ and $(-\infty < s_r < \infty)$. The only property, which is of importance with regard to such a presentation, is that these parameters should be defined on the whole set of integer numbers, in order to include *all* possible numbers, corresponding to a particular presentation; for instance, the term $(2^r t_r + v_r)$ should produce the whole set of the appropriate "stroboscopic" numbers in the range $(-\infty, \infty)$, located at the distance 2^r from each other. As long as this condition is fulfilled, that is such a set can be reproduced, we can make an equivalent substitution for parameters t_r, s_r. For instance, the substitution $t_r = t_{1r} - 2a$ is an equivalent one. Indeed, it preserves the range of definition $(-\infty < t_{1r} < \infty)$, and accordingly produces all numbers, which parameter t_r produces (only with a shift of $(-2a \times 2^r)$ for the same values of t_r and t_{1r}). However, this shift makes no difference with regard to the range of produced numbers, since our range $(-\infty, \infty)$ is infinite in both directions. On the other hand, when $(t_r - s_r) = 0$, we have

$(t_{1r} - s_r) \neq 0$, and vice versa. So, for $(t_r - s_r) = 0$, such a substitution produces an equation with a non-zero left part.

Substituting $t_r = t_{1r} - 2a$ into (22), one obtains the equation

$$(t_{1r} - s_r)A_r = 2aA_r + A_r c/2^r + m_1^N / 2^{r-r_t} \qquad (37)$$

When $(t_r - s_r) = 0$, we have $(t_{1r} - s_r) = 2a \neq 0$. Also, the appearance of the even term $2aA_r$ does not change the parity of the right part, nor the substitution $t_r = t_{1r} - 2a$ changes the parity of the left part (if it is not zero; if it is zero, the substitution still provides an even increment). Thus, with regard to parities, (22) and (37), indeed, are equivalent equations.

If the equivalent equation (37) has no solution, then the original equation (22) has no solution too. The proof is as follows. Let us assume that (37) has no solution, while (22) has a solution, so that

$$(t_r - s_r)A_r = A_r c/2^r + m_1^N / 2^{r-r_t}$$

Adding $2aA_r$ to the left and right parts of this equation, one obtains an equivalent equation, which also should have a solution.

$$(t_r + 2a - s_r)A_r = 2aA_r + A_r c/2^r + m_1^N / 2^{r-r_t}$$

According to the substitution, $t_r = t_{1r} - 2a$, so that $t_r + 2a = t_{1r}$, and the obtained equation transforms to (37), which should also have a solution. However, according to our assumption, it has no solution. The obtained contradiction means that the assumption that (22) has a solution is invalid, and, in fact, it has no solution.

Similarly, we can assume that (37) has a solution, while (22) does not, and show that then (22) should have a solution, which would contradict to the initial assumption.

Although we proved the equivalency of equations with regard to their solution properties in a general case, we need such equivalency only for the case when the left part of equivalent equations is zero (because $(t_r - s_r) = 0$ or $(t_{1r} - s_r) = 0$). The proposed substitution then makes the left part of the equivalent equation a non-zero value, and the inference about the absence of solution or its uncertainty can be made based on parities of the left and right parts. Certainly, one can do an analogous substitution for s_r, or both parameters. This proves the Lemma.

Table 6 shows an example of pairs for the presentation level $(r_t + 3)$. Four initial pairs are from cells (1,1)-(1,4) in Table 5. If (36) has no solution for even $(t_{r_t+2} - s_{r_t+2})$, then these are pairs (1,3), (1,4) in Table 6, which satisfy this condition. Accordingly, pairs (1,1) and (1,2), for which $(t_{r_t+2} - s_{r_t+2})$ is odd, are "uncertain" pairs, which should be used as initial pairs for the next, $(r_t + 4)$, level. If, on the contrary, (36) has no solution for odd $(t_{r_t+2} - s_{r_t+2})$, then (1,1) and (1,2) are the "no solution" pairs, while (1,3), (1,4) become "uncertain" pairs, which should be used as initial pairs for the next level. This way, all new pairs, four per each initial pair, are divided into two halves as before, so that the common ratio of geometrical progression remains equal to 1/2. The case $(t_{r_t+2} - s_{r_t+2}) = 0$ is addressed by Lemma 11 through equivalent equations.

In the same way, as we considered one pair above, we should consider the rest of initial pairs in Table 6 and find out, which two pairs should be used as initial pairs for the next level. Then, the same procedure should be repeated for each initial pair at level $(r_t + 3)$.

Then, the cycle is repeated for the next two levels $(r_t + 4)$ and $(r_t + 5)$, and so forth, to infinity, since there are no anymore threshold values of *r*, at which the right part could change the parity (if it's an integer), and the corresponding equations their form and properties. The following Lemma generalizes the discovered order.

Table 6. Pairs of odd numbers with a factor of 2^{r_t+3}. Initial pairs are (1,1)-(1,4) from Table 5. It is assumed that $r = r_t$.

	0	1	2
	t_{r+2} s_{r+2}	$2t_{r+3}$ $2s_{r+3}+1$	$2t_{r+3}+1$ $2s_{r+3}$
1	$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$
2	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$
3	$2^{r+2}t_{r+2}+v_{r1}$ $2^{r+2}s_{r+2}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+v_{r2}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r2}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$
4	$2^{r+2}t_{r+2}+2^{r+1}+v_{r1}$ $2^{r+2}s_{r+2}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+1}+v_{r2}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r2}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{+1r1}$

Table 6 continued

3	4
$2t_{r+3}$ $2s_{r+3}$	$2t_{r+3}+1$ $2s_{r+3}+1$
$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+v_{r1}$ $2^{r+3}s_{r+3}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^r+v_{r1}$
$2^{r+3}t_{r+3}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+1}+2^r+v_{r1}$	$2^{r+3}t_{r+3}+2^{r+2}+2^{r+1}+v_{r1}$ $2^{r+3}s_{r+3}+2^{r+2}+2^{r+1}+2^r+v_{r1}$

Lemma 12: *From the presentation level (r_t+2), the "no solution" fraction is accumulated across two sequential levels, and then the pattern repeats for each two successive levels, to infinity. Some branches can be completed at level (r_t+2), but otherwise this level provides no explicit division into the "no solution" and "uncertain" groups, as it was the case for the previous levels. Except for the pairs, corresponding to completed branches, the pairs become initial "uncertain" pairs for the next presentation level. At level (r_t+3), all new pairs are divided into the "no solution" and "uncertain" groups (according to odd or even parity of (t_r-s_r) in equation (22)). The "uncertain" pairs become initial pairs for the next presentation level, and the two-level cycle repeats to infinity.*

Proof: Previously, we have seen that the Lemma is true for the paired levels (r_t+2) and (r_t+3). Let us assume that Lemma is true for the previous (r_t+d-1) level, which then supplies initial "uncertain" pairs for the level (r_t+d). We need to prove that Lemma is true for the next two levels (r_t+d) and (r_t+d+1). Initial pairs may have equal and unequal free coefficients.

Let us consider an equation for a pair with free coefficients *v* and *w*.

$$(2^{r_t+d}t_{r_t+d}+v)^N-(2^{r_t+d}s_{r_t+d}+w)^N=2^{N(\mu+1)}m_1^N \quad (38)$$

where $d\geq 2$.

According to Lemma 6, it can be transformed to an equation

$$(t_{r_t+d}-s_{r_t+d})A_{r_t+d}=A_{r_t+d}c/2^{r_t+d}+m_1^N/2^d \quad (39)$$

where $A_{r_t+d}=\sum_{i=0}^{N-1}(2^{r_t+d}t_{r_t+d}+v)^{N-1-i}(2^{r_t+d}s_{r_t+d}+w)^i$, $N = 2n + 1$.

The right part of (39) can be an integer, rational or zero. The left part is an integer (if $(t_{r_t+d}-s_{r_t+d})=0$, the left part can be transformed to an integer, using Lemma 11). When the right part is rational, (39) has no solution for any t_{r_t+d} and s_{r_t+d}, and the branch is completed. If the right part is even or odd, (39) has no solution when $(t_{r_t+d}-s_{r_t+d})$ has the opposite parity. Solution is uncertain for the other parity of $(t_{r_t+d}-s_{r_t+d})$, since both parts of (39) have the same parity in this case. However, at this level, we cannot specify particular parity of $(t_{r_t+d}-s_{r_t+d})$, which should be done at the next presentation level (r_t+d+1). When c = 0, (39) has no solution, since the right part is a rational number, while the left part is an integer or zero, and so the branch is completed.

Table 7. New pairs for the initial pair $[\,2^{r_t+d}t_{r_t+d}+v,\ 2^{r_t+d}s_{r_t+d}+w\,]$ at the presentation level (r_t+d+1) with a factor of 2^{r_t+d+1}.

	0	1	2
0	t_{r_t+d} s_{r_t+d}	$2t_{r_t+d+1}$ $2s_{r_t+d+1}+1$	$2t_{r_t+d+1}+1$ $2s_{r_t+d+1}$
1	$2^{r_t+d}t_{r_t+d}+v$ $2^{r_t+d}s_{r_t+d}+w$	$2^{r_t+d+1}t_{r_t+d+1}+v$ $2^{r_t+d+1}s_{r_t+d+1}+2^{r_t+d}+w$	$2^{r_t+d+1}t_{r_t+d+1}+2^{r_t+d}+v$ $2^{r_t+d+1}s_{r_t+d+1}+w$

Table 7 continued

3	4
$2t_{r+3}$ $2s_{r+3}$	$2t_{r+3}+1$ $2s_{r+3}+1$
$2^{r_t+d+1}t_{r_t+d+1}+v$ $2^{r_t+d+1}s_{r_t+d+1}+w$	$2^{r_t+d+1}t_{r_t+d+1}+2^{r_t+d}+v$ $2^{r_t+d+1}s_{r_t+d+1}+2^{r_t+d}+w$

Even if the branch is completed for some pair, we still can assume that it is "uncertain", and use it as an initial pair at the next presentation level. There, the new pairs, corresponding to this initial pair, are then divided into the "no solution" and "uncertain" groups. The fraction of the former goes to the total "no solution" fraction, while the latter is used as initial pairs for the next level, besides other uncertain pairs. (Such an arrangement, without completed branches, is more convenient for calculation of the total "no solution" fraction.)

Table 7 shows new pairs for the next presentation level for the initial pair from (38). At this level, we can choose the needed parities of pair's terms t_{r_t+d}, s_{r_t+d}, expressed through t_{r_t+d+1}, s_{r_t+d+1}, in order for (39) to have no solution. For instance, if (39) has no solution for even $(t_{r_t+d} - s_{r_t+d})$, then the "no solution" pairs are (1,3), (1,4). Accordingly, solution is uncertain for pairs (1,1), (1,2), since both parts of (39) have the same parity in this case. Consequently, these pairs should be used as initial "uncertain" pairs for the next presentation level.

We can see from Table 7 that when a pair of an actually completed branch is used as an "uncertain" pair for the next level, it produces no new pairs with some specific features, which could prevent their corresponding equations to be transformed into a form (39). We still obtain pairs, satisfying conditions of Lemma 6, to which the same equation (22) is applicable. For instance, when $v = w$, then $c = 0$ in (39), and so the branch is completed. However, if we use it as an initial pair for the next presentation level $(r_t + d + 1)$, then we are free to choose new pairs, corresponding to either even or odd values of $(t_{r_t+d} - s_{r_t+d})$, since the corresponding equations have no solution for both scenarios. Then, the pairs with the opposite parity $(t_{r_t+d} - s_{r_t+d})$ will proceed to the next level as uncertain initial pairs. As before, such a division produces two equal groups of pairs, and so the common ratio of the geometrical progression remains equal to 1/2.

So, with the assumption that Lemma is true for the previous level, we confirmed the same pattern earlier discovered for the coupled levels $[(r_t + 2), (r_t + 3)]$. According to principle of mathematical induction, this means that Lemma is true for any $d \geq 2$. This proves the Lemma.

In this Lemma, we also studied the useful property, considering completed branches as non-completed ones. This property is formulated below as a Corollary.

Corollary 7: *Pairs, corresponding to completed branches, can be considered as regular "uncertain" pairs, which can be passed to the next level as initial pairs, so that such a branch is actually assigned a non-completed status.*

Lemma 13: *At presentation levels above* $(r_t + 1)$*, and in the absence of completed branches, the number of pairs in "no solution" and "uncertain" groups are equal, when such a division takes place.*

Proof: According to Lemma 12 and Corollary 7, all pairs, both regular ones, with "no solution" and "uncertain" components, and the pairs, which could be completed, but continue to participate in the next levels as non-completed pairs, can be presented in a form of Table 7. The solution properties of equations, corresponding to pairs in Table 7, are defined by equation (22), or more particular, by equations in a form (39), whose solution properties depend on the term $(t_{r_t+d} - s_{r_t+d})$. (Unless the right part is rational, in which case equation has no solution for all parities, and the branch is completed. However, according to Corollary 7, we can still consider such a pair as a regular non-completed pair.)

The division of four pairs into two equal "no solution" and "uncertain" groups is based solely on the parity of $(t_{r_t+d} - s_{r_t+d})$, as Lemma 12 showed, with one parity corresponding to a "no solution" group, and with the opposite parity corresponding to "uncertain" group. The number of pairs, corresponding to one parity, is therefore equal to 2π, where π is the number of initial pairs, number two is the number of parity combinations of t_{r_t+d}, s_{r_t+d}, producing the same parity of $(t_{r_t+d} - s_{r_t+d})$, see Table 7. For the opposite parity of $(t_{r_t+d} - s_{r_t+d})$, the number of produced pairs is also 2π. Thus, the number of pairs in "no solution" and "uncertain" groups is the same. This proves the Lemma.

4.8. Calculating the total "no solution" fraction

Using Corollary 7, we consider all levels as if they have no completed branches. Then, according to Lemmas 9 and 10, until the level $(r_t + 2)$, all levels have two equal groups of pair combinations. One corresponds to a "no solution" fraction, and the other to "uncertain" fraction, so that the common ratio q = 1/2. Substituting these values into (32), one obtains

$$F_{r_t+1} = f_2(1-q^{r-1})/(1-q) = 1/2(1-(1/2)^{r_t+1-1})/(1/2) = 1-(1/2)^{r_t} \qquad (40)$$

The "no solution" fraction for the level $(r_t + 1)$ is defined by (31) as follows (the last term of a geometrical progression), taking into account that $f_2 = 1/2$.

$$f_{r_t+1} = f_2 q^{r_t+1-2} = (1/2)^{r_t} \qquad (41)$$

Since in the absence of completed branches the "no solution" and "uncertain" fractions are equal, according to Lemma 10, the "uncertain" fraction of pairs, which is passed to the level $(r_t + 2)$, is the same as the "no solution" fraction (41). This "uncertain" fraction, according to Lemma 13, is equally divided into "no solution" and "uncertain" fractions at each second level, beginning from level $(r_t + 3)$, so that the first term of the geometrical progression, representing the "no solution" fraction of two following coupled levels, is

$$f_{r_t+3} = f_{r_t+1} \times (1/2) \qquad (42)$$

Then, each next two levels add a half of the previous "uncertain" fraction", which is equal to "no solution" fraction. Let $D = \{2L, 2L+1\}$, $L = 1,2...$ This way, $(r_t + D)$ defines the levels' numbers for $r \geq (r_t + 2)$. Levels, at which pairs are divided into two groups, are levels $(r_t + 3)$, $(r_t + 5)$, …, $(r_t + 2L + 1)$, so that the total "no solution" fraction, obtained by summation of "no solution" fractions of all levels above the $(r_t + 1)$ level, is equal to

$$F_{r_t+2,D} = (1/2)^{r_t}[1/2 + (1/2)^2 + (1/2)^3 + ...(1/2)^L] = (1/2)^{r_t}\sum_{i=1}^{L}(1/2)^i = (1/2)^{r_t}(1-(1/2)^L) \qquad (43)$$

when $D = 2L + 1$, and

$$F_{r_t+2,D} = (1/2)^{r_t}[1/2 + (1/2)^2 + (1/2)^3 + ...(1/2)^{L-1}] = (1/2)^{r_t}\sum_{i=1}^{L-1}(1/2)^i = (1/2)^{r_t}(1-(1/2)^{L-1})$$

when $D = 2L$. (44)

In the last case, the division into the "no solution" and "uncertain" groups did not happen yet at the first level of coupled levels, since it occurs at the second level of the couple, as it was earlier discussed. This is why the power is (L - 1), but not L.

The total "no solution" fraction, accordingly, is defined as $F_{r_t+1+D} = F_{r_t+1} + F_{r_t+2,D}$. For $D = 2L + 1$, we have

$$F_{r_t+1+D} = F_{r_t+1} + F_{r_t+2,D} = 1-(1/2)^{r_t} + (1/2)^{r_t} - (1/2)^{r_t+L} = 1-(1/2)^{r_t+L} \qquad (45)$$

It follows form (45) that in the limit

$$\lim_{L->\infty} F_{r_t+1+D} = \lim_{L->\infty}(1-(1/2)^{r_t+L}) = 1 \qquad (46)$$

The same is true for (44). So, when we consider all branches as non-completed, in the limit, equation (15) has no solution for all possible pairs of odd integer numbers. Of course, it may look awkward, considering completed branches as non-completed, but, as Lemma 12 and Corollary 7 showed, this is a legitimate procedure.

Accounting for completed branches. Let us assume that level *r* has *k* completed branches, to which the "no solution" fraction f_{rk} corresponds. Let us assume that these branches were not completed, and consider the pairs, corresponding to these branches, as regular ones, with "no solution" and "uncertain" components, to infinity. In other words, we assume that there are no more completed branches in the following presentations of these *k* pairs, to infinity. (In real situation, if there are such pairs, we can also consider them as non-completed pairs, according to Corollary 7.) In this scenario, the fraction f_{rk} would be divided equally (Lemma 13) between the "no solution" and "uncertain" fractions on each subsequent level (or on the second level in coupled levels beyond the value of $r=(r_t+1)$), so that the total "no solution" fraction, accumulated at level *L*, is defined as follows.

$$F_{r+L} = f_{rk}\sum_{i=1}^{L}(1/2)^i = f_{rk}[1-(1/2)^L] \tag{47}$$

In the limit, (47) transforms to

$$\lim_{L->\infty} F_{r+D} = \lim_{L->\infty} f_{rk}[1-(1/2)^L] = f_{rk} \tag{48}$$

So, in the limit, we obtained in (48) exactly the same "no solution" fraction, which was taken by *k* completed branches. Since, according to (46), in the scenarios with non-completed branches the total "no solution" fraction is equal to one, the result (48) means that accounting for completed branches, in the limit, produces the same "no solution" fraction of one.

Thus, (15) has no solution in integer numbers in case of completed branches too.

4.9. Cases 2 and 3 as equivalent equations

For the case 3, we have $a=2n+1$; $x=2k_1+1$; $y=2p_1+1$. Then, (1) transforms to (14).

$$(2k_1+1)^{2n+1}+(2p_1+1)^{2n+1}=(2m)^{2n+1} \tag{49}$$

Using an approach, similar to one for equation (15), it is possible to prove that it has no solution. The shorter way could be to show the equivalency of (15) and (49) in terms of solution properties. Then, since (15) has no solution, that would mean that (49) has no solution too.

The notion of equivalent equations. It means that for each set of input variables for one equation there is one and only one matching set of corresponding input variables for the other equation, such that the terms in both equations are the same. For instance, with regard to equations (15) and (49), defined on the set of integer numbers, their equivalency would mean that for any combination of terms $(2k+1)$, $(2p+1)$, $2m$ in (15) there is only one combination of terms $(2k_1+1)$, $(2p_1+1)$, $2m_1$ in (49), such, that $(2k+1)=(2k_1+1)$, $(2p+1)=-(2p_1+1)$, $m=m_1$, so that with such a substitution equation (15) becomes equation (49). Similarly, the substitution $(2k_1+1)=(2k+1)$, $(2p_1+1)=-(2p+1)$, $m_1=m$ in (49) produces equation (15). It was proved that (15) has no solution in integer numbers, so that it has no solution for any combination of these terms. However, on the set of all possible pairs of odd numbers, on which both equations are defined, these are equivalent equations, as it will be shown. Then, since (15) has no solution, (49) will have no solution too.

Lemma 14: *Equation (15) is equivalent to equation (49) on the set of integer numbers. If one of these equations has no solution in integer numbers, then the other equation also has no solution.*

Proof: Since the odd power does not change the algebraic sign, we can rewrite (15) as follows.

$$(2k+1)^{2n+1}+(-2p-1)^{2n+1}=(2m)^{2n+1} \quad (50)$$

k, p and k_1, p_1 in (15), (49) are integers defined on the range $(-\infty,+\infty)$. So, we can do a substitution $p=-p_1-1$.

$$(2k_1+1)^{2n+1}+(2p_1+1)^{2n+1}=(2m)^{2n+1} \quad (51)$$

where $k=k_1$. In this transformation, the range of parameters and equations' terms remains the same, that is $(-\infty<p<\infty)$, $(-\infty<p_1<\infty)$, and so $(-\infty<(2p+1)<\infty)$ $(-\infty<(2p_1+1)<\infty)$. Thus, equation (50), which is (15), became equation (51), which is (49). The substitution $p=-p_1-1$ is an equivalent one, because (i) it does not change the range of the substituted parameter, neither it changes the ranges of the terms, defined by these parameters; (ii) this is a one-to-one substitution.

Similarly, we can obtain equation (15) from (49), using substitution $p_1=-p-1$ in (49).

$$(2k_1+1)^{2n+1}+(2(-p-1)+1)^{2n+1}=(2m)^{2n+1} \quad (52)$$

This transforms into equation (15).

$$(2k+1)^{2n+1}-(2p+1)^{2n+1}=(2m)^{2n+1} \quad (53)$$

where $k_1=k$.

Thus, (15) and (49), indeed, are equivalent equations.

Now, we should prove that if one of these equations has no solution, then the other equation also has no solution. For that, let us assume that equation (49) has no solution, while the equivalent equation (15) has a solution for the parameters (k_0,p_0,m_0), that is

$$(2k_0+1)^{2n+1}-(2p_0+1)^{2n+1}=(2m_0)^{2n+1} \quad (54)$$

Doing an equivalent substitution $p_0=-p_1-1$, one obtains

$$(2k_0+1)^{2n+1}+(2p_1+1)^{2n+1}=(2m_0)^{2n+1} \quad (55)$$

Equation (55) (which is the original equation (49)), accordingly, has a solution for the parameters (k_0,p_1,m_0). However, this contradicts to the assumption that (49) has no solution. So, the equivalent equation (15) also has no solution. Similarly, we can assume that (15) has no solution, while (49) has a solution, and find through similar contradiction that (49) has no solution.

This completes the proof of Lemma.

It follows from Lemma 14 that it is suffice to prove that only one of the equations, (15) or (49), has no solution, in order to prove that both equations have no solution. Previously, we found that (15) has no solution in integer numbers. So, according to Lemma 14, (49), which presents case 3 for equation (1), also has no solution.

5. Case 4

In this case, $a=2n$; $x=2p+1$; $y=2m$, $z=2k+1$. Equation (1) can be presented in two forms.

$$(2k+1)^{2n}-(2p+1)^{2n}=(2m)^{2n} \quad (56)$$

$$(2p+1)^{2n}+(2m)^{2n}=(2k+1)^{2n} \quad (57)$$

Because of the even power $a=2n$, we may consider (56) and (57) as defined on the set of integer numbers.

Let us consider (57). It can be rewritten as follows.

$$[(2p+1)^n]^2+[(2m)^n]^2=[(2k+1)^n]^2 \quad (58)$$

We will use Theorem 1 (p. 38) from Chapter 2 in [5]. The Theorem says the following: *All the primitive solutions of the equation* $x^2+y^2=z^2$ *for which y is even number are given by the* *formulae* $x = M^2 - N^2$, $y = 2MN$, $z = M^2 + N^2$, *where M, N are taken to be pairs of relatively prime numbers, one of them even and the other odd and M greater than N.*

All solutions of (58) are defined as follows.

$$(2p+1)^n = (M^2 - N^2)L;\ (2m)^n = 2MNL;\ (2k+1)^n = (M^2 + N^2)L \tag{59}$$

Here, in accordance with the aforementioned Theorem 1, *M* and *N* are pairs of relatively prime natural numbers, one of them even and the other is odd, and *M* > *N*. Substituting (59) into (58), we can see that by dividing both parts by L^2, it can be reduced to an equation, whose terms have no common divisor. So, if a solution of such an equation exists, it can be reduced to a primitive solution, and vice versa - any non-primitive solution can be obtained from a primitive solution. Thus, it is suffice to consider only primitive solutions.

For the primitive solution, using the first and the third formulas from (59), we can write.

$$(2k+1)^n = (M^2 + N^2) \tag{60}$$

$$(2p+1)^n = (M^2 - N^2) \tag{61}$$

Equations (60) and (61) are independent. Indeed, there is no way to obtain one from another by transformations. (Formally, the independence can be proved considering the matrix rank of these equations in a linear representation).

Below, we assume that *M* and *N* are interchangeably equal to $(2c+1)$ and $(2d)$, and *M* > *N*.

5.1. The case of odd n

Lemma 15: Equation

$$(2p+1)^{2n} + (2m)^{2n} = (2k+1)^{2n}$$

has no solution in integer numbers, when *n* is odd.

Proof: Let $n = 2q+1$. We will consider scenario 1 first, when $M = (2c+1)$, $N = (2d)$

$$(2k+1)^n = (M^2 + N^2) \tag{62}$$

Applying binomial expansion to the left part, one obtains.

$$\sum_{i=0}^{2q-1} C_i^{2q+1}(2k)^{2q+1-i} + (2q+1)(2k) + 1 = 4c^2 + 4c + 1 + 4d^2 \tag{63}$$

It transforms into

$$\sum_{i=0}^{2q-1} C_i^{2q+1}(2k)^{2q+1-i} + (2q+1)(2k) = 4(c^2 + c + d^2) \tag{64}$$

Dividing (64) by two, one obtains

$$\sum_{i=0}^{2q-1} C_i^{2q+1} k (2k)^{2q-i} + (2q+1)k = 2(c^2 + c + d^2) \tag{65}$$

The right part is even. The left part is odd when *k* is an odd *integer*, $(-\infty < k < \infty)$. In this case, (65) has no integer solution.

The note that *k* is an integer number is important. Indeed, we can make such an assumption because when *k* is negative, equation (62) has no solution for any *k*, since the left part of (62) in this case is negative, while the right part is positive.

Similarly, we consider equation

$$(2p+1)^n = (M^2 - N^2) \tag{66}$$

$$\sum_{i=0}^{2q-1} C_i^{2q+1}(2p)^{2q+1-i} + (2q+1)(2p) + 1 = 4c^2 + 4c + 1 - 4d^2 \tag{67}$$

Transforming this equation and dividing both parts by two, one obtains

$$\sum_{i=0}^{2q-1} C_i^{2q+1} p(2p)^{2q-i} + (2q+1)p = 2(c^2 + c - d^2) \qquad (68)$$

This equation has no integer solution for odd *integer p*, since the left part is odd, and the right part is even in this case. Similar to *k* in (62), *p* is an integer $(-\infty < p < \infty)$. Indeed, when *p* is negative, (66) has no solution too, since the left part becomes negative, while the right part, due to the condition *M* > *N*, is positive, so that (66) has no solution for negative *p*.

Table 8 presents values of parameters for the first scenario (row 1). Odd values of *k*=2*t*+1 and *p*=2*s*+1 correspond to $(2k+1) = (4t+3)$ and $(2p+1) = (4s+3)$, where *t* and *s* are integers.

Table 8. Values of parameters for the considered scenarios, when (62) and (66) have no solution.

Scen.	*M*	*N*	*k*	*p*	*2k+1*	*2p+1*
1	*2c+1*	*2d*	*2t+1*	*2s+1*	*4t+3*	*4s+3*
2	*2d*	*2c+1*	*2t+1*	*2s*	*4t+3*	*4s+1*

Since equations (62) and (66) are independent, the obtained values of $(4t+3)$ and $(4s+3)$ can be paired in equation (57) with any odd number. (When expressed with a factor of four, these are the numbers $(4s+1)$ and $(4s+3)$ for $(4t+3)$, and $(4t+1)$ and $(4t+3)$ for $(4s+3)$). Three found pairs, for which (57) has no solution, are shown in Table 9 in bold. The missing pair is $[\,(4t+1), (4s+1)\,]$.

Table 9. Found pairs of odd numbers (bold) for scenario 1, for which there is no integer solution.

	0	1	2	3	4
a	*k* *p*	***2t*** ***2s+1***	***2t+1*** ***2s***	*2t* *2s*	***2t+1*** ***2s+1***
b	*2k+1* *2p+1*	***4t+1*** ***4s+3***	***4t+3*** ***4s+1***	*4t+1* *4s+1*	***4t+3*** ***4s+3***

Let us consider scenario 2, when *M* is even and *N* is odd (row 2 in Table 8). Equation (62) has no solution for the odd integer *k* in this case (it is obvious that swapping *M* and *N* in (62) does not influence the previous result), $(-\infty < k < \infty)$. Equation (66) transforms as follows.

$$\sum_{i=0}^{2q-1} C_i^{2q+1} (2p)^{2q+1-i} + (2q+1)(2p) + 1 = 4d^2 - 4c^2 - 4c - 1 \qquad (69)$$

Transforming this equation and dividing both parts by two, one obtains

$$\sum_{i=0}^{2q-1} C_i^{2q+1} p(2p)^{2q-i} + (2q+1)p + 1 = 2(d^2 - c^2 - c) \qquad (70)$$

This equation has no integer solution for even integer *p* $(-\infty < p < \infty)$ The obtained values of odd integer *k* and even integer *p* correspond to numbers $(4t+3)$ and $(4s+1)$, where *t* and *s* are integers. Since these numbers are obtained independently, each can be combined in pair with any odd number. The resulting combinations, shown in Table 10, are in bold. This time, we obtained all possible combinations of odd numbers, expressed with a factor of four.

Table 10. Found pairs of odd numbers for scenario 2 (from Table 8), when (57) has no integer solution.

	0	1	2	3	4
a	*k* *p*	***2t*** ***2s+1***	***2t+1*** ***2s***	***2t*** ***2s***	***2t+1*** ***2s+1***
b	*2k+1* *2p+1*	***4t+1*** ***4s+3***	***4t+3*** ***4s+1***	***4t+1*** ***4s+1***	***4t+3*** ***4s+3***

So, we need to prove that equation (57) has no solution for the pair of odd numbers $[(4t+1),(4s+1)]$ from scenario 1. Let us substitute the found pair $[(4t+3),(4s+3)]$ from Table 8, for which (57) has no solution, into this equation. One obtains

$$(4t+3)^{2n}+(2m)^{2n}=(4s+3)^{2n} \tag{71}$$

Since (71) is defined on the set of integer numbers, we can use Corollary 4 and do equivalent substitutions of $(4t+3)$ by $(-(4t_1+1))$, and $(4s+3)$ by $(-(4s_1+1))$, where t_1 and s_1 are integers, thus obtaining an equivalent equation.

$$(-(4t_1+1))^{2n}+(2m)^{2n}=(-(4s_1+1))^{2n} \tag{72}$$

which, due to even power, can be rewritten as

$$(4t_1+1)^{2n}+(2m)^{2n}=(4s_1+1)^{2n} \tag{73}$$

Since (71) has no solution, and (73) is equivalent to (71), this means that (73) also has no solution. This proves that (57) has no solution for the pair $[(4t+1),(4s+1)]$. Now, we found that (57) has no solution for all possible pairs of odd numbers with a factor of four in Table 9, so that (57) has no solution for both scenarios.

This proves the Lemma.

5.2. Even n

Lemma 16: Equation $(2p+1)^{2n}+(2m)^{2n}=(2k+1)^{2n}$ has no solution in integer numbers when n is even.

Proof: Let $n=2q$. Then, (58) can be presented as follows.

$$[(2k+1)^q]^4-[(2p+1)^q]^4=[(2m)^{2q}]^2 \tag{74}$$

According to Corollary 1 (p. 52) from Chapter 2 in [5], equation (74) has no solutions in natural numbers (because of the even power, this also means that (74) has no solution in integer numbers). The corollary is read as follows: *There are no natural numbers a, b, c such that* $a^4-b^4=c^2$.

Since $(2k+1)^q$, $(2p+1)^q$ and $(2m)^{2q}$ cannot be natural numbers, $(2k+1)$, $(2p+1)$ and $(2m)$ cannot be natural numbers too. Indeed, if one assumes that these are natural numbers, then, raised to appropriate powers, such numbers have to be natural numbers too, which contradicts to the aforementioned Corollary.

So, equations (57), (58) have no solution for even *n*.

The same result can be obtained using the property that there is no Pythagorean triangle, whose sides are squares. Indeed, we can rewrite (74) as follows.

$$[(2p+1)^q]^4+[(2m)^q]^4=[(2k+1)^q]^4 \tag{75}$$

Corollary 2 on p. 53, Chapter 2 in [5], says: *There are no natural numbers x, y, z satisfying the equation* $x^4+y^4=z^4$. This means that (74) and (75), and consequently (57) and (58) for even *n*, have no solution in natural numbers. However, because of the even power, the result is valid for integer numbers too. This proves the Lemma.

Thus, we proved that (56) - (58) have no solution in integer numbers for odd and even *n*.

6. Conclusion

We found that in each of four cases, corresponding to equation (1), the appropriate equations have no solution in integer numbers. This means that (1) has no solution in integer numbers.

Introduced concepts and approaches can be applied to other problems of number theory.

7. Acknowledgements

The author indebted to A. A. Tantsur for the multiple reviews, discussions and thoughtful suggestions, which much contributed to elimination of errors and improvement of the material.

References

1. Wiles, A.: Modular Elliptic Curves and Fermat's Last Theorem. Annals of Mathematics Second Series. 141(3), 443-551 (1995)
2. Shestopaloff, Yu. K.: (2019, May 18). Parity Properties of Equations, Related to Fermat Last Theorem (V. 1). http://doi.org/10.5281/zenodo.2933645; (2020, February 17). Parity properties of some power equations (V. 3). http://doi.org/10.5281/zenodo.3669393
3. Steuding, J.: Probabalistic number theory (2002) https://web.archive.org/web/20111222233654/http://hdebruijn.soo.dto.tudelft.nl/jaar2004/prob.pdf
4. Niven, I.: The asymptotic density of sequences. Bull. Amer. Math. Soc. 57 no. 6, 420-434. https://projecteuclid.org/euclid.bams/1183516304 (1951)
5. Sierpinski, W.: Elementary theory of numbers. PWN - Polish Scientific Publishers, Warszawa. (1988)