On some equations concerning the Casimir Effect Between World-Branes in Heterotic M-Theory and the Casimir effect in spaces with nontrivial topology. Mathematical connections with some sectors of Number Theory.

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Abstract

The present paper is a review, a thesis of some very important contributes of P. Horava, M. Fabinger, M. Bordag, U. Mohideen, V.M. Mostepanenko, Trang T. Nguyen et al. regarding various applications concerning the Casimir Effect.
In this paper in the **Section 1** we have showed some equations concerning the Casimir Effect between two ends of the world in M-Theory, the Casimir force between the boundaries, the Casimir effect on the open membrane, the Casimir form and the Casimir correction to the string tension that is finite and negative. In the **Section 2**, we have described some equations concerning the Casimir effect in spaces with nontrivial topology, i.e. in spaces with non-Euclidean topology, the Casimir energy density of a scalar field in a closed Friedmann model, the Casimir energy density of a massless field, the Casimir contribution and the total vacuum energy density, the Casimir energy density of a massless spinor field and the Casimir stress-energy tensor in the multi-dimensional Einstein equations with regard the Kaluza–Klein compactification of extra dimensions.
Further, in the **Section 1** and **2** we have described some mathematical connections concerning some sectors of Number Theory, i.e. the Palumbo-Nardelli model, the Ramanujan modular equations concerning the physical vibrations of the bosonic strings and the superstrings and the connections of some values contained in the equations with some values concerning the new universal music system based on fractional powers of Phi and Pigreco**.**
In the **Section 3**, we have described some mathematical connections concerning the Riemann zeta function and the zeta-strings. In conclusion, in **Section 4**, we have described some mathematical connections concerning some equations regarding the Casimir effect and vacuum fluctuations. In conclusion (**Appendix A**), we have described some mathematical connections between the equation of the energy negative of the Casimir effect, the Casimir operators and some sectors of the Number Theory, i.e. the triangular numbers, the Fibonacci's numbers, Phi, Pigreco and the partition of numbers.

1. On some equations concerning the Casimir force between the boundaries and the Casimir effect on the open membrane. [1]

We will now demonstrate that the Casimir force at large separation L is attractive. We start with the $E_8 \times \overline{E}_8$ model on $\mathcal{M} = R^{10} \times S^1 / Z_2$ with a flat, direct-product metric

$$ds_0^2 = \eta_{AB} dx^A dx^B + L^2 dz^2 , \quad (1.1)$$

where $x^A, A = 0,...9$ are coordinates on R^{10}, and we have introduced a rescaled coordinate z on the S^1/Z_2 factor such that $z \in [0,1]$. We assume that the distance L between the boundaries is constant and large in Planck units. The geometry (1.1) represents a classical solution of the theory. Quantum fluctuations of the fields on $\mathcal{M}$ generate a non-zero expectation value $\langle T_{MN} \rangle$ of the energy-momentum tensor, which then modifies the classical flat static geometry of $\mathcal{M}$. At large separations L, this effect can be systematically studied in the long-wavelength expansion, i.e., in the perturbation theory in powers of ℓ_{11}/L.

The Poincaré symmetry of the background metric (1.1) implies that $\langle T_{MN} \rangle$ takes the form

$$\langle T_{MN} \rangle dx^M dx^N = -E(z) \eta_{AB} dx^A dx^B + F(z) L^2 dz^2 , \quad (1.2)$$

with $E(z)$ and $F(z)$ are in general some functions of z. The one-loop energy-momentum tensor in the flat background (1.1) has to be traceless. This implies that $F = 10E(z)$, and therefore $E(z) = E_0$ is a constant and the energy-momentum tensor (1.2) takes the following general form,

$$\langle T_{MN} \rangle dx^M dx^N = -E_0 \left(\eta_{AB} dx^A dx^B - 10 L^2 dz^2 \right). \quad (1.3)$$

The remaining constant E_0 plays the role of the vacuum energy density in the eleven-dimensional theory, and can be efficiently determined by Kaluza-Klein reducing the theory from $R^{10} \times S^1/Z_2$ to R^{10}, and calculating the effective one-loop energy-momentum tensor $\langle T_{AB} \rangle_{10}$ of all the KK modes in R^{10}. By Poincaré symmetry, we have

$$\langle T_{AB} \rangle_{10} = -\tilde{E}_0 \eta_{AB} , \quad (1.4)$$

where $\tilde{E}_0$ is the vacuum energy density in ten dimensions, or the one-loop effective cosmological constant. $\tilde{E}_0$ is related to the vacuum energy density E_0 in eleven dimensions by

$$\tilde{E}_0 = L \int dz E_0 = L E_0 . \quad (1.5)$$

The one-loop energy density $\tilde{E}_0$ is conveniently given by

$$\tilde{E}_0 = -\int \frac{1}{(2\pi)^{10}} d^{10} p \sum_{p_i} (-1)^{F_i} \int_0^\infty \frac{1}{2\ell} d\ell e^{-\left(p^2 + p_i^2\right)\ell/2} , \quad (1.6)$$

where the sum over p_i represents the sum over all Kaluza-Klein momenta as well as all possible polarizations in the supergravity multiplet, and $(-1)^{F_i}$ is the fermion number. From the ten-dimensional perspective, the KK reduction gives 128 bosonic polarizations at each mass level $\pi m/L$ for m a positive integer, and 128 fermionic polarizations at each mass level $\pi r/L$ for r a positive odd-half-integer. In addition, 64 out of the original 128 massless bosons also survive the orbifold projection from S^1 to S^1/Z_2. Altogether, (1.6) becomes

$$\tilde{E}_0 = -64\int_0^\infty \frac{d\ell}{2\ell}\frac{1}{(2\pi\ell)^5}\left(\sum_{m\in Z} e^{-m^2\pi^2\ell/2L^2} - \sum_{r\in Z+\frac{1}{2}} e^{-r^2\pi^2\ell/2L^2}\right) = -64\int_0^\infty \frac{d\ell}{2\ell}\frac{1}{(2\pi\ell)^5}\sum_{s\in Z}(-1)^s e^{-s^2\pi^2\ell/8L^2} =$$

$$= -64\int_0^\infty \frac{d\ell}{2\ell}\frac{1}{(2\pi\ell)^5}\theta_4\left(0\middle|i\pi\ell/8L^2\right), \quad (1.7)$$

where $\theta_4(u|t)$ is one of the Jacobi theta functions.

We note that this equation can be connected with the expression (B9) concerning the Palumbo-Nardelli model and the Ramanujan's modular equation. Indeed, we have:

$$\tilde{E}_0 = -8^2\int_0^\infty \frac{d\ell}{2\ell}\frac{1}{(2\pi\ell)^5}\left(\sum_{m\in Z} e^{-m^2\pi^2\ell/2L^2} - \sum_{r\in Z+\frac{1}{2}} e^{-r^2\pi^2\ell/2L^2}\right) = -8^2\int_0^\infty \frac{d\ell}{2\ell}\frac{1}{(2\pi\ell)^5}\sum_{s\in Z}(-1)^s e^{-s^2\pi^2\ell/8L^2} =$$

$$= -8^2\int_0^\infty \frac{d\ell}{2\ell}\frac{1}{(2\pi\ell)^5}\theta_4\left(0\middle|i\pi\ell/8L^2\right) \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (1.7b)$$

Rescaling the loop parameter $\ell \to \tau$ such that all the dependence on L is outside the integral, we thus obtain the following expression for the vacuum energy density per unit area of the boundary,

$$\tilde{E}_0 = -\mathcal{J}\cdot\frac{1}{L^{10}}, \quad (1.8)$$

with the L-independent factor $\mathcal{J}$ given by the integral

$$\mathcal{J} = \frac{1}{2^{15}}\int_0^\infty \frac{d\tau}{\tau^6}\theta_4\left(0\middle|i\tau\right). \quad (1.9)$$

It is easy to demonstrate that $\mathcal{J}$ is convergent and positive. First, change the variables to $t = 1/\tau$, and use the modular properties of the Jacobi theta functions, $\theta_4(0|T) = (-iT)^{-1/2}\theta_2(0|-1/T)$ to obtain

$$\mathcal{I} = \frac{1}{2^{15}} \int_0^{\infty} dt t^{9/2} \theta_2(0|it). \quad (1.10)$$

The theta function $\theta_2(0|it)$ is positive definite for real t, and decays exponentially as $t \to \infty$. Therefore, the integral over τ in (1.9) is convergent and positive. This shows that the vacuum energy density $\tilde{E}_0$ per unit boundary area as given by (1.8) is negative. Thus, we have demonstrated that the Casimir effect between the boundaries of the $E_8 \times \overline{E}_8$ model induces, in the leading order of the long-wavelength approximation, a negative cosmological constant. It is tempting to conclude that the negative ten-dimensional cosmological constant implies an attractive force between the two boundaries.

Using (1.3), (1.8), and (1.5), we obtain the one-loop energy-momentum tensor in eleven dimensions,

$$\langle T_{MN} \rangle dx^M dx^N = \frac{\mathcal{I}}{L^{11}} \left(\eta_{AB} dx^A dx^B - 10 L^2 dz^2 \right). \quad (1.11)$$

The Casimir force $\mathcal{F}$ between the boundaries (per unit boundary area) is given by

$$\mathcal{F} = \langle T_{\hat{z}\hat{z}} \rangle = -\frac{10\mathcal{I}}{L^{11}} < 0, \quad (1.12)$$

where $T_{\hat{z}\hat{z}}$ is the zz component of the energy-momentum tensor (1.11) in the orthonormal vielbein. It is reassuring that in this model the Casimir force $\mathcal{F}$ can also be obtained from the response of the energy density per unit boundary area to changing L,

$$\mathcal{F} = -\frac{\partial \tilde{E}_0}{\partial L} = -\frac{10\mathcal{I}}{L^{11}}. \quad (1.13)$$

We conclude that the leading-order Casimir force exerted on the boundaries in the $E_8 \times \overline{E}_8$ model at large L is indeed attractive. Notice that this force exhibits the typical Casimir-like scaling familiar from the conventional Casimir effect in electrodynamics.

The interaction between a perfectly conducting plate and an atom or molecule with a static polarizibility a is in the limit of large distance R given by

$$\delta E = -\frac{3}{8\pi} \hbar c \frac{a}{R^4}, \quad (1.14)$$

and the interaction between two particles with static polarizibilities a_1 and a_2 is given in that limit by

$$\delta E = -\frac{23}{4\pi} \hbar c \frac{a_1 a_2}{R^7}. \quad (1.15)$$

For very large a the interaction energy is given by the following integral

$$\delta E = \hbar c \frac{L^2}{\pi^2} \cdot \frac{\pi}{2} \left\{ \sum_{(0)1}^{\infty} \int_0^{\infty} \sqrt{\left(n^2 \frac{\pi^2}{a^2} + x^2 \right)} x dx - \int_0^{\infty} \int_0^{\infty} \sqrt{\left(k_z^2 + x^2 \right)} x dx \left(\frac{a}{\pi} dk_z \right) \right\}. \quad (1.16)$$

In order to obtain a finite result it is necessary to multiply the integrands by a function $f(k/k_m)$ *which is unity for* $k << k_m$ *but tends to zero sufficiently rapidly for* $(k/k_m) \to \infty$*, where* k_m *may be defined by* $f(1) = \frac{1}{2}$.

Introducing the variable $u = a^2 x^2 / \pi^2$

$$\delta E = L^2 \hbar c \frac{\pi^2}{4a^3} \left\{ \sum_{(0)1}^{\infty} \int_0^{\infty} \sqrt{n^2+u}\, f\left(\pi\sqrt{n^2+u}/ak_m\right) du - \int_0^{\infty}\int_0^{\infty} \sqrt{n^2+u}\, f\left(\pi\sqrt{n^2+u}/ak_m\right) du\, dn \right\}. \quad (1.17)$$

Now we apply the Euler-Maclaurin formula:

$$\sum_{(0)1}^{\infty} F(n) - \int_0^{\infty} F(n)\,dn = -\frac{1}{12}F'(0) + \frac{1}{24\times 30}F'''(0) + \ldots \quad (1.18)$$

Introducing $w = u + n^2$ *we have*

$$F(n) = \int_{n^2}^{\infty} w^{1/2} f\left(\frac{w\pi}{ak_m}\right) dw, \quad (1.19)$$

whence

$$F'(n) = -2n^2 f\left(\frac{n^2\pi}{ak_m}\right), \quad F'(0) = 0, \quad F'''(0) = -4. \quad (1.20)$$

The higher derivatives will contain powers of (π/ak_m). *Thus we find*

$$\frac{\delta E}{L^2} = -\hbar c \frac{\pi^2}{24\times 30} \cdot \frac{1}{a^3}, \quad (1.21)$$

a formula which holds as long as $ak_m >> 1$. *For the force per cm*^{2} *we find*

$$F = \hbar c \frac{\pi^2}{240} \frac{1}{a^4} = 0{,}013 \frac{1}{a_\mu^4} \text{ dyne/cm}^2 \quad (1.22)$$

where a_μ *is the distance measured in microns.*

There exists an attractive force between two metal plates which is independent of the material of the plates as long as the distance is so large that for wave lengths comparable with that distance the penetration depth is small compared with the distance. This force may be interpreted as a zero point pressure of electromagnetic waves. (Casimir, 1948 "On the attraction between two perfectly conducting plates")

We note that $\frac{3}{8\pi} = 0{,}119366207$ and $\frac{23}{4\pi} = 1{,}830281846$ are connected with 0,11936438 and 1,83121182 values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Also $\frac{\pi^2}{240} = 0{,}041123351$ and $\frac{\pi^2}{720} = 0{,}013707783$ are connected with $0{,}04111130$ and $0{,}01370377$ values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

We will now analyze this response of the metric to the non-zero $\langle T_{MN} \rangle$ of (1.11), in the leading order in the long-wavelength expansion. Consider the following general form of the metric on $R \times R^9 \times S^1 / Z_2$,

$$ds^2 = -dt^2 + a^2(t) g_{ij} dx^i dx^j + L^2(t) dz^2, \quad (1.23)$$

where we have again used the rescaled coordinate z along S^1 / Z_2, with $z \in [0,1]$. The indices $i, j = 1,...9$ parametrize the spacelike slice of the boundary geometry. The metric g_{ij} on R^9 is constrained by the symmetries of the problem to be of constant curvature, i.e., its Ricci tensor $\tilde{R}_{ij}$ satisfies $\tilde{R}_{ij} = k g_{ij}$. The initial configuration at $t = 0$ corresponds to

$$ds^2_{R^9 \times S^1 / Z_2} = g_{ij} dx^i dx^j + L_0^2 dz^2. \quad (1.24)$$

At zeroth-order, the metric is flat and the three-form gauge field C is zero, and we do not have to worry about corrections to Einstein's equations from higher-power curvature terms or the C-dependent terms in the Lagrangian. Thus, the equations of motion at first order in G_{11} are simply

$$R_{MN} = 8\pi G_{11} \langle T_{MN} \rangle. \quad (1.25)$$

Given our one-loop result for the energy-momentum tensor (1.11), we take $\langle T_{MN} \rangle$ in the form

$$\langle T_{MN} \rangle dx^M dx^N = \frac{\mathcal{J}}{L^{11}(t)} \left(-dt^2 + a^2(t) g_{ij} dx^i dx^j - 10 L^2(t) dz^2 \right), \quad (1.26)$$

where L is now allowed to depend on t. Notice that this adiabatic assumption is compatible with the requirement of energy-momentum conservation: the T_{MN} of (1.26) is conserved in the metric given by (1.23). The equations of motion (1.25) for (1.23) and (1.26) lead to

$$-\frac{9\ddot{a}}{a} - \frac{\ddot{L}}{L} = -8\pi G_{11} \frac{\mathcal{J}}{L^{11}}, \quad 8(\dot{a})^2 + a\ddot{a} + \frac{a}{L}\dot{a}\dot{L} + k = 8\pi G_{11} \frac{a^2 \mathcal{J}}{L^{11}}, \quad L\ddot{L} + \frac{L}{a}\dot{a}\dot{L} = -80\pi G_{11} \frac{\mathcal{J}}{L^9}. \quad (1.27)$$

Since we are looking for the leading backreaction of the initial configuration (1.24) to the $\langle T_{MN} \rangle$ given by (1.26) at small $t > 0$, we expand

$$L(t) = L_0 + \frac{1}{2} L_2 t^2 + \ldots, \quad a(t) = 1 + \frac{1}{2} a_2 t^2 + \ldots. \quad (1.28)$$

Plugging this expansion into (1.27) determines

$$k = -\frac{16\pi \mathcal{J} G_{11}}{9 L_0^{11}}, \quad L_2 = -\frac{80\pi \mathcal{J} G_{11}}{L_0^{10}}, \quad a_2 = \frac{88\pi \mathcal{J} G_{11}}{9 L_0^{11}}. \quad (1.29)$$

Thus, we reach the following conclusions:

(1) At leading order in G_{11}, the spacetime geometry responds to the Casimir force by moving the boundaries closer together, i.e., $L(t) < L_0$ for (small) times $t > 0$. At the same time, the metric on the transverse R^9 is rescaled by an increasing conformal factor $a(t) > 1$.

(2) Interestingly, the naïve initial configuration with $k = 0$, corresponding to two flat boundaries at finite distance apart, is incompatible with the constraint part of Einstein's equations. As we adiabatically bring in the second boundary from infinity, the geometry of the transverse R^9 responds by curving with a constant negative curvature given by k in (1.29).

We note that $\frac{16\pi}{9} = 5{,}585053$ and $\frac{88\pi}{9} = 30{,}717794$ are connected with 5,6075849 and 30,843458 values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Consider an open membrane stretched between the two boundaries of spacetime, with worldvolume $\Sigma = R^2 \times S^1 / Z_2$ parametrized by (σ^m, ρ), $m = 0,1,$ and with $\rho \in [0, L]$. In addition to $x^M(\sigma^M, \rho)$, the bulk worldvolume theory contains the spacetime spinor $\theta^\alpha(\sigma^m, \rho)$. All boundary conditions are induced from the Z_2 orbifold action on x^M, θ^α, and Σ. In particular, the fermions satisfy

$$\theta^\alpha(\sigma^m, -\rho) = \pm \Gamma_{10} \theta^\alpha(\sigma^m, \rho). \quad (1.30)$$

In our non-supersymmetric $E_8 \times \overline{E}_8$ model, corresponding to the (+, –) chirality choice, the chiralities of the E_8 current algebras disagree, and each boundary breaks a separate half of the original supersymmetry. We expect a worldvolume analog of the spacetime Casimir effect in the $E_8 \times \overline{E}_8$ model.

Consider an open membrane with worldvolume $\mathbf{R}^2 \times \mathbf{S}^1 / \mathbf{Z}_2$ stretching along $x^1, \ldots x^8 = 0$ between the two boundaries. It will again be more convenient to calculate the correction $\tilde{\tau} = L\tau$ to the vacuum energy density integrated over the compact dimension, i.e., the effective string tension. The first contribution to $\tilde{\tau}$ comes again from the mismatch between the boundary conditions on bulk bosons and bulk fermions on the worldvolume, and does not involve the boundary E_8 current algebras. Taking into account that we have eight fermionic and eight bosonic degrees of freedom at each non-zero mass level, we obtain

$$\tilde{\tau} = -\int \frac{d^2 p}{(2\pi)^2} \sum_{p_i} \int_0^\infty \frac{d\ell}{2\ell} e^{-(p^2 + p_i^2)\ell/2} = -4 \int_0^\infty \frac{d\ell}{2\ell} \frac{1}{2\pi\ell} \theta_4 \left(0 \middle| \frac{i\pi\ell}{8L^2} \right) = -\frac{1}{L^2} \int_0^\infty \frac{dt}{8t^2} \theta_4(0|it). \quad (1.31)$$

Also this equation can be related with the expression (B9) concerning the Palumbo-Nardelli model and the Ramanujan's modular equation. Indeed, we have:

$$\tilde{\tau} = -\int \frac{d^2 p}{(2\pi)^2} \sum_{p_i} \int_0^\infty \frac{d\ell}{2\ell} e^{-(p^2 + p_i^2)\ell/2} = -4 \int_0^\infty \frac{d\ell}{2\ell} \frac{1}{2\pi\ell} \theta_4 \left(0 \middle| \frac{i\pi\ell}{8L^2} \right) = -\frac{1}{L^2} \int_0^\infty \frac{dt}{8t^2} \theta_4(0|it) \Rightarrow$$

$$\Rightarrow -\int d^{26} x \sqrt{g} \left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(\left| F_2 \right|^2 \right) \right] \Rightarrow$$

$$\Rightarrow \frac{1}{3} \frac{4 \left[anti \log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]} . \quad (1.31b)$$

This again has the expected Casimir form, and prove that the Casimir correction $\tilde{\tau}$ to the string tension, as given by (1.31), is finite and negative. This negative Casimir tension competes with the positive bare string tension $\tilde{\tau}_0 \approx L\ell_{11}^{-3}$. While the supergravity approximation breaks down before we reach the regime of $L \approx \ell_{11}$, these results suggest that at distances L smaller than the eleven-dimensional Planck scale, the effective string that corresponds to the stretched open membrane becomes tachionyc.

2. **On some equations concerning the Casimir effect in spaces with nontrivial topology, the Casimir energy density of a scalar field in a closed Friedmann model, the Casimir energy density of a massless field, the Casimir contribution and the total vacuum energy density, the Casimir energy density of a massless spinor field and the Casimir stress-energy tensor in the multi-dimensional Einstein equations with regard the Kaluza–Klein compactification of extra dimensions**. **[2]**

In its simplest case the Casimir effect is the reaction of the vacuum of the quantized electromagnetic field to changes in external conditions like conducting surfaces.
It is well known in classical electrodynamics that both polarizations of the photon field have to satisfy boundary conditions

$$E_{t_{|S}} = H_{n_{|S}} = 0 \quad (2.1)$$

on the surface S of perfect conductors. Here n is the outward normal to the surface. The index t denotes the tangential component which is parallel to the surface S.
We imagine the electromagnetic field as a infinite set of harmonic oscillators with frequencies $\omega_J = c\sqrt{k^2}$. Here the index of the photon momentum in free space is $J = k = (k_1, k_2, k_3)$ where all k_i are continuous. In the presence of boundaries $J = (k_1, k_2, \pi n / a) = (k_\perp, \pi n / a)$, where $k_\perp$ is a two-dimensional vector, n is integer. The frequency results in

$$\omega_J = \omega_{k_\perp, n} = c\sqrt{k_1^2 + k_2^2 + \left(\frac{\pi n}{a} \right)^2} . \quad (2.2)$$

This has to be inserted into the half-sum over frequencies to get the vacuum energy of the electromagnetic field between the plates

$$E_0(a)=\frac{\hbar}{2}\int\frac{dk_1dk_2}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\omega_{k_\perp,n}S\ ,\quad (2.3)$$

where $S\to\infty$ is the area of plates. The expression obtained is ultraviolet divergent for large momenta. Therefore, we have to introduce some regularization. We perform the regularization by introducing a damping function of the frequency which was used in the original paper by Casimir and the modern zeta-functional regularization. We obtain correspondingly

$$E_0(a,\delta)=\frac{\hbar}{2}\int\frac{dk_1dk_2}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\omega_{k_\perp,n}e^{-\delta\omega_{k_\perp,n}}S\quad (2.4)$$

and

$$E_0(a,s)=\frac{\hbar}{2}\sum_{n=-\infty}^{\infty}\int\frac{dk_1dk_2}{(2\pi)^2}\,\omega_{k_\perp,n}^{1-2s}S\ .\quad (2.5)$$

These expressions are finite for $\delta>0$, respectively, for $\mathrm{Re}\,s>\frac{3}{2}$ and the limits of removing the regularization are $\delta\to 0$ and $s\to 0$ correspondingly.

Let us first consider the regularization done by introducing a damping function. The regularized vacuum energy of the electromagnetic field in free Minkowski space-time is given by

$$E_{0M}(-\infty,\infty)=\frac{\hbar}{(2\pi)^3}\int d^3k\omega_k e^{-\delta\omega_k}LS\ ,\quad (2.6)$$

where $L\to\infty$ is the length along the z-axis which is perpendicular to the plates $\omega_k=c|k|=c\sqrt{k_1^2+k_2^2+k_3^2}$, $k=(k_1,k_2,k_3)$. The renormalized vacuum energy is obtained by the subtraction from (2.4) of the Minkowski space contribution in the volume between the plates. After that the regularization can be removed. It is given by

$$E_0^{ren}(a)=\lim_{\delta\to 0}\frac{\hbar}{2}\int\frac{dk_1dk_2}{(2\pi)^2}\left(\sum_{n=-\infty}^{\infty}\omega_{k_\perp,n}e^{-\delta\omega_{k_\perp,n}}-2a\int\frac{dk_3}{2\pi}\omega_k e^{-\delta\omega_k}\right)S=\frac{c\hbar\pi}{a}\lim_{\delta\to 0}\int\frac{dk_1dk_2}{(2\pi)^2}$$
$$\left(\sum_{n=0}^{\infty}\sqrt{\frac{k_\perp^2a^2}{\pi^2}+n^2}e^{\delta\omega_{k_\perp,n}}-\int_0^\infty dt\sqrt{\frac{k_\perp^2a^2}{\pi^2}+t^2}e^{-\delta\omega_k}-\frac{k_\perp a}{2\pi}\right)S\ ,\quad (2.7)$$

where $k_\perp^2\equiv k_1^2+k_2^2, t\equiv ak_3/\pi$.

To calculate (2.7) we apply the following Abel-Plana formula

$$\sum_{n=0}^{\infty}F(n)-\int_0^\infty dtF(t)=\frac{1}{2}F(0)+i\int_0^\infty\frac{dt}{e^{2\pi t}-1}[F(it)-F(-it)],\quad (2.8)$$

and obtain

$$E_0^{ren}(a)=-\frac{c\hbar\pi^2}{a^3}\int_0^\infty ydy\int_y^\infty\frac{1}{e^{2\pi t}-1}\sqrt{t^2-y^2}dtS\ ,\quad (2.9)$$

where $y = k_\perp a/\pi$ is the dimensionless radial coordinate in the (k_1, k_2)-plane. Note that we could put $\delta = 0$ under the sign of the integrals in (2.9) due to their convergence. Also the signs when rounding the branch points $t_{1,2} = \pm iA$ of the function $F(t) = \sqrt{A^2 + t^2}$ by means of

$$F(it) - F(-it) = 2i\sqrt{t^2 - A^2} \quad (t \geq A) \quad (2.10)$$

were taken into account. To calculate (2.9) finally we change the order of integration and obtain

$$E_0^{ren}(a) = -\frac{c\hbar\pi^2}{a^3}\int_y^\infty \frac{dt}{e^{2\pi t}-1}\int_0^t y\sqrt{t^2-y^2}\,dyS = -\frac{c\hbar\pi^2}{3a^3}\frac{1}{(2\pi)^4}\int_0^\infty \frac{dxx^3}{e^x-1}S = -\frac{c\hbar\pi^2}{720a^3}S\,. \quad (2.11)$$

We note that this equation can be related with the eq. (B8) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$E_0^{ren}(a) = -\frac{c\hbar\pi^2}{a^3}\int_y^\infty \frac{dt}{e^{2\pi t}-1}\int_0^t y\sqrt{t^2-y^2}\,dyS = -\frac{c\hbar\pi^2}{3a^3}\frac{1}{(2\pi)^4}\int_0^\infty \frac{dxx^3}{e^x-1}S = -\frac{c\hbar\pi^2}{(24\times 30)a^3}S \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.11b)$$

The force

$$F(a) = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}S \quad (2.12)$$

(the finite force acting between the two parallel neutral plates from the infinite zero-point energy of the quantized electromagnetic field confined in between the plates, were a is the separation between the plates, $S >> a^2$ is their area and c is the speed of light) acting between the plates is obtained as derivative with respect to their distance

$$F(a) = -\frac{\partial E_0^{ren}(a)}{\partial a} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}S\,. \quad (2.13)$$

We note that $\frac{\pi^2}{240} = 0{,}041123351$ and $\frac{\pi^2}{720} = 0{,}013707783$ are connected with $0{,}04111130$ and $0{,}01370377$ values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Furthermore, also $\frac{\pi^2}{3} = 3{,}289868$ is connected with 3,272542 that is a value concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Now, we demonstrate the calculation of the ground state energy in zeta-functional regularization starting from eq. (2.5). Using polar coordinates $(k_\perp, \varphi_k)$ in the plane (k_1, k_2) and performing the substitution $k_\perp = y(\pi n/a)$ we obtain

$$E_0(a,s) = \frac{\hbar c}{2\pi}\int_0^\infty dyy\left(y^2+1\right)^{1/2-s}\sum_{n=1}^{\infty}\left(\frac{\pi n}{a}\right)^{3-2s} S\,. \quad (2.14)$$

Note that we put $s=0$ in the powers of some constants, e.g., c. The integration can be performed easily. The sum reduces to the well known Riemann zeta function $(t = 2s-3)$

$$\zeta_R(t) = \sum_{n=1}^{\infty}\frac{1}{n^t}\,, \quad (2.15)$$

which is defined for $\mathrm{Re}\,t > 1$, i.e., $\mathrm{Re}\,s > \frac{3}{2}$, by this sum.

It can be shown that the use of the value $\zeta_R(-3) = 1/120$ instead of the infinite when $s \to 0$ value (2.14) is equivalent to the renormalization of the vacuum energy under consideration. In this simplest case the value of the analytically continued zeta function can be obtained from the reflection relation

$$\Gamma\left(\frac{z}{2}\right)\pi^{-z/2}\zeta_R(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{(z-1)/2}\zeta_R(1-z)\,, \quad (2.16)$$

where $\Gamma(z)$ is gamma function, taken at $z=4$. Substituting $\zeta_R(-3) = 1/120$ into (2.14) and putting $s=0$ one obtains once more the renormalized physical energy of vacuum (2.11) and attractive force acting between plates (2.13).

We start with a real scalar field $\varphi(t,x)$ defined on an interval $0 < x < a$ and obeying boundary conditions

$$\varphi(t,0) = \varphi(t,a) = 0\,. \quad (2.17)$$

This is the typical case where the Casimir effect arises. The scalar field equation is as usual

$$\frac{1}{c^2}\frac{\partial^2\varphi(t,x)}{\partial t^2} - \frac{\partial^2\varphi(t,x)}{\partial x^2} + \frac{m^2c^2}{\hbar^2}\varphi(t,x) = 0\,, \quad (2.18)$$

where m is the mass of the field.

Let us return to the interval $0 \le x < a$ and impose the following boundary conditions

$$\varphi(t,0) = \varphi(t,a)\,, \quad \partial_x\varphi(t,0) = \partial_x\varphi(t,a)\,, \quad (2.19)$$

which describe the identification of the boundary points $x=0$ and a. As a result we get the scalar field on a flat manifold with topology of a circle S^1. Comparing with (2.17) now solutions are possible with $\varphi \ne 0$ at the points $x = 0, a$. The orthonormal set of solutions to (2.18), (2.19) can be represented in the following form:

$$\varphi_n^{(\pm)}(t,x)=\sqrt{\left(\frac{c}{2a\omega_n}\right)}\exp[\pm i(\omega_n t-k_n x)],\quad \omega_n=\left(\frac{m^2c^4}{\hbar^2}+c^2k_n^2\right)^{1/2},\quad k_n=\frac{2\pi n}{a},\quad n=0,\pm 1,\pm 2,\ldots \quad (2.20)$$

thence

$$\varphi_n^{(\pm)}(t,x)=\sqrt{\left(\frac{c}{2a\sqrt{\left(\frac{m^2c^4}{\hbar^2}+c^2\left(\frac{2\pi n}{a}\right)^2\right)}}\right)}\exp\left[\pm i\left(\sqrt{\left(\frac{m^2c^4}{\hbar^2}+c^2\left(\frac{2\pi n}{a}\right)^2\right)}t-\left(\frac{2\pi n}{a}\right)x\right)\right].\quad (2.20b)$$

Now we note that the standard quantization of the field is performed by means of the expansion

$$\varphi(t,x)=\sum_n\left[\varphi_n^{(-)}(t,x)a_n+\varphi_n^{(+)}(t,x)a_n^+\right].\quad (2.20c)$$

The operator of the energy density is given by the 00-component of the energy-momentum tensor of the scalar field in the two-dimensional space-time

$$T_{00}(x)=\frac{\hbar c}{2}\left\{\frac{1}{c^2}[\partial_t\varphi(x)]^2+[\partial_x\varphi(x)]^2\right\},\quad (2.20d)$$

and thence also

$$\langle 0|T_{00}(x)|0\rangle=\frac{\hbar}{2a}\sum_{n=1}^{\infty}\omega_n-\frac{m^2c^4}{2a\hbar}\sum_{n=1}^{\infty}\frac{\cos 2k_n x}{\omega_n}.\quad (2.20e)$$

Substituting (2.20) into the eqs. (2.20c) and (2.20d) we obtain the vacuum energy density of a scalar field on S^1

$$\langle 0|T_{00}(x)|0\rangle=\frac{\hbar}{2a}\sum_{n=-\infty}^{\infty}\omega_n.\quad (2.21)$$

Here, as distinct from eq. (2.20e), no oscillating contribution is contained. The total vacuum energy is

$$E_0(a,m)=\int_0^a\langle 0|T_{00}(x)|0\rangle dx=\frac{\hbar}{2}\sum_{n=-\infty}^{\infty}\omega_n=\hbar\sum_{n=0}^{\infty}\omega_n-\frac{mc^2}{2}.\quad (2.22)$$

The simplest way to perform the calculation of the renormalized vacuum energy without introducing an explicit renormalization function is the use of the Abel-Plana formula

$$\sum_{n=0}^{\infty}F(n)-\int_0^{\infty}dtF(t)=\frac{1}{2}F(0)+i\int_0^{\infty}\frac{dt}{e^{2\pi t}-1}[F(it)-F(-it)].\quad (2.23)$$

Substituting (2.20), (2.22) and the following equation

$$E_{0M}(a)=\frac{E_{0M}(-\infty,\infty)}{L}a=\frac{\hbar a}{2\pi}\int_0^{\infty}\omega dk\quad (2.24)$$

into the following equation representing the renormalized vacuum energy of the interval $(0,a)$ in the presence of boundary conditions

$$E_0^{ren}(a) \equiv \lim_{\delta \to 0}[E_0(a,\delta) - E_{0M}(a,\delta)] = E(a) = -\frac{\pi\hbar c}{24a}, \quad (2.25)$$

one obtains

$$E_0^{ren}(a,m) = \hbar\left[\sum_{n=0}^{\infty}\omega_n - \frac{a}{2\pi}\int_0^{\infty}\omega(k)dk\right] - \frac{mc^2}{2} = \frac{2\pi\hbar c}{a}\left[\sum_{n=0}^{\infty}\sqrt{A^2+n^2} - \int_0^{\infty}\sqrt{A^2+t^2}dt\right] - \frac{mc^2}{2} \quad (2.26)$$

with $A \equiv amc/(2\pi\hbar)$ and the substitution $t = ak/(2\pi)$ was made. Now we put $F(t) = \sqrt{A^2+t^2}$ into eq. (2.23) and take account of eq. (2.10). Substituting (2.23) into (2.26), we finally obtain

$$E_0^{ren}(a,m) = -\frac{4\pi\hbar c}{a}\int_A^{\infty}\sqrt{\frac{t^2-A^2}{e^{2\pi t}-1}}dt = -\frac{\hbar c}{\pi a}\int_\mu^{\infty}\frac{\sqrt{\xi^2-\mu^2}}{e^{\xi}-1}d\xi, \quad (2.27)$$

where $\xi = 2\pi t$, $\mu \equiv mca/\hbar = 2\pi A$. In the massless case we have $\mu = 0$, and the result corresponding to (2.25) for the interval reads

$$E_0^{ren}(a) = -\frac{\hbar c}{\pi a}\int_0^{\infty}\frac{\xi}{\exp(\xi)-1}d\xi = -\frac{\pi\hbar c}{6a}. \quad (2.28)$$

For $\mu >> 1$ it follows from (2.27)

$$E_0^{ren}(a,m) \approx -\frac{\sqrt{\mu}\hbar c}{\sqrt{2\pi}a}e^{-\mu}, \quad (2.29)$$

i.e., the vacuum energy of the massive field is exponentially small which also happens in the case of flat spaces. Thence, we can obtain the following expression:

$$E_0^{ren}(a,m) = -\frac{4\pi\hbar c}{a}\int_A^{\infty}\sqrt{\frac{t^2-A^2}{e^{2\pi t}-1}}dt = -\frac{\hbar c}{\pi a}\int_\mu^{\infty}\frac{\sqrt{\xi^2-\mu^2}}{e^{\xi}-1}d\xi \approx -\frac{\sqrt{\mu}\hbar c}{\sqrt{2\pi}a}e^{-\mu}. \quad (2.29b)$$

We note that $\frac{\pi}{6} = 0{,}523598$ and $\frac{1}{\sqrt{2\pi}} = 0{,}398942$ are connected with 0,524595 and 0,393446 values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

A plane with the topology of a cylinder $S^1 \times R^1$ is a flat manifold. This topology implies that points with Cartesian coordinates $(x+na, y)$, where $n = 0, \pm1, \pm2, ...$ are identified. For the scalar field φ defined on that manifold the following boundary conditions hold:

$$\varphi(t,0,y) = \varphi(t,a,y), \qquad \partial_x\varphi(t,0,y) = \partial_x\varphi(t,a,y). \quad (2.30)$$

The scalar wave equation in $(N+1)$-dimensional Riemannian space-time is:

$$\left(\nabla_\kappa \nabla^\kappa + \xi R + \frac{m^2c^2}{\hbar^2}\right)\varphi(x) = 0\,, \quad (2.31)$$

where ∇_κ is the covariant derivative, and R is the scalar curvature of the space-time, $\xi = (N-1)/4N$, $x = (x_0, x_1, ..., x_N)$. This is the so called "equation with conformal coupling". For zero mass it is invariant under conformal transformations.
The metric energy-momentum tensor is obtained by varying the Lagrangian corresponding to (2.31) with respect to the metric tensor g^{ik}. Its diagonal components are

$$T_{ii} = \hbar c\left\{(1-2\xi)\partial_i\varphi\partial_i\varphi + \left(2\xi - \frac{1}{2}\right)g_{ii}\partial_k\varphi\partial^k\varphi - \xi(\varphi\nabla_i\nabla_i\varphi + \nabla_i\nabla_i\varphi\varphi) + \right.$$
$$\left. + \left[\left(\frac{1}{2} - 2\xi\right)\frac{m^2c^2}{\hbar^2}g_{ii} - \xi G_{ii} - 2\xi^2 R g_{ii}\right]\varphi^2\right\}, \quad (2.31b)$$

where $G_{ik} = R_{ik} - \frac{1}{2}Rg_{ik}$ is the Einstein tensor, and R_{ik} is the Ricci tensor.
Let $N = 2$, and the curvature be zero as in the case of $S^1 \times R^1$. It is not difficult to find the orthonormalized solutions to eq. (2.31) with the boundary conditions (2.30). Indeed, we have that:

$$E_0^{ren}(a,m) = -\frac{\hbar c}{4\pi a^2}\int_\mu^\infty \frac{\xi^2 - \mu^2}{\exp(\xi) - 1}d\xi L\,, \quad (2.32)$$

where $L \to \infty$ is the normalization length along the y-axis as in the following equation

$$E_{0M}(-\infty,\infty) = \frac{\hbar}{2\pi}\int_0^\infty \omega dk L\,. \quad (2.33)$$

Note that this result is valid for an arbitrary value of ξ and not for $\xi = \frac{1}{8}$ only. In the massless case the integral in (2.32) is easily calculated with the following result

$$E_0^{ren}(a) = -\frac{\hbar c \zeta_R(3)}{2\pi a^2}L\,, \quad (2.34)$$

where $\zeta_R(z)$ is the Riemann zeta function with $\zeta_R(3) \approx 1.202$.
We note that the eq. (2.32) is connected with the eq. (2.29b). Indeed, we have that:

$$E_0^{ren}(a,m) = -\frac{4\pi\hbar c}{a}\int_A^\infty \sqrt{\frac{t^2 - A^2}{e^{2\pi t} - 1}}dt = -\frac{\hbar c}{\pi a}\int_\mu^\infty \frac{\sqrt{\xi^2 - \mu^2}}{e^\xi - 1}d\xi \approx -\frac{\sqrt{\mu}\hbar c}{\sqrt{2\pi}a}e^{-\mu} \Rightarrow$$
$$\Rightarrow E_0^{ren}(a,m) = -\frac{\hbar c}{4\pi a^2}\int_\mu^\infty \frac{\xi^2 - \mu^2}{\exp(\xi) - 1}d\xi L\,, \quad (2.35)$$

and that the eq. (2.34) is connected with the eq. (2.28). Indeed:

$$E_0^{ren}(a) = -\frac{\hbar c}{\pi a}\int_0^\infty \frac{\xi}{\exp(\xi) - 1}d\xi = -\frac{\pi\hbar c}{6a} \Rightarrow E_0^{ren}(a) = -\frac{\hbar c \zeta_R(3)}{2\pi a^2}L\,. \quad (2.36)$$

The non-renormalized vacuum energy density of electromagnetic fields reads

$$E_s(a,d)=\frac{E_0(a,d)}{S}=\frac{\hbar}{2}\int\frac{1}{(2\pi)^2}dk_1dk_2\sum_n\left(\omega^{(1)}_{k_\perp,n}+\omega^{(2)}_{k_\perp,n}\right), \quad (2.37)$$

where, we have separated the proper frequencies of the modes with two different polarizations of the electric field (parallel and perpendicular to the plane formed by $k_\perp$ and z-axis, respectively).
The equations for the determination of the proper frequencies $\omega^{(1)}_{k_\perp,n}$ of the modes with a parallel polarization are the following:

$$\Delta^{(1)}\left(\omega^{(1)}_{k_\perp,n}\right)\equiv e^{-R_2(a+2d)}\left\{\left(r^+_{10}r^+_{12}e^{R_1d}-r^-_{10}r^-_{12}e^{-R_1d}\right)^2e^{R_0a}-\left(r^-_{10}r^+_{12}e^{R_1d}-r^+_{10}r^-_{12}e^{-R_1d}\right)^2e^{-R_0a}\right\}=0\,. \quad (2.38)$$

While the equations for determination of the frequencies $\omega^{(2)}_{k_\perp,n}$ of the perpendicular polarized modes are the following:

$$\Delta^{(2)}\left(\omega^{(2)}_{k_\perp,n}\right)\equiv e^{-R_2(a+2d)}\left\{\left(q^+_{10}q^+_{12}e^{R_1d}-q^-_{10}q^-_{12}e^{-R_1d}\right)^2e^{R_0a}-\left(q^-_{10}q^+_{12}e^{R_1d}-q^+_{10}q^-_{12}e^{-R_1d}\right)^2e^{-R_0a}\right\}=0\,. \quad (2.39)$$

Summation in eq. (2.37) over the solutions of eqs. (2.38) and (2.39) can be performed by applying the following argument theorem:

$$\sum_n\omega^{(1,2)}_{k_\perp,n}=\frac{1}{2\pi i}\left[\int_{i\infty}^{-i\infty}\omega d\ln\Delta^{(1,2)}(\omega)+\int_{C_+}\omega d\ln\Delta^{(1,2)}(\omega)\right], \quad (2.40)$$

where C_+ is a semicircle of infinite radius in the right one-half of the complex ω-plane with a centre at the origin. The second integral on the right hand side of (2.40) is simply calculated with the natural supposition that

$$\lim_{\omega\to\infty}\varepsilon_\alpha(\omega)=1, \qquad \lim_{\omega\to\infty}\frac{d\varepsilon_\alpha(\omega)}{d\omega}=0 \quad (2.41)$$

along any radial direction in complex ω-plane. The result is infinite, and does not depend on a:

$$\int_{C_+}\omega d\ln\Delta^{(1,2)}(\omega)=4\int_{C_+}d\omega\,. \quad (2.42)$$

Now we introduce a new variable $\xi=-i\omega$ in eqs. (2.40) and (2.42). The result is

$$\sum_n\omega^{(1,2)}_{k_\perp,n}=\frac{1}{2\pi}\int_{\infty}^{-\infty}\xi d\ln\Delta^{(1,2)}(i\xi)+\frac{2}{\pi}\int_{C_+}d\xi\,. \quad (2.43)$$

From eqs. (2.38), (2.39) and (2.43) it follows

$$\lim_{a\to\infty}\sum_n\omega^{(1,2)}_{k_\perp,n}=\frac{1}{2\pi}\int_{\infty}^{-\infty}\xi d\ln\Delta^{(1,2)}_\infty(i\xi)+\frac{2}{\pi}\int_{C_+}d\xi\,, \quad (2.44)$$

where the asymptotic behaviour of $\Delta^{(1,2)}$ at $a\to\infty$ is given by

$$\Delta_{\infty}^{(1)} = e^{(R_0 - R_2)a - 2R_2 d}\left(r_{10}^{+} r_{12}^{+} e^{R_1 d} - r_{10}^{-} r_{12}^{-} e^{-R_1 d}\right)^2, \quad \Delta_{\infty}^{(2)} = e^{(R_0 - R_2)a - 2R_2 d}\left(q_{10}^{+} q_{12}^{+} e^{R_1 d} - q_{10}^{-} q_{12}^{-} e^{-R_1 d}\right)^2. \quad (2.45)$$

Now the renormalized physical quantities are found with the help of eqs. (2.43) – (2.45)

$$\left(\sum_n \omega_{k_\perp, n}^{(1,2)}\right)_{ren} \equiv \sum_n \omega_{k_\perp, n}^{(1,2)} - \lim_{a\to\infty} \sum_n \omega_{k_\perp, n}^{(1,2)} = \frac{1}{2\pi}\int_{\infty}^{-\infty} \xi d \ln \frac{\Delta^{(1,2)}(i\xi)}{\Delta_{\infty}^{(1,2)}(i\xi)}. \quad (2.46)$$

They can be transformed to a more convenient form with the help of integration by parts

$$\left(\sum_n \omega_{k_\perp, n}^{(1,2)}\right)_{ren} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \ln \frac{\Delta^{(1,2)}(i\xi)}{\Delta_{\infty}^{(1,2)}(i\xi)}, \quad (2.47)$$

where the term outside the integral vanishes.
To obtain the physical, renormalized Casimir energy density one should substitute the renormalized quantities (2.47) into eq. (2.37) instead of eq. (2.43) with the result

$$E_s^{ren}(a,d) = \frac{\hbar}{4\pi^2}\int_0^{\infty} k_\perp dk_\perp \int_0^{\infty} d\xi \left[\ln Q_1(i\xi) + \ln Q_2(i\xi)\right], \quad (2.48)$$

where we introduced polar coordinates in $k_1 k_2$ plane, and

$$Q_1(i\xi) \equiv \frac{\Delta^{(1)}(i\xi)}{\Delta_{\infty}^{(1)}(i\xi)} = 1 - \left(\frac{r_{10}^{-} r_{12}^{+} e^{R_1 d} - r_{10}^{+} r_{12}^{-} e^{-R_1 d}}{r_{10}^{+} r_{12}^{+} e^{R_1 d} - r_{10}^{-} r_{12}^{-} e^{-R_1 d}}\right)^2 e^{-2R_0 a},$$

$$Q_2(i\xi) \equiv \frac{\Delta^{(2)}(i\xi)}{\Delta_{\infty}^{(2)}(i\xi)} = 1 - \left(\frac{q_{10}^{-} q_{12}^{+} e^{R_1 d} - q_{10}^{+} q_{12}^{-} e^{-R_1 d}}{q_{10}^{+} q_{12}^{+} e^{R_1 d} - q_{10}^{-} q_{12}^{-} e^{-R_1 d}}\right)^2 e^{-2R_0 a}. \quad (2.49)$$

In terms of p, ξ the Casimir energy density (2.48) takes the form

$$E_s^{ren}(a,d) = \frac{\hbar}{4\pi^2 c^2}\int_1^{\infty} p dp \int_0^{\infty} \xi^2 d\xi \left[\ln Q_1(i\xi) + \ln Q_2(i\xi)\right]. \quad (2.50)$$

For the eqs. (2.49), we can rewrite the eq. (2.50) also as follow:

$$E_s^{ren}(a,d) = \frac{\hbar}{4\pi^2 c^2}\int_1^{\infty} p dp \int_0^{\infty} \xi^2 d\xi [\ln(1 - \left(\frac{r_{10}^{-} r_{12}^{+} e^{R_1 d} - r_{10}^{+} r_{12}^{-} e^{-R_1 d}}{r_{10}^{+} r_{12}^{+} e^{R_1 d} - r_{10}^{-} r_{12}^{-} e^{-R_1 d}}\right)^2 e^{-2R_0 a})$$

$$+ \ln(1 - \left(\frac{q_{10}^{-} q_{12}^{+} e^{R_1 d} - q_{10}^{+} q_{12}^{-} e^{-R_1 d}}{q_{10}^{+} q_{12}^{+} e^{R_1 d} - q_{10}^{-} q_{12}^{-} e^{-R_1 d}}\right)^2 e^{-2R_0 a})]. \quad (2.50b)$$

From eq. (2.50) it is easy to obtain the Casimir force per unit area acting between semi-spaces covered with layers:

$$F_{SS}(a,d) = -\frac{\partial E_S^{ren}(a,d)}{\partial a} = -\frac{\hbar}{2\pi^2 c^3}\int_1^{\infty} p^2 dp \int_0^{\infty} \xi^3 d\xi \left[\frac{1 - Q_1(i\xi)}{Q_1(i\xi)} + \frac{1 - Q_2(i\xi)}{Q_2(i\xi)}\right]. \quad (2.51)$$

This expression coincides with Lifshitz result for the force per unit area between semi-spaces with a dielectric permittivity ε_2 if the covering layers are absent. To obtain this limiting case from eq. (2.51) one should put $d = 0$ and $\varepsilon_1 = \varepsilon_2$

$$F_{SS}(a) = -\frac{\hbar}{2\pi^2 c^3}\int_1^\infty p^2 dp \int_0^\infty \xi^3 d\xi \left\{ \left[\left(\frac{K_2 + \varepsilon_2 p}{K_2 - \varepsilon_2 p} \right)^2 e^{2(\xi/c)pa} - 1 \right]^{-1} + \left[\left(\frac{K_2 + p}{K_2 - p} \right)^2 e^{2(\xi/c)pa} - 1 \right]^{-1} \right\}. \quad (2.52)$$

The corresponding quantity for the energy density follows from eq. (2.50)

$$E_S^{ren}(a) = \frac{\hbar}{4\pi^2 c^2}\int_1^\infty pdp \int_0^\infty \xi^2 d\xi \left\{ \ln\left[1 - \left(\frac{K_2 - \varepsilon_2 p}{K_2 + \varepsilon_2 p} \right)^2 e^{-2(\xi/c)pa} \right] + \ln\left[1 - \left(\frac{K_2 - p}{K_2 + p} \right)^2 e^{-2(\xi/c)pa} \right] \right\}. \quad (2.53)$$

It is well known that eqs. (2.52) and (2.53) contain the limiting cases of both van der Waals and Casimir forces and energy densities. At small distances $a << \lambda_0$ these equations take the simplified form

$$F_{SS}(a) = -\frac{3\hbar}{48\pi^2 a^3}\int_0^\infty x^2 dx \int_0^\infty d\xi \left[\left(\frac{\varepsilon_2 + 1}{\varepsilon_2 - 1} \right)^2 e^x - 1 \right]^{-1}, \quad (2.54)$$

$$E_S^{ren}(a) = -\frac{3\hbar}{2 \cdot 48\pi^2 a^2}\int_0^\infty x^2 dx \int_0^\infty d\xi \left[\left(\frac{\varepsilon_2 + 1}{\varepsilon_2 - 1} \right)^2 e^x - 1 \right]^{-1}, \quad (2.55)$$

where we have introduced the Hamaker constant.
In the opposite case of large distances $a >> \lambda_0$ the dielectric permittivity can be represented by their static values at $\xi = 0$. Introducing in eq. (2.52) the variable x (now instead of ξ) one obtains

$$F_{SS}(a) = -\frac{5\hbar c}{160 a^4 \pi^2}\int_1^\infty \frac{dp}{p^2}\int_0^\infty x^3 dx \left\{ \left[\left(\frac{K_{20} + p}{K_{20} - p} \right)^2 e^x - 1 \right]^{-1} + \left[\left(\frac{K_{20} + p\varepsilon_{20}}{K_{20} - p\varepsilon_{20}} \right)^2 e^x - 1 \right]^{-1} \right\}, \quad (2.56)$$

$$E_S^{ren}(a) = -\frac{5\hbar c}{480 a^3 \pi^2}\int_1^\infty \frac{dp}{p^2}\int_0^\infty x^3 dx \left\{ \left[\left(\frac{K_{20} + p}{K_{20} - p} \right)^2 e^x - 1 \right]^{-1} + \left[\left(\frac{K_{20} + p\varepsilon_{20}}{K_{20} - p\varepsilon_{20}} \right)^2 e^x - 1 \right]^{-1} \right\}, \quad (2.57)$$

and $K_{20} = \left(p^2 - 1 + \varepsilon_{20}\right)^{1/2}, \varepsilon_{20} = \varepsilon_2(0)$.
We note that the eqs. (2.56) – (2.57) can be related with the eqs. (B8) and (B9) concerning the P-N model and the Ramanujan's modular equations. Indeed, we have:

$$F_{SS}(a) = -\frac{5\hbar c}{(8 \times 20) a^4 \pi^2}\int_1^\infty \frac{dp}{p^2}\int_0^\infty x^3 dx \left\{ \left[\left(\frac{K_{20} + p}{K_{20} - p} \right)^2 e^x - 1 \right]^{-1} + \left[\left(\frac{K_{20} + p\varepsilon_{20}}{K_{20} - p\varepsilon_{20}} \right)^2 e^x - 1 \right]^{-1} \right\} \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=$$

$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]\Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

$$E_S^{ren}(a)=-\frac{5\hbar c}{(24\times 20)a^3\pi^2}\int_1^\infty\frac{dp}{p^2}\int_0^\infty x^3dx\left\{\left[\left(\frac{K_{20}+p}{K_{20}-p}\right)^2e^x-1\right]^{-1}+\left[\left(\frac{K_{20}+p\varepsilon_{20}}{K_{20}-p\varepsilon_{20}}\right)^2e^x-1\right]^{-1}\right\}\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=$$

$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]\Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Furthermore, we observe that the function Ψ is defined by

$$\Psi(\varepsilon_{20})=\frac{5}{16\pi^3}\int_1^\infty\frac{dp}{p^2}\int_0^\infty x^3dx\left\{\left[\left(\frac{K_{20}+p}{K_{20}-p}\right)^2e^x-1\right]^{-1}+\left[\left(\frac{K_{20}+p\varepsilon_{20}}{K_{20}-p\varepsilon_{20}}\right)^2e^x-1\right]^{-1}\right\}. \quad (2.57b)$$

And we note that also this equation can be related with the eq. (B9) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$\Psi(\varepsilon_{20})=\frac{5}{(8\times 2)\pi^3}\int_1^\infty\frac{dp}{p^2}\int_0^\infty x^3dx\left\{\left[\left(\frac{K_{20}+p}{K_{20}-p}\right)^2e^x-1\right]^{-1}+\left[\left(\frac{K_{20}+p\varepsilon_{20}}{K_{20}-p\varepsilon_{20}}\right)^2e^x-1\right]^{-1}\right\}\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x(-G)^{1/2} e^{-2\Phi}\left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(\left|F_2\right|^2\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

If both bodies are ideal metals the dielectric permittivity $\varepsilon_2(i\xi) \to \infty$ for all ξ including $\xi \to 0$. Putting $\varepsilon_{20} \to \infty$, $\Psi(\varepsilon_{20}) \to \pi/24$ we obtain the Casimir result for the force per unit area and energy density

$$F_{SS}^{(0)}(a) = -\frac{\pi^2}{240} \frac{\hbar c}{a^4}, \qquad E_S^{(0)ren}(a) = -\frac{\pi^2}{720} \frac{\hbar c}{a^3}. \quad (2.58)$$

Let us apply the Epstein zeta function method to calculate the Casimir energy and force for the electromagnetic vacuum inside a rectangular box with the side lengths a_1, a_2, and a_3. The box faces are assumed to be perfect conductors. Imposing the boundary conditions of eq. (2.1) on the faces, the proper frequencies are found to be

$$\omega_{n_1 n_2 n_3}^2 = \pi^2 c^2 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2} \right). \quad (2.59)$$

There are no oscillations with two or three indices equal to zero because in such cases electromagnetic field vanishes. As a consequence, the non-renormalized vacuum energy of electromagnetic field inside a box takes the form

$$E_0(a_1, a_2, a_3) = \frac{\hbar}{2}\left(2 \sum_{n_1, n_2, n_3 = 1}^{\infty} \omega_{n_1 n_2 n_3} + \sum_{n_2, n_3 = 1}^{\infty} \omega_{0 n_2 n_3} + \sum_{n_1, n_3 = 1}^{\infty} \omega_{n_1 0 n_3} + \sum_{n_1, n_2 = 1}^{\infty} \omega_{n_1 n_2 0} \right). \quad (2.60)$$

We regularize this quantity with the help of the Epstein zeta function which for a simple case under consideration is defined by

$$Z_3\left(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}; t\right) = \sum_{n_1, n_2, n_3 = -\infty}^{\infty}{}' \left[\left(\frac{n_1}{a_1}\right)^2 + \left(\frac{n_2}{a_2}\right)^2 + \left(\frac{n_3}{a_3}\right)^2 \right]^{-t/2}. \quad (2.61)$$

This series is convergent if $t > 3$. The prime near sum indicates that the term for which all $n_i = 0$ is to be omitted. At first, eq. (2.38) should be transformed identically to

$$E_0(a_1, a_2, a_3) = \frac{\hbar}{8} \sum_{n_1, n_2, n_3 = -\infty}^{\infty}{}' \omega_{n_1 n_2 n_3} \left(1 - \delta_{n_1,0}\delta_{n_2,0} - \delta_{n_1,0}\delta_{n_3,0} - \delta_{n_2,0}\delta_{n_3,0}\right). \quad (2.62)$$

Introducing the regularization parameter s and using definitions (2.61), (2.15) one obtains

$$E_0(a_1,a_2,a_3;s)=\frac{\hbar}{8}\sum_{n_1,n_2,n_3=-\infty}^{\infty}{}'\omega_{n_1n_2n_3}^{1-2s}\left(1-\delta_{n_1,0}\delta_{n_2,0}-\delta_{n_1,0}\delta_{n_3,0}-\delta_{n_2,0}\delta_{n_3,0}\right)=$$
$$=\frac{\hbar\pi c}{8}\left[Z_3\left(\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3};2s-1\right)-2\zeta_R(2s-1)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)\right]. \quad (2.63)$$

To remove regularization ($s\to 0$) we need the values of Epstein and Riemann zeta functions at $t=-1$. Both of them are given by the analytic continuation of these functions. As to $\zeta_R(t)$ the eq. (2.16) should be used. For Epstein zeta function the reflection formula analogical to (2.16) is

$$\Gamma\left(\frac{t}{2}\right)\pi^{-t/2}Z_3(a_1,a_2,a_3;t)=(a_1a_2a_3)^{-1}\Gamma\left(\frac{3-t}{2}\right)\pi^{(t-3)/2}Z_3\left(\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3};3-t\right), \quad (2.64)$$

where $\Gamma(z)$ is gamma function. The results of their application are

$$Z_3\left(\frac{1}{a_1},\frac{1}{a_2},\frac{1}{a_3};-1\right)=-\frac{a_1a_2a_3}{2\pi^3}Z_3(a_1,a_2,a_3;4), \qquad \zeta_R(-1)=-\frac{1}{12}. \quad (2.65)$$

Substituting the obtained finite values into eq. (2.63) in the limit $s\to 0$ we obtain the renormalized (by means of zeta function regularization) vacuum energy

$$E_0^{ren}(a_1,a_2,a_3)=-\frac{\hbar c a_1a_2a_3}{16\pi^2}Z_3(a_1,a_2,a_3;4)+\frac{\hbar c\pi}{48}\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right). \quad (2.66)$$

As usual, the forces acting upon the opposite pairs of faces and directed normally to them are

$$F_i(a_1,a_2,a_3)=-\frac{\partial E_0^{ren}(a_1,a_2,a_3)}{\partial a_i}, \quad (2.67)$$

so that the total vacuum force is

$$F(a_1,a_2,a_3)=-\nabla E_0^{ren}(a_1,a_2,a_3). \quad (2.68)$$

According to energy sign forces (2.67) can be both repulsive or attractive depending on the relationship between the lengths of the sides $a_1,a_2,$ and a_3. Let us start with a massless scalar field in a two-dimensional box $0\le x\le a_1$, $0\le y\le a_2$ for which the non-renormalized vacuum energy is expressed by

$$E_0(a_1,a_2)=\frac{\hbar}{2}\sum_{n_1,n_2=1}^{\infty}\omega_{n_1n_2}, \qquad \omega_{n_1n_2}^2=\pi^2c^2\left(\frac{n_1^2}{a_1^2}+\frac{n_2^2}{a_2^2}\right). \quad (2.69)$$

To perform the summation in (2.69) we apply twice the Abel-Plana formula (2.8). The explicit introduction of the dumping function is not necessary. After the first application one obtains

$$S_{n_1} \equiv \sum_{n_2=1}^{\infty}\left(\frac{n_1^2}{a_1^2}+\frac{n_2^2}{a_2^2}\right)^{1/2} = -\frac{n_1}{2a_1}+\int_0^{\infty}dt\left(\frac{n_1^2}{a_1^2}+\frac{t^2}{a_2^2}\right)-2\int_{n_1a_2/a_1}^{\infty}\left(\frac{t^2}{a_2^2}-\frac{n_1^2}{a_1^2}\right)^{1/2}\frac{1}{e^{2\pi}-1}dt\,, \quad (2.70)$$

where the last integral uses the fact that the difference of the radicals is non-zero only above the branch point. The result of the second application is

$$\sum_{n_1=1}^{\infty}S_{n_1} = -\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\int_0^{\infty}tdt+\int_0^{\infty}dt\int_0^{\infty}dv\left(\frac{v^2}{a_1^2}+\frac{t^2}{a_2^2}\right)^{1/2}+\frac{1}{24a_1}-\frac{a_2}{8\pi^2a_1^2}\zeta_R(3)+\frac{2a_2}{a_1^2}G\left(\frac{a_2}{a_1}\right), \quad (2.71)$$

where

$$G(x) = -\int_1^{\infty}ds\sqrt{s^2-1}\sum_{n_1=1}^{\infty}\frac{n_1^2}{e^{2\pi x n_1 s}-1}. \quad (2.72)$$

Renormalization of quantity (2.71) is equivalent to the omission of first two integrals in the right hand side.
As a result, the renormalized vacuum energy is

$$E_0^{ren}(a_1,a_2) = \frac{\hbar\pi c}{2}\left(\sum_{n_1=1}^{\infty}S_{n_1}\right) = \hbar c\left[\frac{\pi}{48a_1}-\frac{\zeta_R(3)a_2}{16\pi a_1^2}+\frac{\pi a_2}{a_1^2}G\left(\frac{a_2}{a_1}\right)\right]. \quad (2.73)$$

Thence, from the eq. (2.71) we can rewrite the eq. (2.73) also as follows:

$$E_0^{ren}(a_1,a_2) = \frac{\hbar\pi c}{2}\left[-\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\int_0^{\infty}tdt+\int_0^{\infty}dt\int_0^{\infty}dv\left(\frac{v^2}{a_1^2}+\frac{t^2}{a_2^2}\right)^{1/2}+\frac{1}{24a_1}-\frac{a_2}{8\pi^2a_1^2}\zeta_R(3)+\frac{2a_2}{a_1^2}G\left(\frac{a_2}{a_1}\right)\right] =$$
$$= \hbar c\left[\frac{\pi}{48a_1}-\frac{\zeta_R(3)a_2}{16\pi a_1^2}+\frac{\pi a_2}{a_1^2}G\left(\frac{a_2}{a_1}\right)\right]. \quad (2.73b)$$

We note that also this equation can be related with the eq. (B8) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$E_0^{ren}(a_1,a_2) = \frac{\hbar\pi c}{2}\left[-\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\int_0^{\infty}tdt+\int_0^{\infty}dt\int_0^{\infty}dv\left(\frac{v^2}{a_1^2}+\frac{t^2}{a_2^2}\right)^{1/2}+\frac{1}{24a_1}-\frac{a_2}{8\pi^2a_1^2}\zeta_R(3)+\frac{2a_2}{a_1^2}G\left(\frac{a_2}{a_1}\right)\right] =$$
$$= \hbar c\left[\frac{\pi}{(24\times 2)a_1}-\frac{\zeta_R(3)a_2}{(8\times 2)\pi a_1^2}+\frac{\pi a_2}{a_1^2}G\left(\frac{a_2}{a_1}\right)\right] \Rightarrow$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^{\infty}\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.73c)$$

It can be easily shown from eq. (2.71) that for $a_1 = a_2$ the contribution of $G(1)$ to the vacuum energy is of order 1% and for $a_2 > a_1$ is exponentially small. It is seen from this expression that the energy is positive if

$$1 \le \frac{a_2}{a_1} < \frac{\pi^2}{3\zeta_R(3)} \approx 2.74, \quad (2.74)$$

and negative if $a_2 > 2.74a_1$ (we remind that $\zeta_R(3) \approx 1.202$).
We note that 2,74 and 1,202 are connected with 2,746817 and 1,198543 values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**
After three applications of the Abel-Plana formula and renormalization, the result is the following:

$$E_0^{ren}(a_1,a_2,a_3) = \hbar c\left[-\frac{\pi^2 a_2 a_3}{720a_1^3} - \frac{\zeta_R(3)}{16\pi}\frac{a_3}{a_2^2} + \frac{\pi}{48}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) + H\left(\frac{a_2}{a_1},\frac{a_3}{a_1},\frac{a_3}{a_2}\right)\right], \quad (2.75)$$

where function H is exponentially small in all its arguments if $a_1 \le a_2 \le a_3$. If there is a square section $a_1 = a_2 \le a_3$ and the small integral sums contained in H are neglected the result is

$$E_0^{ren}(a_1,a_3) \approx \frac{\hbar c}{a_1}\left[\frac{\pi}{24} - \left(\frac{\pi^2}{720} + \frac{\zeta_R(3)}{16\pi}\right)\frac{a_3}{a_1}\right]. \quad (2.76)$$

In the opposite case $a_1 = a_2 > a_3$ the vacuum energy is

$$E_0^{ren}(a_1,a_3) \approx \frac{\hbar c}{a_1}\left[\frac{\pi}{48} - \frac{\zeta_R(3)}{16\pi} + \frac{\pi}{48}\frac{a_1}{a_3} - \frac{\pi^2}{720}\left(\frac{a_1}{a_3}\right)^3\right]. \quad (2.77)$$

We note that $\frac{\pi}{24} = 0{,}1308996$ and $\frac{\pi}{48} = 0{,}0654498$ are connected with 0,131148 and 0,0657780 values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

For the case $a_1 = a_2$, one can easily obtain from (2.75) that the energy is positive if

$$0.408 < \frac{a_3}{a_1} < 3.48 \quad (2.78)$$

and passes through zero at the ends of this interval. Outside interval (2.78) vacuum energy of electromagnetic field inside a box is negative. For the cube $a_1 = a_2 = a_3$ the vacuum energy takes the value

$$E_0^{ren}(a_1) \approx 0.0916 \frac{\hbar c}{a_1}. \quad (2.79)$$

Exactly the same results as in (2.78) and (2.79) are obtained from eq. (2.66) by numerical computation.
We note that 0,0916 0,408 and 3,48 are connected with 0,091844 0,407430 and 3,490711 values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

The closed Friedmann model is topologically nontrivial and naturally incorporates the Casimir effect. We are interested here by the local characteristics of a vacuum. We are interested only in the energy density

$$\varepsilon(\eta) = \langle 0 | T_0^0(x) | 0 \rangle, \quad (2.80)$$

where η is a conformal time variable, and the stress-energy tensor is defined in eq. (2.31b). Metric of the closed Friedmann model has the form

$$ds^2 = a^2(\eta)\left(d\eta^2 - dl^2\right), \qquad dl^2 = d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (2.81)$$

where $a(\eta)$ is a scale factor with dimensions of length, $0 \le \chi \le \pi$, θ and φ are the usual spherical angles. Furthermore, we have also that:

$$ds^2 = a^2(\eta)\left[d\eta^2 - \left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)\right]. \quad (2.81b)$$

Taking into account that $N = 3$, $\xi = 1/6$ and the scalar curvature $R = 6(a'' + a)/a^3$, where prime denotes differentiation with respect to conformal time η, we rewrite eq. (2.31) in the form

$$\varphi''(x) + 2\frac{a'}{a}\varphi'(x) - \Delta^{(3)}\varphi(x) + \left(\frac{m^2c^2a^2}{\hbar^2} + \frac{a''}{a} + 1\right)\varphi(x) = 0, \quad (2.82)$$

where $\Delta^{(3)}$ is the angular part of the Laplacian operator on a 3-sphere, $x = (\eta, \chi, \theta, \varphi)$. The orthonormal set of solutions to eq. (2.82) can be represented as

$$\varphi_{\lambda lM}^{(+)}(x) = \frac{1}{\sqrt{2}a(\eta)} g_\lambda(\eta)\Phi_{\lambda lM}^*(\chi, \theta, \varphi), \qquad \varphi_{\lambda lM}^{(-)}(x) = \left[\varphi_{\lambda lM}^{(+)}(x)\right]^*, \quad (2.83)$$

where the eigenfunctions of the Laplacian operator are defined according to

$$\Phi_{\lambda lM}(\chi, \theta, \varphi) = \frac{1}{\sqrt{\sin\chi}} \sqrt{\frac{\lambda(\lambda + l)!}{(\lambda - l + 1)!}} P_{\lambda - 1/2}^{-l-1/2}(\cos\chi) Y_{lM}(\theta, \varphi), \quad (2.84)$$

$\lambda = 1,2,...,$ $l = 0,1,...,\lambda - 1$, Y_{lM} are the spherical harmonics and $P_\mu^\nu(z)$ are the adjoint Legendre functions on the cut. The discrete quantity λ has the sense of a dimensionless momentum, the physical momentum being $\hbar\lambda / a$. The time dependent function g_λ satisfies the oscillatory equation

$$g''_\lambda(\eta) + \omega_\lambda^2(\eta) g_\lambda(\eta) = 0, \qquad \omega_\lambda^2(\eta) = \lambda^2 + \frac{m^2c^2a^2(\eta)}{\hbar^2} \quad (2.85)$$

with the time dependent frequency and initial conditions fixing the frequency sign at the initial time

$$g_\lambda(\eta_0) = \frac{1}{\sqrt{\omega_\lambda(\eta_0)}}, \qquad g'_\lambda(\eta_0) = i\sqrt{\omega_\lambda(\eta_0)}. \quad (2.86)$$

Eigenfunctions (2.83) – (2.86) define the vacuum state at a moment η_0. In the homogeneous isotropic case one may put $\eta_0 = 0$. Substituting the field operator expanded in terms of functions (2.83), (2.84) into the 00-component of (2.31b) and calculating the mean value in the initial vacuum state according to (2.80) one obtains the non-renormalized vacuum energy density

$$\varepsilon^{(0)}(\eta) = \frac{\hbar c}{4\pi^2 a^4(\eta)} \sum_{\lambda=1}^{\infty} \lambda^2 \omega_\lambda(\eta)[2s_\lambda(\eta) + 1], \qquad s_\lambda(\eta) = \frac{1}{4\omega_\lambda}\left(|g'_\lambda|^2 + \omega_\lambda^2 |g_\lambda|^2 - 2\omega_\lambda\right). \quad (2.87)$$

The corresponding vacuum energy density in tangential Minkowski space at a given point is

$$\varepsilon_M^{(0)}(\eta) = \frac{\hbar c}{4\pi^2 a^4(\eta)} \int_0^\infty \lambda^2 d\lambda\, \omega_\lambda(\eta). \quad (2.88)$$

Subtracting (2.88) from (2.87) with the help of Abel-Plana formula (2.8) we come to the result

$$\varepsilon_{ren}^{(0)}(\eta) = E^{(0)}(\eta) + \frac{\hbar c}{2\pi^2 a^4(\eta)} \sum_{\lambda=1}^{\infty} \lambda^2 \omega_\lambda(\eta) s_\lambda(\eta), \quad (2.89)$$

where the Casimir energy density of a scalar field in a closed Friedmann model is

$$E^{(0)}(\eta) = \frac{\hbar c}{2\pi^2 a^4(\eta)} \int_{mca(\eta)/\hbar}^{\infty} \frac{\lambda^2 d\lambda}{e^{2\pi\lambda} - 1} \left[\lambda^2 - \frac{m^2c^2a^2(\eta)}{\hbar^2}\right]^{1/2}. \quad (2.90)$$

We note that can be rewrite the eq. (2.89) also as follows:

$$\varepsilon_{ren}^{(0)}(\eta) = \frac{\hbar c}{2\pi^2 a^4(\eta)} \int_{mca(\eta)/\hbar}^{\infty} \frac{\lambda^2 d\lambda}{e^{2\pi\lambda} - 1} \left[\lambda^2 - \frac{m^2c^2a^2(\eta)}{\hbar^2}\right]^{1/2} + \frac{\hbar c}{2\pi^2 a^4(\eta)} \sum_{\lambda=1}^{\infty} \lambda^2 \omega_\lambda(\eta) s_\lambda(\eta). \quad (2.90b)$$

We note that $\frac{1}{2\pi^2} = 0{,}050660591 \cong 0{,}05065904$, value concerning the **new universal music system based on fractional powers of Phi and Pigreco.**
The second term on the right hand side of eq. (2.89) is the subject of two additional renormalizations in accordance with the general structure of infinities of Quantum Field Theory in

curved space-time. As a result of these renormalizations the total vacuum energy of massless scalar field in closed Friedmann model takes the form

$$\varepsilon_{ren}^{(0)}(\eta) = E_0^{(0)}(\eta) + \frac{\hbar c}{960\pi^2 a^4(\eta)}\left(2b''b - b'^2 - 2b^4\right), \quad (2.91)$$

where $b = b(\eta) \equiv a'(\eta)/a(\eta)$. The Casimir energy density of a massless field which appears in this expression is obtained from eq. (2.90) for both constant and variable a as

$$E_0^{(0)}(\eta) = \frac{\hbar c}{2\pi^2 a^4(\eta)}\int_0^\infty \frac{d\lambda\lambda^3}{e^{2\pi\lambda} - 1} = \frac{\hbar c}{480\pi^2 a^4(\eta)}. \quad (2.92)$$

Also here we can rewrite the eq. (2.91) as follows:

$$\varepsilon_{ren}^{(0)}(\eta) = \frac{\hbar c}{2\pi^2 a^4(\eta)}\int_0^\infty \frac{d\lambda\lambda^3}{e^{2\pi\lambda} - 1} = \frac{\hbar c}{480\pi^2 a^4(\eta)} + \frac{\hbar c}{960\pi^2 a^4(\eta)}\left(2b''b - b'^2 - 2b^4\right). \quad (2.92b)$$

Also this equation can be related with the eq. (B8) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$\varepsilon_{ren}^{(0)}(\eta) = \frac{\hbar c}{2\pi^2 a^4(\eta)}\int_0^\infty \frac{d\lambda\lambda^3}{e^{2\pi\lambda} - 1} = \frac{\hbar c}{(24\times 20)\pi^2 a^4(\eta)} + \frac{\hbar c}{(24\times 40)\pi^2 a^4(\eta)}\left(2b''b - b'^2 - 2b^4\right) \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.92c)$$

In the case of massive fields eq. (2.90) can be integrated analytically in the limit $mca(\eta)/\hbar >> 1$ both for the constant and variable a with the result

$$E^{(0)}(\eta) \approx \frac{(mca(\eta)/\hbar)^{5/2}\hbar c}{8\pi^3 a^4(\eta)}e^{-2\pi mca(\eta)/\hbar}. \quad (2.93)$$

Thence, we can obtain the following expression:

$$E^{(0)}(\eta)=\frac{\hbar c}{2\pi^2 a^4(\eta)}\int_{mca(\eta)/\hbar}^{\infty}\frac{\lambda^2 d\lambda}{e^{2\pi\lambda}-1}\left[\lambda^2-\frac{m^2c^2a^2(\eta)}{\hbar^2}\right]^{1/2}\approx\frac{(mca(\eta)/\hbar)^{5/2}\hbar c}{8\pi^3 a^4(\eta)}e^{-2\pi mca(\eta)/\hbar}. \quad (2.93b)$$

Also here, we can to connect this equation with the eq. (B9) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$E^{(0)}(\eta)=\frac{\hbar c}{2\pi^2 a^4(\eta)}\int_{mca(\eta)/\hbar}^{\infty}\frac{\lambda^2 d\lambda}{e^{2\pi\lambda}-1}\left[\lambda^2-\frac{m^2c^2a^2(\eta)}{\hbar^2}\right]^{1/2}\approx\frac{(mca(\eta)/\hbar)^{5/2}\hbar c}{8\pi^3 a^4(\eta)}e^{-2\pi mca(\eta)/\hbar}\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]\Rightarrow$$

$$\Rightarrow\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.93c)$$

All the considerations on the Casimir energy density of scalar field in the closed Friedmann model can be extended for the case of quantized spinor field. After the calculation of the vacuum energy density and subtraction of the tangential Minkowski space contribution the result similar to eq. (2.89) is

$$\varepsilon_{ren}^{(1/2)}(\eta)=E^{(1/2)}(\eta)+\frac{2\hbar c}{\pi^2 a^4(\eta)}\sum_{\lambda=3/2}^{\infty}\left(\lambda^2-\frac{1}{2}\right)\omega_\lambda(\eta)s_\lambda(\eta), \quad (2.94)$$

where $s_\lambda(\eta)$ is expressed in terms of the solution to the oscillatory equation with a complex frequency obtained from the Dirac equation after the separation of variables. The Casimir contribution in the right hand side is

$$E^{(1/2)}(\eta)=\frac{2\hbar c}{\pi^2 a^4(\eta)}\int_{mca(\eta)/\hbar}^{\infty}\frac{d\lambda}{e^{2\pi\lambda}+1}\left(\lambda^2+\frac{1}{4}\right)\left[\lambda^2-\frac{m^2c^2a^2(\eta)}{\hbar^2}\right]^{1/2}. \quad (2.95)$$

In the non-stationary case $a'(\eta)\neq 0$ two additional renormalizations are needed to obtain the total physical energy density of a vacuum. The result in a massless case is given by

$$\varepsilon_{ren}^{(1/2)}(\eta)=E_0^{(1/2)}(\eta)+\frac{\hbar c}{480\pi^2 a^4(\eta)}\left(6b''b-3b'^2-\frac{7}{2}b^4+5b^2\right), \quad (2.96)$$

where the Casimir energy density of a massless spinor field is

$$E_0^{(1/2)}(\eta)=\frac{2\hbar c}{\pi^2 a^4(\eta)}\int_0^\infty\left(\lambda^2+\frac{1}{4}\right)\frac{\lambda d\lambda}{e^{2\pi\lambda}+1}=\frac{17\hbar c}{960\pi^2 a^4(\eta)}. \quad (2.97)$$

Thence, we can rewrite the eq. (2.96) also as follows:

$$\varepsilon_{ren}^{(1/2)}(\eta)=\frac{2\hbar c}{\pi^2 a^4(\eta)}\int_0^\infty\left(\lambda^2+\frac{1}{4}\right)\frac{\lambda d\lambda}{e^{2\pi\lambda}+1}=\frac{17\hbar c}{960\pi^2 a^4(\eta)}+\frac{\hbar c}{480\pi^2 a^4(\eta)}\left(6b''b-3b'^2-\frac{7}{2}b^4+5b^2\right). \quad (2.97b)$$

Also this equation can be related with the eq. (B8) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$\varepsilon_{ren}^{(1/2)}(\eta)=\frac{2\hbar c}{\pi^2 a^4(\eta)}\int_0^\infty\left(\lambda^2+\frac{1}{4}\right)\frac{\lambda d\lambda}{e^{2\pi\lambda}+1}=\frac{17\hbar c}{(24\times 40)\pi^2 a^4(\eta)}+$$

$$+\frac{\hbar c}{(24\times 20)\pi^2 a^4(\eta)}\left(6b''b-3b'^2-\frac{7}{2}b^4+5b^2\right)\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]\Rightarrow$$

$$\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.97c)$$

The second contribution on the right hand side of (2.96), as well as in (2.91) for a scalar case, is interpreted as a vacuum polarization by the non-stationary gravitational field. In the case of a massive field, **the effect of fermion pair creation in vacuum by the gravitational field is possible**. At last for the quantized electromagnetic field the Casimir energy density in the closed Friedmann model is

$$E_0^{(1)}(\eta)=\frac{11\hbar c}{240\pi^2 a^4(\eta)}. \quad (2.98)$$

We note that we have in the eqs. (2.95) – (2.98) the following values: $\frac{2}{\pi^2}=0{,}202642367$; $\frac{17}{960\pi^2}=0{,}001794229294$; $\frac{11}{240\pi^2}=0{,}00464388$ all values very near to 0,20225425 0,00179942 0,00464401 values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

In Kaluza-Klein compactification the manifold K is presumed to be stationary. Because of this, the Casimir vacuum energy density is important for the determination of the parameters of K and its stability. Representing the Casimir stress-energy tensor by T_{AB}^{ren}, the multi-dimensional Einstein equations take the form

$$R_{AB} - \frac{1}{2} R_d g_{AB} + \Lambda_d g_{AB} = -\frac{8\pi G_d}{c^4} T_{AB}^{ren}, \quad (2.99)$$

where $A, B = 0,1,\ldots,d-1$, G_d and Λ_d are the gravitational and cosmological constants in d dimensions.
We are looking for the solution of eq. (2.99) which are Poincaré invariant in four dimensions. What this means is the metrical tensor g_{AB} and Ricci tensor R_{AB} have the block structure

$$g_{AB} = \begin{pmatrix} \eta_{nm} & 0 \\ 0 & h_{ab}(u) \end{pmatrix}, \qquad R_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & R_{ab}(u) \end{pmatrix}, \quad (2.100)$$

where η_{mn} $(m, n = 0,1,2,3)$ is the metric tensor in Minkowski space-time M^4, and $h_{ab}(u)$ is the metric tensor on a manifold S^N with coordinates u $(a, b = 4,5,\ldots,d-1)$. It is clear that the scalar curvature R_d coincides with the one calculated from the metrical tensor $h_{ab}(u)$. The Casimir stress-energy tensor also has the block structure

$$T_{mn}^{ren} = T_1 \eta_{mn}, \qquad T_{ab}^{ren}(u) = T_2 h_{ab}(u). \quad (2.101)$$

Note that $T_{1,2}$ do not depend on u due to space homogeneity. The Ricci tensor on an N-dimensional sphere is

$$R_{ab}(u) = -\frac{N-1}{a^2} h_{ab}(u), \quad (2.102)$$

where a is a sphere radius. To find T_1 and T_2 we remind that the Casimir stress-energy tensor T_{ab}^{ren} can be expressed in terms of the effective potential V by variation with respect to the metric

$$T_{ab}^{ren} = -\frac{2}{\sqrt{|\det h_{ab}|}} \frac{\delta V(h)}{\delta h^{ab}}. \quad (2.103)$$

The variation of the metric tensor h_{ab} can be considered as a change in the sphere radius a. Multiplying both sides of (2.103) by h^{ab}, summing over a and b, and integrating over the volume of S^N, one obtains with the use of (2.101)

$$\frac{1}{2} T_2 N \Omega_N = -\int d^N u h^{ab} \frac{\delta V(h)}{\delta h^{ab}} = -a^2 \frac{dV}{da^2}, \quad (2.104)$$

where the volume of the sphere is

$$\Omega_N = \int d^N u \sqrt{|\det h_{ab}|}. \quad (2.105)$$

Thence, we can rewrite the eq. (2.104) also as follows:

$$\frac{1}{2} T_2 N \int d^N u \sqrt{|\det h_{ab}|} = -\int d^N u h^{ab} \frac{\delta V(h)}{\delta h^{ab}} = -a^2 \frac{dV}{da^2}. \quad (2.105b)$$

To express T_1 in terms of the effective potential a similar trick is used. It is provisionally assumed that the Minkowski tensor is of the form $g_{mn} = \lambda^2 \eta_{mn}$, where λ is varied. The result is

$$T_1 \Omega_N = -V . \quad (2.106)$$

Now we rewrite the Einstein equations (2.99) separately for the subspaces M^4 and S^N using eqs. (2.100) – (2.102) and (2.104), (2.106). The result is

$$\frac{N(N-1)}{2a^2} + \Lambda_d = \frac{8\pi G}{c^4} V(a), \quad -\frac{N-1}{a^2} + \frac{N(N-1)}{2a^2} + \Lambda_d = \frac{8\pi G}{c^4 N} a \frac{dV(a)}{da} . \quad (2.107)$$

Subtracting the second equation from the first, one obtains

$$\frac{c^4 (N-1)}{8\pi G a^2} = V(a) - \frac{a}{N} \frac{dV(a)}{da} , \quad (2.108)$$

where the usual gravitational constant G is connected with the d-dimensional one by the equality $G_d = \Omega_N G$. From dimensional considerations for the massless field we have $T_{1,2} \approx a^{-d}$ in a d-dimensional space-time. With account of eqs. (2.104), (2.106) and $\Omega_N \approx a^N$ this leads to

$$V(a) = \frac{\hbar c C_N}{a^4} . \quad (2.109)$$

Here C_N is a constant whose values depend on the dimensionality of a compact manifold. Substituting this into eq. (2.108) we find the self-consistent value of the radius of the sphere

$$a^2 = \frac{8\pi C_N (N+4) G\hbar}{N(N-1)c^3} . \quad (2.110)$$

Then from (2.107) the cosmological constant is

$$\Lambda_d = -\frac{N^2 (N-1)^2 (N+2) c^3}{16\pi C_N (N+4)^2 G\hbar} . \quad (2.111)$$

Thence, the second eq. (2.107) can be written also as follows:

$$-\frac{N-1}{a^2} + \frac{N(N-1)}{2a^2} - \frac{N^2 (N-1)^2 (N+2) c^3}{16\pi C_N (N+4)^2 G\hbar} = \frac{8\pi G}{c^4 N} a \frac{dV(a)}{da} . \quad (2.111b)$$

Thus, the self-consistent radii are possible when $N > 1$, and $C_N > 0$. In that case the multidimensional cosmological constant is negative. It is seen from eq. (2.110) that $a \approx l_{P1}$, and the value of the coefficient in this dependence is determined by the value of C_N. Generally speaking, **one should take into account not of one field but of all kinds of boson and fermion fields contributing to the Casimir energy**. From this point of view the self-consistent radii are expressed by

$$a = \left[\frac{8\pi(N+4)}{N(N-1)} C_N \right]^{1/2} l_{P1}, \qquad C_N = n_B C_B^N + n_F C_F^N \ , \quad (2.112)$$

where C_B^N and C_F^N are the dimensionless constants in the eq. (2.109) written for each field separately, n_B **and** n_F **are the numbers of boson and fermion massless field**. It is important that $C_N \geq 1$. Thence, we can rewrite the expression (2.112) also as follows:

$$a = \sqrt{\frac{8\pi(N+4)}{N(N-1)} \left(n_B C_B^N + n_F C_F^N \right) l_{P1}}, \quad (2.112b)$$

and the eq. (2.105b) as follows

$$\frac{1}{2} T_2 N \int d^N u \sqrt{|\det h_{ab}|} = -\int d^N u h^{ab} \frac{\delta V(h)}{\delta h^{ab}} = -\left[\frac{8\pi(N+4)}{N(N-1)} \left(n_B C_B^N + n_F C_F^N \right) l_{P1} \right] \frac{dV}{da^2}. \quad (2.112c)$$

Furthermore, we can rewrite the eq. (2.105b) connecting it with the eq. (2.110), and obtain also the following expression:

$$\frac{1}{2} T_2 N \int d^N u \sqrt{|\det h_{ab}|} = -\int d^N u h^{ab} \frac{\delta V(h)}{\delta h^{ab}} = -a^2 \frac{dV}{da^2} \Rightarrow \frac{8\pi C_N (N+4) G\hbar}{N(N-1)c^3}. \quad (2.113)$$

We note that this equation can be related with the eq. (B9) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$\frac{1}{2} T_2 N \int d^N u \sqrt{|\det h_{ab}|} = -\int d^N u h^{ab} \frac{\delta V(h)}{\delta h^{ab}} = -a^2 \frac{dV}{da^2} \Rightarrow \frac{8\pi C_N (N+4) G\hbar}{N(N-1)c^3} \Rightarrow$$

$$\Rightarrow -\int d^{26} x \sqrt{g} \left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr\left(G_{\mu\nu} G_{\rho\sigma} \right) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10} x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v \left(|F_2|^2 \right) \right] \Rightarrow$$

$$\Rightarrow \frac{1}{3} \frac{4 \left[anti \log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]}. \quad (2.113b)$$

It is seen from the following Table that the values of all coefficients are rather small, being especially small for the conformally coupled scalar field. The values of C_N from eq. (2.112) are positive, e.g., for $N = 3$ or 7 regardless of the number of different fields. For $N = 5,9$ the number of fermions and conformally coupled scalar fields should not be too large comparing the number of minimally coupled scalar fields in order to assure the condition $C_N \geq 0$. In all these cases eq. (2.112) provides the self-consistent values of a compactification radius. To reach the value $C_N \approx 1$ a large number of fields is needed, however. For example, even for $N = 11$ where the value of $C_B^N \approx 133 \times 10^{-5}$ is achieved, one needs to have 752 scalar fields with minimal coupling to have $C_N \approx 1$. The enormous number of light matter fields required to get the reasonable size of the compactification radius (no smaller than a Planckean one) is the general characteristic feature of the spontaneous compactification mechanisms based on the Casimir effect.

Table

N	$10^5 C_B^N$ (min.)	$10^5 C_B^N$ (min.)	$10^5 C_F^N$
3	7.56870	0.714589	19.45058
5	42.8304	-0.078571	-11.40405
7	81.5883	0.007049	5.95874
9	113.389	-0.000182	-2.99172
11	132.932	-0.000157	1.47771

We note that the following values: 19.45058 11.40405 5.95874 2.99172 1.47771 are very near to the following: 19.416 11.423 5.933 2.996 1.4754 values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Now let us apply the following equation

$$F_{SS}^T = -\frac{\hbar}{\beta}\sum_l \int \frac{dk_\perp}{(2\pi)^2} q_l \left\{ \left[\left(\frac{\varepsilon(i\xi_l)q_l + k_l}{\varepsilon(i\xi_l)q_l - k_l} \right)^2 e^{2aq_l} - 1 \right]^{-1} + \left[\left(\frac{q_l + k_l}{q_l - k_l} \right)^2 e^{2aq_l} - 1 \right]^{-1} \right\}, \quad (2.114)$$

to ideal metals of infinitely high conductivity in order to find the temperature correction to the Casimir force $F_{SS}^{(0)}(a)$ between perfect conductors. To do this we use the prescription by Schwinger, DeRaad and Milton that the limit $\varepsilon \to \infty$ should be taken before setting $l = 0$. Then, introducing a new variable $y = 2aq_l$ in eq. (2.114) instead of $|k_\perp| = k$ we arrive at

$$F_{SS}^T(a) = -\frac{k_B T}{4\pi a^3} \sum_{l=0}^{\infty}{}' \int_{2a\xi_l / c}^{\infty} \frac{y^2 dy}{e^y - 1}. \quad (2.115)$$

This expression can be put in a form

$$F_{SS}^T(a) = F_{SS}^{(0)}(a) \left\{ 1 + \frac{30}{\pi^4} \sum_{n=1}^{\infty} \left[\left(\frac{T}{T_{eff}} \right)^4 \frac{1}{n^4} - \pi^3 \frac{T}{T_{eff}} \frac{1}{n} \cosh\left(\pi n \frac{T_{eff}}{T} \right) \sinh^{-3}\left(\pi n \frac{T_{eff}}{T} \right) \right] \right\}, \quad (2.116)$$

where the effective temperature is defined as $k_B T_{eff} = \hbar c / (2a)$. Note that the quantity in square brackets is always positive.
Note that we have:

$\frac{1}{4\pi} = 0{,}0795774 \cong 0{,}07982$; $\quad \frac{30}{\pi^4} = 0{,}307979 \cong 0{,}3099$; $\quad \pi^3 = 31{,}00627 \cong 31{,}0662$ all values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Thence, we can rewrite the eq. (2.115) also as follows:

$$F_{SS}^{T}(a) = -\frac{k_B T}{4\pi a^3} \sum_{l=0}^{\infty}{}' \int_{2a\xi_l/c}^{\infty} \frac{y^2 dy}{e^y - 1} =$$

$$= F_{SS}^{(0)}(a) \left\{ 1 + \frac{30}{\pi^4} \sum_{n=1}^{\infty} \left[\left(\frac{T}{T_{eff}} \right)^4 \frac{1}{n^4} - \pi^3 \frac{T}{T_{eff}} \frac{1}{n} \cosh\left(\pi n \frac{T_{eff}}{T} \right) \sinh^{-3}\left(\pi n \frac{T_{eff}}{T} \right) \right] \right\}, \quad (2.116b)$$

At low temperatures $T << T_{eff}$ it follows from (2.116)

$$F_{SS}^{T}(a) \approx F_{SS}^{(0)}(a) \left[1 + \frac{1}{3} \left(\frac{T}{T_{eff}} \right)^4 \right]. \quad (2.117)$$

At high temperature limit $T >> T_{eff}$

$$F_{SS}^{T}(a) \approx -\frac{k_B T}{4\pi a^3} \zeta_R(3). \quad (2.118)$$

Note that the corrections to the above asymptotic results are exponentially small as $\exp(-2\pi T_{eff}/T)$ at low temperatures and as $\exp(-2\pi T/T_{eff})$ at high temperatures. As a consequence, the asymptotic regime is even achieved when the temperature is only two times lower (higher) than the effective temperature value.
The expression for the free energy density in the configuration of two dielectric plates at a temperature T, is obtained by the integration of $-F_{SS}^{T}(a)$ from eq. (2.114) with respect to a. The result is

$$F_E(a) = \frac{k_B T}{4\pi} \sum_{l=-\infty}^{\infty} \int_0^{\infty} k_\perp dk_\perp \left\{ \ln\left[1 - \left(\frac{\varepsilon(i\xi_l) q_l - k_l}{\varepsilon(i\xi_l) q_l + k_l} \right)^2 e^{-2aq_l} \right] + \ln\left[1 - \left(\frac{q_l - k_l}{q_l + k_l} \right)^2 e^{-2aq_l} \right] \right\}. \quad (2.119)$$

The Casimir (van der Waals) force acting between a disk and a sphere (lens) is:

$$F_{d\ell}(a) = 2\pi R E_S^{ren}(a) \quad (2.120)$$

The non-zero temperature Casimir force in configuration of a sphere (lens) above a disk (plate) is given by eq. (2.120) as follows:

$$F_{d\ell}^{T}(a) = 2\pi R F_E(a). \quad (2.121)$$

The general expression for the Casimir and van der Waals force between two semispaces made of material with a frequency dependent dielectric permittivity ε_2 is given by eq. (2.52). It is convenient to use the notation ε instead of ε_2, introduce the new variable $x = 2\xi p a/c$ instead of ξ, and change the order of integration. As a result the force equation takes a form

$$F_{SS}^{C}(a) = -\frac{\hbar c}{32\pi^2 a^4}\int_0^\infty x^3 dx \int_1^\infty \frac{dp}{p^2}\left\{\left[\frac{(K+p\varepsilon)^2}{(K-p\varepsilon)^2}e^x - 1\right]^{-1} + \left[\frac{(K+p)^2}{(K-p)^2}e^x - 1\right]^{-1}\right\}. \quad (2.122)$$

We have that: $\frac{1}{32\pi^2} = 0{,}003166286 \cong 0{,}00319896$ value concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

The Casimir force between metallic plates with finite conductivity corrections up to the fourth power in relative penetration depth is given from the following equation

$$F_{SS}^{C}(a) = F_{SS}^{(0)}(a)\left[1 - \frac{16}{3}\frac{\delta_0}{a} + 24\frac{\delta_0^2}{a^2} - \frac{640}{7}\left(1 - \frac{\pi^2}{210}\right)\frac{\delta_0^3}{a^3} + \frac{2800}{9}\left(1 - \frac{163\pi^2}{7350}\right)\frac{\delta_0^4}{a^4}\right]. \quad (2.123)$$

We have that: $\frac{\pi^2}{210} = 0{,}046998 \cong 0{,}04644008$; $\frac{163\pi^2}{7350} = 0{,}218876941 \cong 0{,}21884705$ values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Thence, we can rewrite the eq. (2.122) also as follows:

$$F_{SS}^{C}(a) = -\frac{\hbar c}{32\pi^2 a^4}\int_0^\infty x^3 dx \int_1^\infty \frac{dp}{p^2}\left\{\left[\frac{(K+p\varepsilon)^2}{(K-p\varepsilon)^2}e^x - 1\right]^{-1} + \left[\frac{(K+p)^2}{(K-p)^2}e^x - 1\right]^{-1}\right\} =$$

$$= F_{SS}^{(0)}(a)\left[1 - \frac{16}{3}\frac{\delta_0}{a} + 24\frac{\delta_0^2}{a^2} - \frac{640}{7}\left(1 - \frac{\pi^2}{210}\right)\frac{\delta_0^3}{a^3} + \frac{2800}{9}\left(1 - \frac{163\pi^2}{7350}\right)\frac{\delta_0^4}{a^4}\right]. \quad (2.123b)$$

Also this equation can be related with the eq. (B9) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$F_{SS}^{C}(a) = -\frac{\hbar c}{(8\times 4)\pi^2 a^4}\int_0^\infty x^3 dx \int_1^\infty \frac{dp}{p^2}\left\{\left[\frac{(K+p\varepsilon)^2}{(K-p\varepsilon)^2}e^x - 1\right]^{-1} + \left[\frac{(K+p)^2}{(K-p)^2}e^x - 1\right]^{-1}\right\} =$$

$$= F_{SS}^{(0)}(a)\left[1 - \frac{16}{3}\frac{\delta_0}{a} + 24\frac{\delta_0^2}{a^2} - \frac{(8^2\times 10)}{7}\left(1 - \frac{\pi^2}{210}\right)\frac{\delta_0^3}{a^3} + \frac{(8\times 350)}{9}\left(1 - \frac{163\pi^2}{7350}\right)\frac{\delta_0^4}{a^4}\right] \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.123c)$$

We consider now the plasma model perturbation approach in configuration of a sphere (lens) above a disk. The sphere (lens) radius R is suggested to be much larger than the sphere-disk separation a. Owing to this the Proximity Force Theorem is valid and the Casimir force is given by eqs. (2.120) and (2.53). Introducing once more the variable $x = 2\xi pa / c$ the following result is obtained:

$$F_{d\ell}^{C}(a) = \frac{\hbar cR}{16\pi a^3}\int_0^\infty x^2 dx \int_1^\infty \frac{dp}{p^2}\left\{ \ln\left[1 - \frac{(K - p\varepsilon)^2}{(K + p\varepsilon)^2} e^{-x}\right] + \ln\left[1 - \frac{(K - p)^2}{(K + p)^2} e^{-x}\right]\right\}. \quad (2.124)$$

Bearing in mind the need to do perturbative expansions it is convenient to perform in (2.124) an integration by parts with respect to x. The result is

$$F_{d\ell}^{C}(a) = -\frac{\hbar cR}{48\pi a^3}\int_0^\infty x^3 dx \int_1^\infty \frac{dp}{p^2}\left[\frac{(K - p\varepsilon)^2 - (K + p\varepsilon)^2 \frac{\partial}{\partial x}\frac{(K - p\varepsilon)^2}{(K + p\varepsilon)^2}}{(K + p\varepsilon)^2 e^x - (K - p\varepsilon)^2} + \frac{(K - p)^2 - (K + p)^2 \frac{\partial}{\partial x}\frac{(K - p)^2}{(K + p)^2}}{(K + p)^2 e^x - (K - p)^2}\right]$$

(2.125)

Also this equation can be related to the eq. (B8) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$F_{d\ell}^{C}(a) = -\frac{\hbar cR}{(24\times 2)\pi a^3}\int_0^\infty x^3 dx \int_1^\infty \frac{dp}{p^2}\left[\frac{(K - p\varepsilon)^2 - (K + p\varepsilon)^2 \frac{\partial}{\partial x}\frac{(K - p\varepsilon)^2}{(K + p\varepsilon)^2}}{(K + p\varepsilon)^2 e^x - (K - p\varepsilon)^2} + \frac{(K - p)^2 - (K + p)^2 \frac{\partial}{\partial x}\frac{(K - p)^2}{(K + p)^2}}{(K + p)^2 e^x - (K - p)^2}\right] \Rightarrow$$

(2.125)

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2} e^{-2\Phi}\left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu\left(\left|F_2\right|^2\right)\right] \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4}\right)}\right]}. \quad (2.125b)$$

With regard the eqs. (2.124) – (2.125), we have that: $\frac{1}{16\pi} = 0,0198943 \cong 0,01973343$; $\frac{1}{48\pi} = 0,00663145 \cong 0,00659023 - 0,0067000$ values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Consider two semi-spaces modelled by plates which are made of a material with a static dielectric permittivity ε_0 bounded by surfaces with small deviations from plane geometry. The approximate expression for the Casimir energy in this configuration is given by the following equation

$$U(a) \equiv \frac{U^{add}(a)}{K} = -\hbar c \Psi(\varepsilon_{20}) \int_{V_1} d^3 r_1 \int_{V_2} d^3 r_2 |r_2 - r_1|^{-7}, \quad (2.126)$$

where the function Ψ was defined in (2.57), and we now change the notation ε_{20} for ε_0. Let us describe the surface of the first plate by the equation

$$z_1^{(s)} = A_1 f_1(x_1, y_1) \quad (2.127)$$

and the surface of the second plate by

$$z_2^{(s)} = a + A_2 f_2(x_2, y_2), \quad (2.128)$$

where a is the mean value of the distance between the plates. The non-normalized potential of one atom at a height z_2 over the plate P_1 is given by

$$U_A(x_2, y_2, z_2) = -CN \int_{-L}^{L} dx_1 \int_{-L}^{L} dy_1 \int_{-D}^{A_1 f_1(x_1, y_1)} dz_1 \times \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{-7/2}, \quad (2.129)$$

where C is an interaction constant from the following equation

$$U(r_{12}) = -\frac{C}{r_{12}^7}, \qquad C \equiv \frac{23}{4\pi} \hbar c \alpha^2(0), \quad (2.130)$$

N is the number of atoms per unit volume of plates P_1 and P_2. The result of the expansion up to the fourth order with respect to the small parameter A_1 / z_2 may be written in the form

$$U_A(x_2, y_2, z_2) = -CN \left\{ \frac{\pi}{10 z_2^4} + \int_{-L}^{L} dx_1 \int_{-L}^{L} dy_1 \times \left[\frac{z_2 f_1(x_1, y_1)}{X^{7/2}} \left(\frac{A_1}{z_2} \right) + \frac{7 z_2^3 f_1^2(x_1, y_1)}{2 X^{9/2}} \left(\frac{A_1}{z_2} \right)^2 + \right. \right.$$

$$\left. \left. + \frac{7 z_2^3}{6 X^{9/2}} \left(\frac{9 z_2^2}{X} - 1 \right) f_1^3(x_1, y_1) \left(\frac{A_1}{z_2} \right)^3 + \frac{21 z_2^5}{8 X^{11/2}} \left(\frac{11 z_2^2}{X} - 3 \right) f_1^4(x_1, y_1) \left(\frac{A_1}{z_2} \right)^4 \right] \right\} \quad (2.131)$$

with $X = (x_1 - x_2)^2 + (y_1 - y_2)^2 + z_2^2$. Here, the limit $L \to \infty$ is performed in the first item, which describes the perfect plates without deviations from planar case.

We have that for the eq. (2.130): $\frac{23}{4\pi} = 1{,}830281846 \cong 1{,}83343685$; and for the eq. (2.131): $\frac{\pi}{10} = 0{,}314159265 \cong 0{,}31475730$ values concerning the **new universal music system based on fractional powers of Phi and Pigreco.** Furthermore, in the eq. (2.131) the values 21 and 8 are Fibonacci's numbers.

Now, we start from the Lifshitz formula (2.114):

$$F_{SS}^{T}=-\frac{\hbar}{\beta}\sum_{l}\int\frac{dk_{\perp}}{(2\pi)^{2}}q_{l}\left\{\left[\left(\frac{\varepsilon(i\xi_{l})q_{l}+k_{l}}{\varepsilon(i\xi_{l})q_{l}-k_{l}}\right)^{2}e^{2aq_{l}}-1\right]^{-1}+\left[\left(\frac{q_{l}+k_{l}}{q_{l}-k_{l}}\right)^{2}e^{2aq_{l}}-1\right]^{-1}\right\},$$

and rewrite it more conveniently in the form

$$F_{SS}^{T}(a)=-\frac{k_{B}T}{2\pi}\sum_{l=-\infty}^{\infty}\int_{0}^{\infty}k_{\perp}dk_{\perp}q_{l}\left\{\left[r_{1}^{-2}(\xi_{l},k_{\perp})e^{2aq_{l}}-1\right]^{-1}+\left[r_{2}^{-2}(\xi_{l},k_{\perp})e^{2aq_{l}}-1\right]^{-1}\right\},\quad (2.132)$$

where $r_{1,2}$ are the reflection coefficients with parallel (perpendicular) polarization, respectively, given by

$$r_{1}^{-2}(\xi_{l},k_{\perp})=\left[\frac{\varepsilon(i\xi_{l})q_{l}+k_{l}}{\varepsilon(i\xi_{l})q_{l}-k_{l}}\right]^{2},\qquad r_{2}^{-2}(\xi_{l},k_{\perp})=\left(\frac{q_{l}+k_{l}}{q_{l}-k_{l}}\right)^{2}.\quad (2.133)$$

In terms of dimensionless variables

$$y=2aq_{l}=2a\sqrt{\frac{\xi_{l}^{2}}{c^{2}}+k_{\perp}^{2}},\qquad \tilde{\xi}_{l}=2a\frac{\xi_{l}}{c}.\quad (2.134)$$

Eqs. (2.132) and (2.133) can be rearranged to

$$F_{SS}^{T}(a)=-\frac{k_{B}T}{16\pi a^{3}}\sum_{l=-\infty}^{\infty}\int_{|\tilde{\xi}_{l}|}^{\infty}y^{2}dy\left\{\left[r_{1}^{-2}(\tilde{\xi}_{l},y)e^{y}-1\right]^{-1}+\left[r_{2}^{-2}(\tilde{\xi}_{l},y)e^{y}-1\right]^{-1}\right\}.\quad (2.135)$$

According to Poisson summation formula if $c(\alpha)$ is the Fourier transform of a function $b(x)$

$$c(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}b(x)e^{-i\alpha x}dx\quad (2.136)$$

then it follows

$$\sum_{l=-\infty}^{\infty}b(l)=2\pi\sum_{l=-\infty}^{\infty}c(2\pi l).\quad (2.137)$$

Let us apply this formula to eq. (2.135) using the identification

$$b_{SS}(l)\equiv-\frac{k_{B}T}{16\pi a^{3}}\int_{|l|\tau}^{\infty}y^{2}dyf_{SS}(l\tau,y),\qquad \tau\equiv\frac{4\pi ak_{B}T}{\hbar c},\quad (2.138)$$

where $\tilde{\xi}_{l}=\tau l$ and

$$f_{SS}(l\tau,y)=f_{SS}^{(1)}(l\tau,y)+f_{SS}^{(2)}(l\tau,y)\equiv\left(r_{1}^{-2}e^{y}-1\right)^{-1}+\left(r_{2}^{-2}e^{y}-1\right)^{-1}\quad (2.139)$$

is an even function of l.
Then the quantity $c_{SS}(\alpha)$ from eq. (2.136) is given by

$$c_{SS}(\alpha) = -\frac{k_B T}{16\pi^2 a^3}\int_0^\infty dx \cos\alpha x \int_{x\tau}^\infty y^2 dy f_{SS}(x\tau, y). \quad (2.140)$$

Using eqs. (2.135), (2.137) and (2.140) one finally obtains the new representation of Lifshitz formula

$$F_{SS}^T(a) = \sum_{l=-\infty}^{\infty} b_{SS}(l) = -\frac{\hbar c}{16\pi^2 a^4}\sum_{l=0}^{\infty}{}' \int_0^\infty d\tilde{\xi} \cos\left(l\tilde{\xi}\frac{T_{eff}}{T}\right)\int_{\tilde{\xi}}^\infty y^2 dy f_{SS}(\tilde{\xi}, y), \quad (2.141)$$

where the continuous variable $\tilde{\xi} = \tau x$.

Also this equation can be related with the eq. (B9) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$F_{SS}^T(a) = \sum_{l=-\infty}^{\infty} b_{SS}(l) = -\frac{\hbar c}{(8\times 2)\pi^2 a^4}\sum_{l=0}^{\infty}{}' \int_0^\infty d\tilde{\xi} \cos\left(l\tilde{\xi}\frac{T_{eff}}{T}\right)\int_{\tilde{\xi}}^\infty y^2 dy f_{SS}(\tilde{\xi}, y) \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.141b)$$

Note that in the representation (2.141) the $l = 0$ term gives the force at zero temperature.

From the eq. (2.118), for high temperatures, we can rewrite the eq. (2.141) also as follows:

$$F_{SS}^T(a) = \sum_{l=-\infty}^{\infty} b_{SS}(l) = -\frac{\hbar c}{16\pi^2 a^4}\sum_{l=0}^{\infty}{}' \int_0^\infty d\tilde{\xi} \cos\left(l\tilde{\xi}\frac{T_{eff}}{T}\right)\int_{\tilde{\xi}}^\infty y^2 dy f_{SS}(\tilde{\xi}, y) \approx -\frac{k_B T}{4\pi a^3}\zeta_R(3). \quad (2.141c)$$

For real metals eq. (2.141) can be transformed to the form

$$F_{SS}^T(a) = -\frac{k_B T}{16\pi a^3}\left\{\int_0^\infty y^2 dy\left[f_{SS}^{(1)}(0, y) + f_{SS}^{(2)}(y, y)\right] + 2\sum_{n=1}^{\infty}\int_{\tilde{\xi}_n}^\infty y^2 dy f_{SS}(\tilde{\xi}_n, y)\right\}. \quad (2.142)$$

To obtain perturbation expansion of eq. (2.141) in terms of a small parameter of the plasma model δ_0 / a it is useful to change the order of integration and then rewrite it in terms of the new variable $v \equiv \tilde{\xi} / y$ instead of $\tilde{\xi}$

$$F_{SS}^{T}(a)=-\frac{\hbar c}{16\pi^2 a^4}\sum_{l=0}^{\infty}{}'\int_0^{\infty} y^3 dy\int_0^1 dv\cos\left(vyl\frac{T_{eff}}{T}\right)f_{ss}(v,y). \quad (2.143)$$

We note that

$$\frac{1}{16\pi}=0{,}019894367\cong 0{,}01987597\cong 0{,}01973343;$$
$$\frac{1}{16\pi^2}=0{,}0063325739\cong 0{,}00632662\cong 0{,}00628125;$$

values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Expanding the quantity f_{SS} up to the first order in powers of δ_0/a one obtains

$$f_{SS}(v,y)=\frac{2}{e^y-1}-2\frac{ye^y}{(e^y-1)^2}\left(1+v^2\right)\frac{\delta_0}{a}.$$

Substituting this into eq. (2.143) we come to the Casimir force including the effect of both the nonzero temperature and finite conductivity

$$F_{SS}^{T,C}(a)=F_{SS}^{(0)}(a)\left\{1+\frac{30}{\pi^4}\sum_{n=1}^{\infty}\left[\frac{1}{t_n^4}-\frac{\pi^3}{t_n}\frac{\cosh(\pi t_n)}{\sinh^3(\pi t_n)}\right]-\frac{16}{3}\frac{\delta_0}{a}-60\frac{\delta_0}{a}\times\right.$$
$$\left.\times\sum_{n=1}^{\infty}\left[\frac{2\cosh^2(\pi t_n)+1}{\sinh^4(\pi t_n)}-\frac{2\cosh(\pi t_n)}{\pi t_n\sinh^3(\pi t_n)}-\frac{1}{2\pi^2 t_n^2\sinh^2(\pi t_n)}-\frac{\coth(\pi t_n)}{2\pi^3 t_n^3}\right]\right\}, \quad (2.144)$$

where $t_n\equiv nT_{eff}/T$. The first summation in (2.144) is exactly the temperature correction in the case of ideal metals. In the limit of low temperatures $T<<T_{eff}$ one has from (2.144) up to exponentially small corrections

$$F_{SS}^{T,C}(a)\approx F_{SS}^{(0)}(a)\left\{1+\frac{1}{3}\left(\frac{T}{T_{eff}}\right)^4-\frac{16}{3}\frac{\delta_0}{a}\left[1-\frac{45\zeta_R(3)}{8\pi^3}\left(\frac{T}{T_{eff}}\right)^3\right]\right\}. \quad (2.145)$$

In the limit of high temperatures $T>>T_{eff}$ eq. (2.144) leads to

$$F_{SS}^{T,C}(a)\approx-\frac{k_B T}{4\pi a^3}\zeta_R(3)\left(1-3\frac{\delta_0}{a}\right) \quad (2.146)$$

up to exponentially small corrections. For $\delta_0=0$ one obtains from (2.146) the known result (2.118) for perfect conductors.
Also here, the eq. (2.123b), for high temperatures, with good approximation, can be connected with eq. (2.146) to give the following expression:

$$F_{SS}^{C}(a)=-\frac{\hbar c}{32\pi^2 a^4}\int_0^{\infty}x^3dx\int_1^{\infty}\frac{dp}{p^2}\left\{\left[\frac{(K+p\varepsilon)^2}{(K-p\varepsilon)^2}e^x-1\right]^{-1}+\left[\frac{(K+p)^2}{(K-p)^2}e^x-1\right]^{-1}\right\}=$$

$$= F_{SS}^{(0)}(a)\left[1 - \frac{16}{3}\frac{\delta_0}{a} + 24\frac{\delta_0^2}{a^2} - \frac{640}{7}\left(1 - \frac{\pi^2}{210}\right)\frac{\delta_0^3}{a^3} + \frac{2800}{9}\left(1 - \frac{163\pi^2}{7350}\right)\frac{\delta_0^4}{a^4}\right] \approx -\frac{k_B T}{4\pi a^3}\zeta_R(3)\left(1 - 3\frac{\delta_0}{a}\right). \quad (2.146b)$$

With regard the eq. (2.144) – (2.145) we have that:

$$\frac{30}{\pi^4} = 0{,}307979 \cong 0{,}3090169 \cong 0{,}3147573\,; \quad \pi^3 = 31{,}00627668 \cong 31{,}4164078\,;$$

$$\frac{1}{2\pi^2} = 0{,}050660591 \cong 0{,}05065904 \cong 0{,}05166278\,; \quad \frac{2}{\pi} = 0{,}6366197 \cong 0{,}636610\,;$$

$$\frac{1}{2\pi^3} = 0{,}0161257 \cong 0{,}0160800 \cong 0{,}01626124\,; \quad \frac{45}{8\pi^3} = 0{,}1814148 \cong 0{,}18033989 \cong 0{,}18237254\,;$$

values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Now we start from the temperature Casimir force of eqs. (2.119) and (2.121) acting in the abovementioned configurations. In terms of the reflection coefficients introduced in eq. (2.133) this force can be represented as

$$F_{d\ell}^T(a) = \frac{k_B TR}{2}\sum_{n=-\infty}^{\infty}\int_0^\infty k_\perp dk_\perp \left\{\ln\left[1 - r_1^2(\xi_n, k_\perp)e^{-2aq_n}\right] + \ln\left[1 - r_2^2(\xi_n, k_\perp)e^{-2aq_n}\right]\right\}. \quad (2.147)$$

Introducing the dimensionless variables $\tilde{\xi}_n = 2a\xi_n / c$ and $y = 2aq_n$ we rewrite eq. (2.147) in the form

$$F_{d\ell}^T(a) = \frac{k_B TR}{8a^2}\sum_{n=-\infty}^{\infty}\int_{|\tilde{\xi}_n|}^\infty ydy \left\{\ln\left[1 - r_1^2(\tilde{\xi}_n, y)e^{-y}\right] + \ln\left[1 - r_2^2(\tilde{\xi}_n, y)e^{-y}\right]\right\}. \quad (2.148)$$

We rewrite eq. (2.148) in the form analogical to (2.141). This is achieved by the Poisson summation formula of eqs. (2.136) and (2.137). The final result is the following:

$$F_{d\ell}^T(a) = \frac{\hbar cR}{8\pi a^3}\sum_{n=0}^{\infty}{}' \int_0^\infty d\tilde{\xi}\cos\left(n\tilde{\xi}\frac{T_{eff}}{T}\right)\int_{\tilde{\xi}}^\infty ydyf_{d\ell}(\tilde{\xi}, y), \quad (2.149)$$

where

$$f_{d\ell}(\tilde{\xi}, y) = f_{d\ell}^{(1)}(\tilde{\xi}, y) + f_{d\ell}^{(2)}(\tilde{\xi}, y) \equiv \ln\left(1 - r_1^2 e^{-y}\right) + \ln\left(1 - r_2^2 e^{-y}\right). \quad (2.150)$$

Thence, we can rewrite the eq. (2.149) also as follows:

$$F_{d\ell}^T(a) = \frac{\hbar cR}{8\pi a^3}\sum_{n=0}^{\infty}{}' \int_0^\infty d\tilde{\xi}\cos\left(n\tilde{\xi}\frac{T_{eff}}{T}\right)\int_{\tilde{\xi}}^\infty ydy\,\left[\ln\left(1 - r_1^2 e^{-y}\right) + \ln\left(1 - r_2^2 e^{-y}\right)\right]. \quad (2.150b)$$

We note that: $\frac{1}{8\pi} = 0{,}039788735 \cong 0{,}03946685\,;$ value concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Also this equation can be related with the eq. (B9) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$F_{d\ell}^{T}(a)=\frac{\hbar cR}{8\pi a^{3}}\sum_{n=0}^{\infty}{}'\int_{0}^{\infty}d\tilde{\xi}\cos\left(n\tilde{\xi}\frac{T_{eff}}{T}\right)\int_{\tilde{\xi}}^{\infty}ydy\,[\ln\left(1-r_1^2e^{-y}\right)+\ln\left(1-r_2^2e^{-y}\right)]\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=$$

$$=\int_{0}^{\infty}\frac{1}{2\kappa_{10}^{2}}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_{\mu}\Phi\partial^{\mu}\Phi-\frac{1}{2}\left|\tilde{H}_{3}\right|^{2}-\frac{\kappa_{10}^{2}}{g_{10}^{2}}Tr_{v}\left(\left|F_{2}\right|^{2}\right)\right]\Rightarrow$$

$$\Rightarrow\frac{1}{3}\frac{4\left[anti\log\frac{\int_{0}^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^{2}w'}dx}{e^{-\frac{\pi t^{2}}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^{2}w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.150c)$$

Eq. (2.149) can be represented in the form analogical to (2.142)

$$F_{d\ell}^{T}(a)=\frac{k_{B}TR}{8a^{2}}\left\{\int_{0}^{\infty}ydy\left[f_{d\ell}^{(1)}(0,y)+f_{d\ell}^{(2)}(y,y)\right]+2\sum_{n=1}^{\infty}\int_{\tilde{\xi}_{n}}^{\infty}ydyf_{d\ell}\left(\tilde{\xi}_{n},y\right)\right\}. \quad (2.151)$$

Let us first calculate the temperature Casimir force (2.149) in the framework of the plasma model. Changing the order of integration and introducing the new variable $v=\tilde{\xi}/y$ instead of $\tilde{\xi}$ one obtains

$$F_{d\ell}^{T}(a)=\frac{\hbar cR}{8\pi a^{3}}\sum_{n=0}^{\infty}{}'\int_{0}^{\infty}y^{2}dy\int_{0}^{1}dv\cos\left(n\frac{T_{eff}}{T}vy\right)f_{d\ell}(v,y). \quad (2.152)$$

The expansion of $f_{d\ell}$ up to first order in the small parameter δ_0/a is

$$f_{d\ell}(v,y)=2\ln\left(1-e^{-y}\right)+2\frac{y}{e^{y}-1}\left(1+v^{2}\right)\frac{\delta_{0}}{a}.$$

Substitution of this into eq. (2.152) leads to result

$$F_{d\ell}^{T,C}(a)=F_{d\ell}^{(0)}(a)\left\{1+\frac{45}{\pi^{3}}\sum_{n=1}^{\infty}\left[\frac{1}{t_{n}^{3}}\coth(\pi t_{n})+\frac{\pi}{t_{n}^{2}\sinh^{2}(\pi t_{n})}\right]-\frac{1}{t_{1}^{4}}-4\frac{\delta_{0}}{a}+\right.$$

$$\left.+\frac{180}{\pi^{4}}\frac{\delta_{0}}{a}\sum_{n=1}^{\infty}\left[\frac{\pi\coth(\pi t_{n})}{2t_{n}^{3}}-\frac{2}{t_{n}^{4}}+\frac{\pi^{3}\cosh(\pi t_{n})}{t_{n}\sinh^{3}(\pi t_{n})}+\frac{\pi^{2}}{2t_{n}^{2}\sinh^{2}(\pi t_{n})}\right]\right\}. \quad (2.153)$$

Remind that $t_n\equiv nT_{eff}/T$. In the case of low temperatures $T<<T_{eff}$

$$F_{d\ell}^{T,C}(a) \approx F_{d\ell}^{(0)}(a)\left\{1+\frac{45\zeta_R(3)}{\pi^3}\left(\frac{T}{T_{eff}}\right)^3-\left(\frac{T}{T_{eff}}\right)^4-4\frac{\delta_0}{a}\left[1-\frac{45\zeta_R(3)}{2\pi^3}\left(\frac{T}{T_{eff}}\right)^3+\left(\frac{T}{T_{eff}}\right)^4\right]\right\}. \quad (2.154)$$

We note that:

$$\frac{45}{\pi^3}=1{,}45131905\cong 1{,}454463\cong 1{,}45898034\,; \quad \frac{180}{\pi^4}=1{,}8478768\cong 1{,}848361\cong 1{,}85410197\,;$$

$$\frac{\pi}{2}=1{,}570796327\cong 1{,}57082039\cong 1{,}57378652\,; \quad \frac{\pi^2}{2}=4{,}9348022\cong 4{,}928964\cong 4{,}94427191\,;$$

$\frac{45}{2\pi^3}=0{,}725659524\cong 0{,}72723166\cong 0{,}72135955$; values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Let an atom of a sphere be situated at a point with the coordinates (x_A, y_A, z_A). Integrating the interatomic potential $U=-C/r_{12}^7$ over the volume of corrugated plate (r_{12} is a distance between this atom and the atoms of a plate) and calculating the lateral force projection according to $-\partial U/\partial x_A$ one obtains

$$F_x^{(A)}(x_A, y_A, z_A)=\frac{4\pi^2 N_p C}{5z_A^5}\frac{A}{z_A}\frac{z_A}{L}\left[\cos\frac{2\pi x_A}{L}+\frac{5}{2}\frac{A}{z_A}\sin\frac{4\pi x_A}{L}\right], \quad (2.155)$$

where N_p is the atomic density of a corrugated plate. Eq. (2.155) is obtained by perturbation expansion of the integral (up to second order) in small parameter A/z_A. The lateral Casimir force acting upon a sphere is calculated by the integration of (2.155) over the sphere volume and subsequent division by the normalization factor $K=24CN_pN_s/(\pi\hbar c)$ obtained by comparison of additive and exact results for the configuration of two plane parallel plates

$$F_x(x_0, y_0, z_0)=\frac{N_s}{K}\int_{V_s} d^3r F_x^{(A)}(x_0+x, y_0+y, z_0+z), \quad (2.156)$$

where N_s is the atomic density of sphere metal. Let us substitute eq. (2.155) into eq. (2.156) neglecting the small contribution of the upper semisphere which is of order $z_0/R<4\times 10^{-3}$ comparing to unity. In a cylindrical coordinate system the lateral force acting upon a sphere rearranges to the form

$$F_x(x_0, y_0, z_0)=\frac{\pi^3\hbar c}{30}\frac{A}{L}\left[\cos\frac{2\pi x_0}{L}\int_0^R \rho d\rho\int_0^{R-\sqrt{R^2-\rho^2}}\frac{dz}{(z_0+z)^5}\int_0^{2\pi}d\varphi\cos\left(\frac{2\pi\rho}{L}\cos\varphi\right)+\right.$$
$$\left.+\frac{5}{2}A\sin\frac{4\pi x_0}{L}\int_0^R \rho d\rho\int_0^{R-\sqrt{R^2-\rho^2}}\frac{dz}{(z_0+z)^6}\int_0^{2\pi}d\varphi\cos\left(\frac{4\pi\rho}{L}\cos\varphi\right)\right]. \quad (2.157)$$

Preserving only the lowest order terms in small parameter $x_0/R<10^{-2}$ we arrive at

$$F_x(x_0, y_0, z_0)=-\frac{\pi^4\hbar c}{60z_0^4}\frac{A}{L}\left[\cos\frac{2\pi x_0}{L}\int_0^R \rho d\rho J_0\left(\frac{2\pi\rho}{L}\right)+2\frac{A}{z_0}\sin\frac{4\pi x_0}{L}\int_0^R \rho d\rho J_0\left(\frac{4\pi\rho}{L}\right)\right], \quad (2.158)$$

where $J_n(z)$ is Bessel function.
We note, from the eqs. (2.155), (2.157) and (2.158), that:

$\frac{4\pi^2}{5} = 7,89568352 \cong 7,91083506 \cong 7,85410197$; $\frac{\pi^3}{30} = 1,033542556 \cong 1,03005665$;
$\frac{\pi^4}{60} = 1,62348485 \cong 1,61803399 \cong 1,63627124$; all values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Integrating in ρ the final result is obtained

$$F_x(x_0, y_0, z_0) = 3F_{d\ell}^{(0)}(z_0)\frac{A}{z_0}\left[\cos\frac{2\pi x_0}{L}J_1\left(\frac{2\pi R}{L}\right) + \frac{A}{z_0}\sin\frac{4\pi x_0}{L}J_1\left(\frac{4\pi R}{L}\right)\right], \quad (2.159)$$

where the vertical Casimir force $F_{d\ell}^{(0)}$ for ideal metal is defined in the following equation:

$$F_{d\ell}^{(0)}(a) = -\frac{\pi^3\hbar cR}{360a^3}. \quad (2.160)$$

Thence, the eq. (2.159) can be rewritten also as follows:

$$F_x(x_0, y_0, z_0) = 3\left(-\frac{\pi^3\hbar cR}{360a^3}\right)(z_0)\frac{A}{z_0}\left[\cos\frac{2\pi x_0}{L}J_1\left(\frac{2\pi R}{L}\right) + \frac{A}{z_0}\sin\frac{4\pi x_0}{L}J_1\left(\frac{4\pi R}{L}\right)\right]. \quad (2.159b)$$

In conclusion, we have the following expression:

$$F_x(x_0, y_0, z_0) = -\frac{\pi^4\hbar c}{60z_0^4}\frac{A}{L}\left[\cos\frac{2\pi x_0}{L}\int_0^R \rho d\rho J_0\left(\frac{2\pi\rho}{L}\right) + 2\frac{A}{z_0}\sin\frac{4\pi x_0}{L}\int_0^R \rho d\rho J_0\left(\frac{4\pi\rho}{L}\right)\right] \Rightarrow$$
$$\Rightarrow 3\left(-\frac{\pi^3\hbar cR}{360a^3}\right)(z_0)\frac{A}{z_0}\left[\cos\frac{2\pi x_0}{L}J_1\left(\frac{2\pi R}{L}\right) + \frac{A}{z_0}\sin\frac{4\pi x_0}{L}J_1\left(\frac{4\pi R}{L}\right)\right]. \quad (2.160)$$

We note that 360 = 24*15 and 60 = 12*5, with 24 (and 12 = 24/2) that is the number of physical vibrations of the bosonic strings. Thence, we can to connect this expression with the eq. (B8) concerning the P-N model and the Ramanujan's modular equation. Indeed, we have:

$$F_x(x_0, y_0, z_0) = -\frac{\pi^4\hbar c}{60z_0^4}\frac{A}{L}\left[\cos\frac{2\pi x_0}{L}\int_0^R \rho d\rho J_0\left(\frac{2\pi\rho}{L}\right) + 2\frac{A}{z_0}\sin\frac{4\pi x_0}{L}\int_0^R \rho d\rho J_0\left(\frac{4\pi\rho}{L}\right)\right] \Rightarrow$$
$$\Rightarrow 3\left(-\frac{\pi^3\hbar cR}{360a^3}\right)(z_0)\frac{A}{z_0}\left[\cos\frac{2\pi x_0}{L}J_1\left(\frac{2\pi R}{L}\right) + \frac{A}{z_0}\sin\frac{4\pi x_0}{L}J_1\left(\frac{4\pi R}{L}\right)\right] \Rightarrow$$
$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (2.160b)$$

3. On some mathematical connections concerning the zeta strings, the Palumbo-Nardelli model and the Ramanujan's modular functions applied to the string theory. [3]

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (3.1)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L = \sum_{n\geq 1} C_n \mathcal{L}_n = \sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1} n^{-\frac{\Box}{2}}\phi + \sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (3.2)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (3.3)$$

Employing usual expansion for the logarithmic function and definition (3.3) we can rewrite (3.2) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (3.4)$$

where $|\phi| < 1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2+\varepsilon, \quad (3.5)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)} \phi(x) dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \quad (3.6)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t} \zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0) dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (3.7)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk} \zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k) dk = \sum_{n\geq 1} \theta^{\frac{n(n-1)}{2}} \phi^n , \quad (3.8)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk} \zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k) dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (3.9)$$

and one can easily see trivial solution $\phi = \theta = 0$.

In the **Section 2**, we have various equations that can be related with the eq. (3.7) concerning the equation of motion for the zeta strings.

We note that the eq. (2.36) can be related with the eq. (3.7) as follows:

$$E_0^{ren}(a) = -\frac{\hbar c}{\pi a}\int_0^\infty \frac{\xi}{\exp(\xi)-1} d\xi = -\frac{\pi \hbar c}{6a} \Rightarrow E_0^{ren}(a) = -\frac{\hbar c \zeta_R(3)}{2\pi a^2} L \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t} \zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0) dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (3.10)$$

The eq. (2.71) can be related as follows:

$$\sum_{n_1=1}^\infty S_{n_1} = -\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\int_0^\infty t dt + \int_0^\infty dt \int_0^\infty dv \left(\frac{v^2}{a_1^2}+\frac{t^2}{a_2^2}\right)^{1/2} + \frac{1}{24a_1} - \frac{a_2}{8\pi^2 a_1^2}\zeta_R(3) + \frac{2a_2}{a_1^2} G\left(\frac{a_2}{a_1}\right) \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t} \zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0) dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (3.11)$$

The eq. (2.73b) can be related as follows:

$$E_0^{ren}(a_1,a_2)=\frac{\hbar\pi c}{2}\left[-\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\int_0^\infty tdt+\int_0^\infty dt\int_0^\infty dv\left(\frac{v^2}{a_1^2}+\frac{t^2}{a_2^2}\right)^{1/2}+\frac{1}{24a_1}-\frac{a_2}{8\pi^2 a_1^2}\zeta_R(3)+\frac{2a_2}{a_1^2}G\left(\frac{a_2}{a_1}\right)\right]=$$

$$=\hbar c\left[\frac{\pi}{48a_1}-\frac{\zeta_R(3)a_2}{16\pi a_1^2}+\frac{\pi a_2}{a_1^2}G\left(\frac{a_2}{a_1}\right)\right]\Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (3.12)$$

The eq. (2.141b) can be related as follows:

$$F_{SS}^T(a)=\sum_{l=-\infty}^{\infty}b_{SS}(l)=-\frac{\hbar c}{16\pi^2 a^4}\sum_{l=0}^{\infty}{}'\int_0^\infty d\tilde{\xi}\cos\left(l\tilde{\xi}\frac{T_{eff}}{T}\right)\int_{\tilde{\xi}}^\infty y^2dyf_{SS}(\tilde{\xi},y)\approx-\frac{k_BT}{4\pi a^3}\zeta_R(3)\Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (3.13)$$

The eq. (2.146b) can be related as follows:

$$F_{SS}^C(a)=-\frac{\hbar c}{32\pi^2a^4}\int_0^\infty x^3dx\int_1^\infty\frac{dp}{p^2}\left\{\left[\frac{(K+p\varepsilon)^2}{(K-p\varepsilon)^2}e^x-1\right]^{-1}+\left[\frac{(K+p)^2}{(K-p)^2}e^x-1\right]^{-1}\right\}=$$

$$=F_{SS}^{(0)}(a)\left[1-\frac{16}{3}\frac{\delta_0}{a}+24\frac{\delta_0^2}{a^2}-\frac{640}{7}\left(1-\frac{\pi^2}{210}\right)\frac{\delta_0^3}{a^3}+\frac{2800}{9}\left(1-\frac{163\pi^2}{7350}\right)\frac{\delta_0^4}{a^4}\right]\approx-\frac{k_BT}{4\pi a^3}\zeta_R(3)\left(1-3\frac{\delta_0}{a}\right)\Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (3.14)$$

4. On some equations concerning the Casimir Effect and vacuum fluctuations [4]

With regard the Casimir effect and vacuum fluctuations, the Hamiltonian of the electromagnetic field can be written in a normal-mode decomposition as:

$$\hat{H}=\sum_k\sum_\lambda\hbar\omega_k\left(\hat{a}_{k\lambda}^+\hat{a}_{k\lambda}+\frac{1}{2}\right). \quad (4.1)$$

Define the vacuum state as the state with no photons in any mode. Thus the vacuum energy is:

$$E_0=\tfrac{1}{2}\sum_k\sum_\lambda\hbar\omega_k=\sum_k\hbar\omega_k . \quad (4.2)$$

First consider a one-dimensional system where two conducting reflecting mirrors are placed a distance L apart. The presence of the cavity allows only discrete modes, with a density of modes $k=\frac{\nu\pi}{L}$. Thus, we can write for the energy inside the cavity

$$E_0=\sum_k \hbar\omega_k=\sum_k \hbar ck=\frac{\pi\hbar c}{L}\sum_{\nu=1}^{\infty}\nu\,. \quad (4.3)$$

The vacuum energy in the same space without the mirrors is

$$E_{free}=\frac{\pi\hbar c}{L}\int_0^{\infty}\nu d\nu\,. \quad (4.4)$$

Both of these energies are infinite. Thus, the change in energy produced by the presence of the cavity is

$$\Delta E=E_0-E_{free}=\frac{\pi\hbar c}{L}\left[\sum_{\nu=1}^{\infty}\nu-\int_0^{\infty}\nu d\nu\right]. \quad (4.5)$$

Using the Euler-Maclaurin summation formula and conversion factor, $\lim_{\varepsilon\to\infty}e^{-\varepsilon\nu}$, we have

$$\Delta E=-\frac{\pi\hbar c}{12L}\,. \quad (4.6)$$

Therefore, there is an attractive force between the two mirrors

$$F=\frac{\partial\Delta E}{\partial L}-\frac{\pi\hbar c}{12L^2}\,. \quad (4.7)$$

We now consider three dimensions. If we consider a box with two sides (x and y) of length D, and the third (z) of length L, where L<<D, the sums for x and y can be replaced by integrals, and the energy difference can be written by:

$$\Delta E=\frac{D^2\hbar c}{\pi^2}\left[\sum_{\nu=1}^{\infty}\int_0^{\infty}dk_x\int_0^{\infty}dk_y\left(k_x^2+k_y^2+\frac{\nu^2\pi^2}{L^2}\right)^{1/2}-\frac{L}{\pi}\int_0^{\infty}dk_x\int_0^{\infty}dk_y\int_0^{\infty}dk_z\left(k_x^2+k_y^2+k_z^2\right)^{1/2}\right]. \quad (4.8)$$

Using the third derivative in the Euler-Maclaurin summation formula, we have

$$\Delta E=-\left(\frac{\pi^2\hbar c}{720L^3}\right)D^2\,. \quad (4.9)$$

Thence, the eq. (4.8) can be rewritten also as follows:

$$\Delta E=\frac{D^2\hbar c}{\pi^2}\left[\sum_{\nu=1}^{\infty}\int_0^{\infty}dk_x\int_0^{\infty}dk_y\left(k_x^2+k_y^2+\frac{\nu^2\pi^2}{L^2}\right)^{1/2}-\frac{L}{\pi}\int_0^{\infty}dk_x\int_0^{\infty}dk_y\int_0^{\infty}dk_z\left(k_x^2+k_y^2+k_z^2\right)^{1/2}\right]=-\left(\frac{\pi^2\hbar c}{720L^3}\right)D^2.$$

(4.9b)

Therefore, the force F_{cas} and energy E_{cas} can be written as:

$$F_{cas} = \frac{\pi^2}{240}\frac{\hbar c}{L^4}A; \qquad E_{cas} = \frac{\pi^2}{720}\frac{\hbar c}{L^3}A. \quad (4.10)$$

Where $\hbar$ is Planck's constant, "c" is speed of light and "A" is area of the mirrors. The signs correspond to a convention opposite to the standard convention of thermodynamics: the force is attractive and corresponds to a negative pressure; meanwhile, the energy is binding energy corresponding to a mean energy density slightly smaller inside the cavity than in the outside vacuum. Note that the energy density and pressure obey the equation of state of pure radiation. An important feature of the Casimir effect is that even though it is quantum in nature, it predicts a force between macroscopic bodies. For two plane-parallel metallic plates of area $A = 1cm^2$ separated by large distance (on the atomic scale) of $L = 1\mu m$ the value of the attractive is $F_{cas} \approx 1.3 \times 10^{-7}$ N.

From the eqs. (4.7) and (4.8), we note that:

$\frac{\pi}{12} = 0{,}261799387 \cong 0{,}26180340 \cong 0{,}26229775$; $\frac{1}{\pi} = 0{,}318309886 \cong 0{,}31830501$;

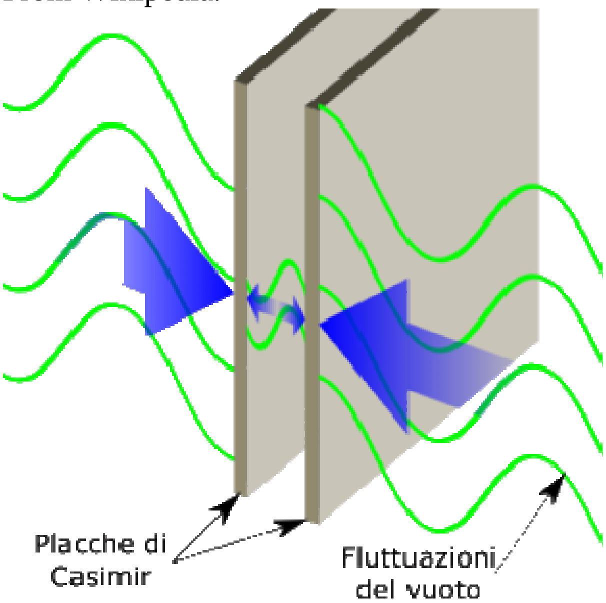
$\frac{1}{\pi^2} = 0{,}101321183 \cong 0{,}10112712 \cong 0{,}10018883$; all values concerning the **new universal music system based on fractional powers of Phi and Pigreco.**

Appendix A

On some mathematical connections between the equation of the energy negative of the Casimir effect, the Casimir operators and some sectors of the Number Theory.

Casimir Effect

From Wikipedia.



In physics, the Casimir effect is the force exerted between two extended bodies located in the vacuum and not due to the action of a gravitational or electromagnetic field, but to the presence - in the space surrounding the bodies - of a quantum field, known as "zero point". Because of the Heisenberg uncertainty principle, the energy of this quantum field (vacuum energy) is subject to fluctuations - described in terms of virtual particles - which occur, at the macroscopic level, in the interaction between the two bodies as a result of a force. The phenomenon is named from the physicist Hendrik Casimir, who theorized it in 1948, based on considerations of quantum mechanics, in the course of his research on the origin of the viscous forces in colloidal solutions….

Mathematical formulation

In the original formulation, Casimir has calculated the effect for two flat parallel metal plates, each a few microns apart, between which a vacuum was created and which were not subject to any electromagnetic field. The theory predicts that only the virtual particles whose wavelength was a sub-multiple integer of the distance between the plates contribute to the energy of the vacuum, in other words, can exist between the plates only these particles, the interaction with the inner walls of the apparatus results a 'push' out no more exactly balanced by those who are outside. The result is a non-zero net force that tends to push the plates against each other and that can be measured. The Casimir force per unit area (Fc / A), in the ideal case of metal plates perfectly conducting between which a vacuum was created, is calculated as:

$$\frac{F_c}{A} = -\frac{d}{dA}\frac{\langle E\rangle}{A} = -\frac{\hbar c\pi^2}{240a^4} = -\frac{hc\pi}{480a^4},$$

where

$$\hbar = \frac{h}{2\pi}$$

is the reduced Planck's constant, "h" is the Planck's constant, "c" is the speed of light, "a" is the distance between the two plates, A is the area of the plates.
The value of the force is negative and indicates that its nature is attractive: the energy density decreases, in fact, approaching the plates.
For example, in the case of plates placed at a distance of 1 micron (μm), the resulting force per unit area is 0.0013 N/m^2. The presence of $\hbar$ shows how small is Fc / A and shows the origin quantum-mechanical of the strength.
Finally we would like to add something on the above formula of the negative energy produced from the Casimir effect.
A denominator of this formula

$$\frac{F_c}{A} = -\frac{d}{dA}\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{240 a^4} = -\frac{hc\pi}{480 a^4},$$

we can see that there are the numbers **240** and **480**: we have found some connections with triangular numbers T, the Fibonacci's numbers and π, the latter present in the numerator.

a) are both multiples of the triangular number 120, indeed 240 = 2 * 120 and thus also of form 2T, sum of the first **15** even numbers, and the numbers 2T are, with the addition of 1, the Lie's numbers

$$L(n) = n^2 + n + 1$$

to the basis of the five exceptional Lie's groups, including E8 = 248 = 31 * 8 very important in the string theories and perhaps even in the TOE, and linked to the number 15 by the formula of Lie

$$\mathbf{15^2} + \mathbf{15} + 8 = \mathbf{240} + 8 = 248 = E8$$

Since **480** = **240** * 2, **480** is also connected to the numbers 2T, with **480** = 120 * 4 = 4T.
(We note that 480 and 240 are 240 = 24 * 10 and 480 = 24 * 20, where 24 is a very important number, because it is the number of the physical vibrations of the bosonic strings).
We recall, that the Fibonacci's numbers F(n) and the partition of numbers p(n) are near to 2T, and the numbers of the Lie algebra L(n) are always halfway between a square and the next, being 2n +1 the distance between two successive square: **Nature seems to choose "his" numbers (Fibonacci, Lie and partitions), for some reason still mysterious, in the central area between a square and the other**

At this "choice" does not seem to escape even the numbers **240** and **480** of the formula of the negative energy produced from the Casimir effect, which, also if they aren't themselves triangular numbers T (but only 2T and 4T with T = 120), are almost exactly, and, respectively, the average of the triangular numbers 231 and 253, because (231 +253) / 2 = 484 / 2 = 242 ≈ **240**, and between 465 and 496 because (465 +493) / 2 = 958 / 2 = 479 ≈ **480**

b) about the **Fibonacci**'s numbers, we still have the number **15**, as the average between **13** and 17, in fact, (**13** +17) / 2 = **15**, with 17 that is the average between **13** and **21**, since (**13** +**21**) / 2 = 17, and, as we have seen in a),

240 = $\mathbf{15^2}$ + **15**, and **480** = 2 ($\mathbf{15^2}$ +**15**)

c) about π, we find that 480 = π^6 / 2, in fact, $(3.14 ...)^6$ / 2 = 961.38 / 2 = **480**.69, furthermore π^6 / 4 = 961.38 / 4 = **240**.34 with very good approximation.

So the **final part** of the formula for the Casimir effect could be written as

$$-\frac{hc\pi}{\frac{\pi^6}{2}a^4}$$

eliminating the number 480 and maintaining a good approximation in the final result.
Recall, from the Basilea's problem, that

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

where we have another connection between π and the number 6, but with exponents and denominators reversed : $\frac{\pi^6}{2}$ and $\frac{\pi^2}{6}$. There might be a connection between the two formulas?

(We note only that $\frac{\pi^6}{2}$ = 480,69… and $\frac{\pi^2}{6}$ = 1,64493. We have that 480,69 ≈ 481,49; 1,64493 ≈ 1,64809 where 481,49 and 1,64809 are both values (frequencies) connected with the **new universal music system based on fractional powers of Phi and Pigreco**)

The formula of the Basilea's problem, we remember, involves the zeta function for exponent n that is even, in this case n = 2, while in the Riemann zeta function n = prime numbers p and their exponent "s" is a complex number.

For the exponent n = 4, we have $\frac{\pi^4}{90}$ and for n = 6 we have $\frac{\pi^6}{945}$ with 90 / 6 = **15** and 945/63 = **15**, thence also here we have the number **15** ... (furthermore, we note that 63 +1 = 64 = 8^2, with 8 that is the number of the physical vibrations of the superstrings, thence very important in the string theories).
Furthermore, we have that $\frac{\pi^4}{90}$ = 1,0823232 and $\frac{\pi^6}{945}$ = 1,017343062, both very near to the following values: 1,082039 and 1,018576 connected with the **new universal music system based on fractional powers of Phi and Pigreco.**

Finally, we note that 240 = 15 * 16 and 480 = 15 * 32, and also 120 = 15 * 8, all multiples 2^n of **15**, with **15** that is the fifth triangular number, and 30 number 2T for T = 15. But also 240 and 480 are divisible by 30, being 240 / 30 = 8 and 480/30 = 16, also here we note that are powers of 2, precisely 3 and 4 (and also here we note that is present the number 8, i.e. the number of the physical vibrations of the superstrings).

Thence, we return to the connection a) with the triangular numbers, using the powers of 2 as factors of 30, 60, 120, **240** and **480**, and for **15** we have 15 = 1 *15 = 2^0 *15.

Thence, we have that the triangular numbers T (binomial coefficients and sums of the first n natural numbers) and 2T (sum of first n even numbers), and 2T + 1 = Lie's numbers, related to the Fibonacci's numbers and the partitions of numbers * , are always present in Nature, also in the energy negative formula produced from the Casimir effect (and also by a magnetic generator) and connected to the teleporter recently patented in the USA.
This could be also the our small mathematics contribution aimed at a better understanding of this new technology, previously considered only science fiction.

* See the previously mentioned our paper: "Una teoria aritmetica, o aritmetica-geometrica, per la TOE (Il principio aritmetico per le teorie di stringa, PATS, complementare al PGTS)”, Francesco Di Noto – Michele Nardelli, and final references.

The Casimir Operators

In the paper of Ahmet Canoglu: "Construction of Casimir Operators of the Group SU(N)" (Journal of Marmara for Pure and Applied Sciences, 18 (2002) 55- 61 Marmara University, Printed in Turkey), the relationship between the Weyl generators on the real and complex representation of the chain of Lie group SU(N) is studied. And using these relations, the connections between the Casimir operators of degree n and independent Casimir operators of these groups are obtained. For example, the Casimir operators of SU(3) and SU(4) are obtained.
Casimir operators and group cohomology are studied extensively in mathematics and physics. Casimir operators commute with all of the generators of the Lie group. The number of independent Casimir operators of each group is equal to the rank of the group. Hence, SU(N) have N independent Casimir operators.
In this work, general procedure for SU(N) is given and the Casimir operators of SU(3) and SU(4) are obtained in terms of independent Casimir operators, respectively.

From the various equations of this paper, we have obtained the following three series of numbers that we have analyzed in terms of mathematical connections with number theory.

3, 4, 10, 15, 28, 35, 45, 56, 120, 126, 210 first series

3, 4, 5, 7, 16, 20, 26, 27, 31, 64, 486, 815 second series

2, 4, 5, 9, 11, 13, 25, 33, 78, 156, 213, 292, 1041, 2016 third series

TABLE UNIFIED

First series	**Second series**	**Third series**	**Fibonacci**	**Lie's Numbers**	**Partitions of numbers**	**Numbers Triangular**	**Observations:small differences ≈ Fibonacci's numbers**
3	**3**	**2**	**2, 3**	**3**	**2**	**3 ; 3**	
4	**4**	**4**	**3+1= 4; 4; 4**		3; 3+1= 5 -1 = **4**; **4**; **4**	3+1= **4**: **4**; **4**	**1**
10	**5**	**5**	**5**; **5**;8+2=**10**		11	6-1 = **5; 5** **10**	**1;2**
15	**7**	**9**	8-1=**7** 8+1=**9** 13+2=**15**		**7**; 7+2 = **9** **15**	6 +1 = **7** 6 +3 = **9** 10 -1= **9** **15**	**1,,2,3**
28	**16**	**11**	13-2=**11**		15+1 =**16**; **11**,	10 +1 =**11** 15 +1=**16** **28**	**1; 2**
35	**20**	**13**	**13**, 21-1=**20** 34+1=**35**	**13** 21-1=**20** 31+4=**35**	15 -2=**13** 22-2=**20**; 30+5=**35**	10 +3=**13** 21 -1 = **20** 36-1=**35**	**1,2,3.4≈5;5**
45	**26**	**25**	21+5=**26** 21+4=**25**	43+2=**45**	22 +3=**25** 22+4=**26** 42+3=**45**	28 -2=**26** 28-3=**25** **45**	**2, 3**; 4≈5; **5**

56	**27**	**33**	55 +1=**56** 34-1=**33** (21+34)/2= **27,5 ≈ 27**	57-1=**56** 31+2=**33**	22+5=**27** 30+3=**33** **56**	28-1=**27** 36-3=**33** 55+1=**56**	**1,2,3,5**
120	**31**	**78**	34-3=**31**	**31,** 73+5= **78** 111+9=**120**	30+1=**31** 77+1=**78** 135-15=**120**	28+3=**31** **78** **120**	**1,3,5**, 9 ≈ **8** 15≈**13**
126	**64**	**156**		157-1=**156** 111+15 =**126** 57+7=**64**	56+8=**64** 77-13=**64** 135+21=**156**	66-2=**64** 120+6=**126** 153+3=**156**	**1,2,3;**6≈**5;8**, **13**;15≈**13**; **21**
210	**486**	**213**	**233**-20 =**210**	211-1 = **210** 211+2=**213** 507-21= **486**	231-21 =**210** 490-4=**486**	**210** 210+3=**213** 465+21=**486** 496-10= **486**	**1, 2,3,4≈5:** 10≈**8,** 20≈**21; 21**
	815	**292**	233+59=**292**	307-15=**292** 813+2= **815**	297-5=**292** 792+23=**815**	300 -8=**292** 820-5=**815**	**2; 5**; 15 ≈ **13**; 23≈**21**; 59 ≈ **55**
		1041	987 +54 = **1041**	1057-16=**1041**	1002+39= **1041**	1035+6= **1041**	6≈**5**: 16 ≈ **13** 39≈**34**; 54 ≈ **55**
		2016		1981+35 =**2016**	1958+58= **2016**	**2016**	35 ≈ **34** 58 ≈ **55**

We can easily note that all the numbers of the three series are very near to the Fibonacci's numbers, numbers of Lie, (2T+1), partitions of numbers and triangular numbers T; in particular, the numbers of the first series are either triangular numbers themselves, or with difference 1.

All the other differences (last column, in absolute value, i.e. negative or positive), or are Fibonacci's numbers, or numbers very near to them. These proximity do not seem, as usual, completely random, meaning that also in this phenomenon, the numbers involved are in the area Fibonacci, triangular, etc. and therefore regulated by the ubiquitous golden section (and golden ratio), of the exceptional Lie groups (based on the numbers of Lie = n^2+n+1) and their natural symmetries.

(See also: http://www.carlosantagata.it/doc/casimirita.pdf)

Appendix B

In the work of Ramanujan, [i.e. the modular functions,] the number 24 (8 x 3) appears repeatedly. This is an example of what mathematicians call magic numbers, which continually appear where we least expect them, for reasons that no one understands. Ramanujan‘s function also appears in string theory. Modular functions are used in the mathematical analysis of Riemann surfaces. Riemann surface theory is relevant to describing the behavior of strings as they move through space-time. When strings move they maintain a kind of symmetry called "conformal invariance". Conformal invariance (including "scale invariance") is related to the fact that points on the surface of a string's world sheet need not be evaluated in a particular order. As long as all points on the surface are taken into account in any consistent way, the physics should not change. Equations of how strings must behave when moving involve the Ramanujan function. When a string moves in space-time by splitting and recombining a large number of mathematical identities must be satisfied. These are the identities of Ramanujan's modular function. The KSV loop diagrams of interacting strings can be described using modular functions. The "Ramanujan function" (an elliptic modular function that satisfies the need for "conformal symmetry") has 24 "modes" that correspond to the physical vibrations of a bosonic string. When the Ramanujan function is generalized, 24 is replaced by 8 (8 + 2 = 10) for fermionic strings.

Palumbo (2001) ha proposed a simple model of the birth and of the evolution of the Universe. Palumbo and Nardelli (2005) have compared this model with the theory of the strings, and translated it in terms of the latter obtaining:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=$$
$$=\int_0^{\infty}\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_{\mu}\Phi\partial^{\mu}\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right], \qquad \text{(B1)}$$

A general relationship that links bosonic and fermionic strings acting in all natural systems.

It is well-known that the series of Fibonacci’s numbers exhibits a fractal character, where the forms repeat their similarity starting from the reduction factor $1/\phi$ = 0,618033 = $\frac{\sqrt{5}-1}{2}$ (Peitgen et al. 1986). Such a factor appears also in the famous fractal Ramanujan identity (Hardy 1927):

$$0{,}618033=1/\phi=\frac{\sqrt{5}-1}{2}=R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}, \qquad \text{(B2)}$$

and

$$\pi=2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right], \qquad \text{(B3)}$$

where

$$\Phi=\frac{\sqrt{5}+1}{2}.$$

Furthermore, we remember that π arises also from the following identities (Ramanujan's paper: "Modular equations and approximations to π" Quarterly Journal of Mathematics, 45 (1914), 350-372.):

$$\pi = \frac{12}{\sqrt{130}} \log\left[\frac{\left(2+\sqrt{5}\right)\left(3+\sqrt{13}\right)}{\sqrt{2}}\right], \text{(B4a)}$$

and

$$\pi = \frac{24}{\sqrt{142}} \log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \text{(B4b)}$$

From (B4b), we have that

$$24 = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad \text{(B5)}$$

Now, we note that the number 8, and thence the numbers $64 = 8^2$ and $32 = 2^2 \times 8$, are connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad \text{(B6)}$$

Furthermore, with regard the number 24 (12 = 24 / 2 and 32 = 24 + 8) they are related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad \text{(B7)}$$

Thence, we can obtain the following mathematical connection between the eq. (B5) and the formula of P-N model, i.e. the bosonic strings:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr\left(G_{\mu\nu} G_{\rho\sigma}\right) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(\left| F_2 \right|^2 \right) \right] \Rightarrow$$

$$\Rightarrow \frac{4 \left[anti \log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]} . \quad \text{(B8)}$$

And between the eq. (B6) and the formula of P-N model, thence the superstrings:

$$- \int d^{26}x \sqrt{g} \left[- \frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr \left(G_{\mu\nu} G_{\rho\sigma} \right) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(\left| F_2 \right|^2 \right) \right] \Rightarrow$$

$$\Rightarrow \frac{1}{3} \frac{4 \left[anti \log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log \left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]} . \quad \text{(B9)}$$

In the **Section 1 and 2**, we have various equations that can be related with the eqs. (B7) and (B8).

Appendix C

From “Scienza news” (Le Scienze, January 2012)

Physics

Light from nothing

An experiment has obtained the dynamic Casimir effect, in which virtual photons become real

A mirror that oscillates at the speed of light, generates quanta of light from nothing. This is, in short, is the result of an experiment published in "Nature", which represents the first experimental demonstration of a quantum phenomenon called "dynamic Casimir effect."

The demonstration was obtained by the group of Christopher Wilson, of the Swedish Chalmers University of Technology. According to quantum mechanics, even the vacuum escapes to the uncertainty principle, and therefore can not be said that it is completely empty: it is filled instead of "virtual particles" that emerge and disappear at any time.
These "vacuum fluctuations" are essential to explain the expansion of the universe and the evaporation of blacks holes, among other phenomena. The theory predicts that the presence of virtual particles can have real consequences. The Casimir effect is that the virtual photons of the vacuum can exert pressure on stationary objects.
In 1970 Gerald Moore theorized a dynamic version of this effect: bodies moving accelerated can induce the reaction of real photons from vacuum fluctuations.
The main problem in the experimental demonstration of this effect is that the movement must reach speeds close to that of light to produce measurable results.
Wilson's group to find the solution to this problem, building a system very similar to a mirror which vibrates at speeds necessary for the detection of the phenomenon: a superconducting circuit that produces an electric surface similar to a mirror in vibration.
Measurements made on this system identify a radiation-induced from the movement that is consistent with theoretical predictions dynamic Casimir effect.

Researchers have tried to exclude causes that may generate this effect in a spurious, not due to vacuum fluctuations.

Among these, some possible defects or background noise generated by the apparatus experimental.

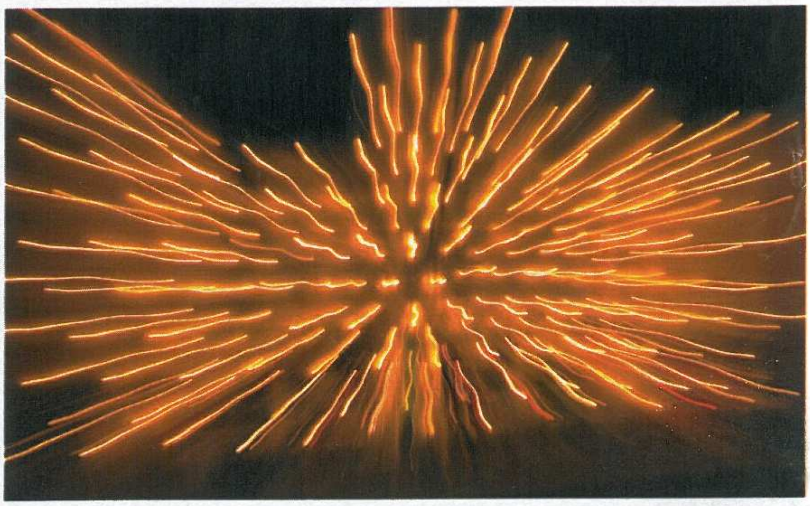
The creation of photons out of nothing does not violate the first law of thermodynamics, since it takes more energy to move the mirror compared to that obtained by light quanta

Michele Catanzaro

FISICA

Luce dal nulla

Un esperimento ha ottenuto l'effetto Casimir dinamico, in cui fotoni virtuali diventano reali



Uno specchio che oscilli alla velocità della luce genera quanti luce dal nulla. È questo, in estrema sintesi, il risultato di un esperimento pubblicato su «Nature», che rappresenta la prima dimostrazione sperimentale di un fenomeno quantistico chiamato «effetto Casimir dinamico». La dimostrazione è stata ottenuta dal gruppo di Christopher Wilson, della svedese Chalmers University of Technology.

Secondo la meccanica quantistica, neanche il vuoto sfugge al principio di indeterminazione, e quindi non si può dire che è completamente vuoto: è invece pervaso di «particelle virtuali» che emergono e scompaiono in ogni momento. Queste «fluttuazioni del vuoto» svolgono un ruolo essenziale per spiegare l'espansione dell'universo e l'evaporazione dei buchi neri, fra gli altri fenomeni. La teoria prevede che la presenza di particelle virtuali può avere conseguenze reali. L'effetto Casimir consiste nel fatto che i fotoni virtuali del vuoto possono esercitare una pressione su oggetti stazionari. Nel 1970 Gerald Moore teorizzò una versione dinamica di questo effetto: corpi in movimento accelerato possono indurre la reazione di fotoni reali a partire dalle fluttuazioni del vuoto. Il principale problema nella dimostrazione sperimentale di questo effetto è che il movimento deve raggiungere velocità prossime a quelle della luce per produrre risultati misurabili.

Il gruppo di Wilson ha aggirato questo problema costruendo un sistema del tutto analogo a uno specchio che vibra alle velocità necessarie per la rilevazione del fenomeno: un circuito superconduttore che produce una superficie elettrica simile a uno specchio in vibrazione. Le misurazioni realizzate su questo sistema identificano una radiazione indotta dal movimento consistente con le previsioni teoriche dell'effetto Casimir dinamico. I ricercatori hanno cercato di escludere le cause che potrebbero generare questo effetto in maniera spuria, non attribuibile a fluttuazioni del vuoto. Fra queste, alcuni possibili difetti o il rumore di fondo generati dall'apparato sperimentale. La creazione di fotoni dal nulla non viola la prima legge della termodinamica, visto che è necessaria più energia per muovere lo specchio rispetto a quella ottenuta dai quanti di luce.

Michele Catanzaro

Appendix D

In the new musical system based on Phi and Pi.greco, we will find the connections with sigma, Pi.greco and the most important harmonic relationships. There is also a column 10/x. In this system, all the notes have this connection as well as the factor 1,2 (which creates the connection with Pi.greco). There are other harmonics connections less important that, however, may be connected with the values concerning the equations of the string theory. The system is perfectly framed: begins with the factor 0,75 and ends with the connection 12. The corresponding to the factor 0,75 is 13,33333 but no longer enters the system because the notes are completed. There are 36 notes for a range Phi that applied to the circle of 360 degrees creates 10 subdivisions (a decagon, the DNA, etc…).
This was the initial intuition that meant that the Eng. Christian Lange has reconsidered the musical system based on Phi and Pi.greco.

This is the link to see the Table with all the values of the various frequencies concerning the musical system based on the aurea section (Φ) and π.

http://nardelli.xoom.it/virgiliowizard/sites/default/files/sp_wizard/docs/Sistema%20833333%20su%20833333_PiGreco_Terzitoni%2040Hz_0.pdf

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