On the existence of prime numbers in constant gaps

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"Entia non sunt multiplicanda praeter necessitatem" (Ockam, W.)

Abstract

This paper studies the existence of prime numbers on the constant gaps defined as

$$G_{a,b} := (ab, (a+1)\,b)$$

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1 Proposition of the problem

Definition 1.0.1. *Let two positive integer numbers be a and b. Then, being b a constant, we define a constant gap as*

$$G_{a,b} := (ab, (a+1)\,b)$$

Some interesting question that arises and which this paper addresses is: assuming b is constant, is there necessarily a prime number in a given $G_{a,b}$?

From the definition, and looking at the literature, it can be found that if we let b be a constant and $a = 1$, the answer is yes. This result is called Bertrand's Theorem. The following question of interest that arises is to study what occurs when b remains constant and $a = 2$. And what if $a = 3$?

In other words: which is the highest value for a, such that there is necessarily a prime number in a given constant gap $G_{a,b}$?

2 Lower bound for a

In this Section we will adress the question posed, trying to find the highest posible lower bound for a, such that there is not necessarily a prime number in a given constant gap $G_{a,b}$.

2.1 Previous considerations.

- It is worth noting that every gap $G_{a,b}$ contains exactly $b-1$ numbers, whereas the number of prime and composite numbers less than b is $b-2$, as the number 1 is neither prime nor composite.
- One can also observe that any positive integer n in a given gap $G_{a,b}$ is composite if and only if $n = pq$, where p is some prime number less than $\sqrt{(a+1)\,b}$.
- Finally, it can also be noted that $p < b$ for $a \leq b+1$, since $p_{n+1} = b+1$ implies that

  $$p_{n+1}^2 = (b+1)^2$$

  and

  $$(b+1)^2 > (b+2)\,b$$

2.2 Important Lemmas

Consider the gap $G_1 = [p_k, np_k]$ where p_k is some prime number and $np_k < b$, and the gap $G_m = [mp_k, (m+n)p_k]$ with p_k being the same prime number and $(m-1)p_k < ab < mp_k$.

It is trivial the fact that $G_1 \subset G_{0,b}$, so there are exactly n multiples of prime number p_k less than b.

It is also trivial the fact that $G_m \subset G_{a,b}$, but there can be n or $n+1$ multiples of prime number p_k contained in the gap $G_{a,b}$, as showed by the following:

Lemma 2.1.1. *Let it be $G_m \subset G_{a,b}$. Then, if $(m+n)p_k - ab < b$, the gap $G_{a,b}$ has exactly $n+1$ multiples of prime number p_k, whereas if $(m+n)p_k - ab \geq b$, the gap $G_{a,b}$ has exactly n multiples of prime number p_k.*

Proof. If $(m+n)p_k - ab < b$, then $G_m = [mp_k, (m+n)p_k]$, which contains exactly $n+1$ multiples of prime number p, whereas if $(m+n)p_k - ab \geq b$, then $G_m = [mp_k, (m+n-1)p_k]$, which contains exactly n multiples of prime number p.

It can be seen that $(m+n)p_k - ab < b$ when $mp_k - ab < b - np_k$, and $(m+n)p_k - ab > b$ when $mp_k - ab \geq b - np_k$.

Lemma 2.1.2. *Gap $G_{a,b}$ has exactly the same number of multiples of 2 as the number of multiples of 2 (including itself) which are less than b.*

Proof. As the inequality $mp_k - ab \geq b - np_k$ stated in the proof of Lemma 2.1.1. holds for $p = 2$ independently of the values of m, b and n, then there are exactly the same number of multiples of 2 as the number of multiples of 2 (including itself) which are less than b.

Lemma 2.1.3. *Let it be A_k the set of composite numbers multiples of some p_k of any gap $G_{a,b}$. The inclusion-exclusion principle guarantees that the number of composite numbers of any gap $G_{a,b}$ is equal to*

$$\sum_{i=1}^{k} \mid A_i \mid - \sum_{1 \leq i_1 < i_2 \leq k} \mid A_{i_1} \cap A_{i_2} \mid + ... + (-1)^{j+1} \sum_{1 \leq i_1 < ... < i_j \leq k} \mid A_{i_1} \cap ... \cap A_{i_j} \mid + ...$$

$$... + (-1)^{k+1} \mid A_1 \cap A_2 \cap ... \cap A_k \mid$$

Lemma 2.1.4. *The number of intersections (I) of some prime p_k of any gap $G_{a,b}$ such that $a > 0$ is always equal or greater than the number of intersections of p_k in the gap $G_{0,b}$.*

Proof. Some number is not an intersection only if is a power of some prime number (including the prime number itself). In gap $G_{0,b}$ there are all the prime numbers which are prime factors in any other gap $G_{a,b}$. Finally, the number of powers of some prime p_k in gap $G_{a,b}$ can be at most equal to the number of powers of p_k in $G_{0,b}$, and this only could happen if the only power of p_k in $G_{0,b}$ is p_k itself.

Lemma 2.1.5. *For $p_k > \frac{b}{2}$, odd, any gap $G_{a,b}$ has at most 2 multiples of prime number p_k, and one of them is multiple of 2. Therefore, gap $G_{a,b}$ has at most 1 multiple of some prime number $p_k > \frac{b}{2}$ which is not an intersection.*

Proof. If $p_k > \frac{b}{2}$, then there can be no multiple of p_k less than b; therefore, by Lemma 2.1.1., the gap $G_{a,b}$ has at most 2 multiples mp_k or $(m+1)p_k$. When picking two consecutive positive integers m and $m+1$ randomly, then m or $m+1$ is even. Thus, mp_k or $(m+1)p_k$ is multiple of 2. Excepting the case of this odd multiple being a perfect power of p_k or multiple of some prime number $p_i > b$, the other multiple would be also an intersection. This implies that, if the gap $G_{0,b}$ has n multiples of prime number p_k and of them I are intersections, and $G_{a,b}$ has $n+1$ multiples of prime number p_k, then $I+1$ of this multiples are intersections.

Lemma 2.1.6. *For $p_k < \frac{b}{2}$, odd, and $a \leq b+1$, if the gap $G_{0,b}$ has n multiples of prime number p_k and of them I are intersections, and $G_{a,b}$ has $n+1$ multiples of prime number p_k, this additional element of the gap $G_k = [mp_k, (m+n)p_k]$ is an intersection. Thus, from the $n+1$ multiples of prime number p_k in the gap $G_{a,b}$, $I+1$ are intersections.*

Proof. As per Lemma 2.1.2, gap $G_{a,b}$ has exactly the same number of multiples of 2 as the number of multiples of 2 (including itself) which are less than b. Thus, if the gap $G_{a,b}$ has exactly $n+1$ multiples of prime number p_k, then the "extra" multiple must be necessarily odd.

If $p_k < \frac{b}{2}$, then $|G_m| \geq 3$, and thus any composite number $c \in G_m$ must be multiple of p_k and some other odd prime number less than b, even if there is other composite number being some power of p_k. Per Lemma 2.1.2. there are at least two odd composite numbers,so at least there is one of them multiple of p_k and some other odd prime number less than b *which is not p_k itself.* Thus, the additional element of the gap $G_m = [mp_k, (m+n)p_k]$ is an intersection.

2.3 Main result

Now it is possible to enunciate and prove the main result of the paper.

Theorem 2.3.1. *Letting $a, b \in \mathbb{N}$, $b \geq 2$ constant and $a \leq b+1$, there exists at least one prime number in the gap $G_{a,b}$.*

Proof.

Applying the inclusion-exclusion principle, and considering the Lemmas exposed, the number of composite numbers of any gap $G_{a,b}$ can be at most the sum of all the prime and composite numbers of gap $G_{0,b}$, which, as stated at the Previous considerations section, account $b-2$ numbers.

Since every $G_{a,b}$ has exactly $b-1$ numbers, we can conclude that at least there is necessarily a prime number in a given $G_{a,b}$ for an arbitrary $a \leq b+1$ and b constant.

2.4 Further considerations

From the theorem and the Lemmas used, we find that there is a necessary condition to be satisfied in order to state that there is not necessarily a prime in a given $G_{a,b}$:

Necessary condition.

- There must exist at least some composite number $n = pq$ in the given $G_{a,b}$, where p and q are prime numbers or multiples of prime numbers greater than b.

As the minimum composite number $n = pq$ where p and q are prime numbers or multiples of prime numbers greater than b, is $p = q = b + 1$, then, assuming that the application of the inclusion-exclusion principle yields that the number of composite numbers of any gap $G_{a,b}$ equals to the sum of all the prime and composite numbers of gap $G_{0,b}$, the value $a = b + 2$ becomes the solution to the question of which is the lower bound for a, so there is not necessarily a prime number in a given $G_{a,b}$.

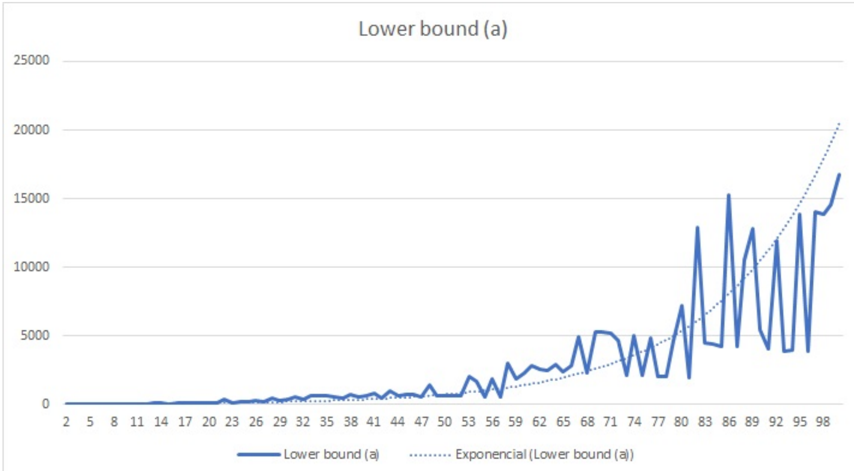
This lower bound could be much higher, since the lower bound $a = b+2$ assumes that the application of the inclusion-exclusion principle yields that the number of composite numbers of any gap $G_{a,b}$ equals to the sum of all the prime and composite numbers of gap $G_{0,b}$, when empirically we find that this ocurr in very few cases. Moreover, the difference between the number of intersections in gap $G_{a,b}$ and in gap $G_{0,b}$ widens when b increases, principally because the difference between the number of powers of some prime number p_k in gap $G_{0,b}$ and in any gap $G_{a,b}$ increases for bigger values of b.

As it is showed in the table below, which contains the different lower bounds of a for the values of b between 0 and 100, it seems that the growth of the lower bound grows exponentially compared to the growth of b. In fact, the established lower bound $a = b + 2$ is the best possible only for $b = 2$ and for $b = 6$.

Also, it can be noted that there is an increasing oscilatory trend, suggesting that for bigger values of b, the difference between the lower bound of a for b and the lower bound of a for $b + 1$ grows exponentially too.

b	Lower bound (a)	Beginning of first gap Ga,b with no prime numbers
2	4	8
3	8	24
4	6	24
5	18	90
6	15	90
7	17	119
8	25	200
9	13	117
10	20	200
11	29	319
12	44	528
13	87	1131
14	81	1134
15	35	525
16	83	1328
17	79	1343
18	74	1332
19	70	1330
20	67	1340
21	118	2478
22	330	7260
23	58	1334
24	223	5352
25	172	4300
26	229	5954
27	179	4833
28	471	13188
29	292	8468
30	360	10800
31	506	15686
32	367	11744
33	586	19338
34	577	19618
35	645	22575
36	545	19620
37	424	15688
38	743	28234
39	503	19617
40	637	25480
41	766	31406
42	467	19614
43	937	40291
44	579	25476
45	698	31410
46	683	31418
47	542	25474
48	1443	69264
49	641	31409
50	628	31400

51	616	31416
52	604	31408
53	2026	107378
54	1661	89694
55	571	31405
56	1834	102704
57	551	31407
58	2989	173362
59	1820	107380
60	2242	134520
61	2842	173362
62	2515	155930
63	2475	155925
64	2938	188032
65	2399	155935
66	2849	188034
67	4960	332320
68	2293	155924
69	5227	360663
70	5290	370300
71	5215	370265
72	4695	338040
73	2136	155928
74	5004	370296
75	2079	155925
76	4872	370272
77	2025	155925
78	1999	155922
79	4687	370273
80	7210	576800
81	1925	155925
82	12852	1053864
83	4461	370263
84	4408	370272
85	4243	360655
86	15273	1313478
87	4256	370272
88	10544	927872
89	12809	1140001
90	5468	492120
91	4069	370279
92	11944	1098848
93	3878	360654
94	3939	370266
95	13826	1313470
96	3857	370272
97	13992	1357224
98	13849	1357202
99	14589	1444311
100	16718	1671800



3 Corollaries

As corollaries of the theorem stated before, there can proved many important conjectures in number theory, (e.g., Oppermann's, Andrica's, Brocard's, and Legendre's).

3.1 First corollary: Oppermann's Conjecture

Oppermann's Conjecture [1] can be expressed as follows:

$$\forall n > 1 \in \mathbb{N}, \exists P_a, P_b / n^2 - n < P_a < n^2 < P_b < n^2 + n$$

This is equivalent to the Conjecture proved, for the cases $a = b - 1$ and $a = b$ put together, so the Conjecture proof implies directly Oppermann's Conjecture proof.

3.2 Second corollary: Legendre's Conjecture

Legendre's Conjecture[2] states that for every natural number n, exists at least a prime number p such that $n^2 < p < (n+1)^2$.

As $(n+1)^2 = n^2 + 2n + 1$, and according to Oppermann's Conjecture proved, we know that:

$$n^2 < P_a < n^2 + n < P_b < (n+1)^2$$

Therefore,

$$n^2 < P_a < P_b < (n+1)^2$$

Subsequently, it is demonstrated Legendre's Conjecture.

3.3 Third corollary: Brocard's Conjecture

Brocard's Conjecture[3] states that, if p_n and p_{n+1} are two consecutive prime numbers greater than two, then between p_n^2 and p_{n+1}^2 exist at least four prime numbers.

According to the conjecture's statement,

$$2 < p_n < p_{n+1}$$

As the minimum distance between primes is two, we can state that:

$$p_n < M < p_{n+1}$$

Where M is some natural number between p_n and p_{n+1}. Subsequently,

$$p_n^2 < M^2 < p_{n+1}^2$$

As $M \geq p_n + 1$, and according to the demonstrated Oppermann's conjecture,

$$p_n^2 < P_a < p_n^2 + p_n < P_b < M^2$$

Idem, as $p_{n+1} \geq M + 1$, and according to Oppermann's Conjecture proved,

$$M^2 < P_c < M^2 + M < P_d < p_{n+1}^2$$

Therefore,

$$p_n^2 < P_a < P_b < P_c < P_d < p_{n+1}^2$$

Subsequently, it is demonstrated Brocard's Conjecture.

3.4 Fourth corollary: Andrica's Conjecture

Andrica's Conjecture[4] states that for every pair of consecutive prime numbers p_n and p_{n+1}, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$

According to the demonstrated Oppermann's Conjecture, the maximum distance between p_n and p_{n+1} is:

$$n^2 + n + 1 \leq P_n < (n+1)^2 < p_{n+1} \leq n^2 + 3n + 1$$

It is easily verifiable that:

$$\sqrt{n^2 + 3n + 1} - \sqrt{n^2 + n + 1} < 1$$

For every value of n. As $n^2 + 3n + 1 \geq p_{n+1}$, and $P_n \geq n^2 + n + 1$, then $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$

Therefore, it is demonstrated Andrica's Conjecture.

3.5 Fifth corollary: a new maximum interval between every natural number and the nearest prime number

According to the exposed in the fourth corollary, it can be stated that the maximum distance between every natural number and the nearest prime number will be:

$$n^2 + 3n - (n^2 + n + 1) = 2n - 1$$

Therefore, and stating that:

$$n = \sqrt{n^2 + n + 1}$$

It can be determined that:

$$\forall n \in \mathbb{N}, \exists P_a, P_b / (n - (2\sqrt{n} - 1)) \leq P_a \leq n \leq P_b \leq (n + (2\sqrt{n} - 1))$$

And therefore, we can define a new maximum interval between every natural number and the nearest prime number as:

$$\forall n \in \mathbb{N}, \exists P / n \leq P \leq (n + (2\sqrt{n} - 1))$$

3.6 Sixth corollary: the existence of infinite prime numbers of the form $n^2 \pm k/0 < k < n$

According to the demonstrated Oppermann's Conjecture, it can be stated that every prime number p_i will be of the following form:

$$p_i = n^2 \pm k/0 < k < n$$

Subsequently, as it is widely proved the existence of infinite prime numbers, and every prime number can be expressed as $n^2 \pm k/0 < k < n$, then it is proved the existence of infinite prime numbers of the form $n^2 \pm k/0 < k < n$.

References

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