On some equations concerning the cusp anomalous dimension from a TBA equation and generalized quark-antiquark potential at weak and strong coupling; some equations concerning the complete four-loop four-point amplitude of N = 4 super-Yang-Mills theory. Mathematical connections with some sectors of Number Theory

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Abstract

In the present paper in the **Section 1**, we have described some equations concerning the cusp anomalous dimension in the planar limit of N = 4 super Yang-Mills from a Thermodynamic Bethe Antsaz (TBA) system, the Luscher correction at strong coupling and the strong coupling expansion of the function F. In the **Section 2**, we have described some equations concerning a two-parameter family of Wilson loop operators in N = 4 supersymmetric Yang-Mills theory which interpolates smoothly between the 1/2 BPS line or circle, principally some equations concerning the one-loop determinants. In the **Section 3**, we have described some results and equations of the mathematician Ramanujan concerning some definite integrals and an infinite product and some equations concerning the development of derivatives of order n (n positive integer) of various trigonometric functions and divergent series. Thence, we have described some mathematical connections between some equations concerning this Section and the Sections 1 and 2. In the **Section 4**, we have described some equations concerning the relationship between Yang-Mills theory and gravity and, consequently, the complete four-loop four-point amplitude of N = 4 super-Yang-Mills theory including the nonplanar contributions regarding the gauge theory and the gravity amplitudes. In conclusion, in the **Appendix A** and **B**, we have described a new possible method of factorization of a number and various mathematical connections with some sectors of Number Theory (Fibonacci's numbers, Lie's numbers, triangular numbers, Phi, Pigreco, etc...).

1. On some equations concerning the cusp anomalous dimension in the planar limit of N = 4 super Yang-Mills from a Thermodynamic Bethe Antsaz (TBA) system, the Luscher correction at strong coupling and the strong coupling expansion of the function F

The bulk excitations are in a fundamental representation of each of the two $s\tilde{u}(2|2)$ factors of the $s\tilde{u}(2|2)^2$ symmetry of the Z-vacuum. We can think of them as particles with two indices $\Psi_{A,\dot{B}}$, where A labels the fundamental of the first $s\tilde{u}(2|2)$ and $\dot{B}$ labels the fundamental of the second $s\tilde{u}(2|2)$ factor of the $s\tilde{u}(2|2)^2$ symmetry of the infinite chain. This central extension determines the dispersion relation for the excitations

$$\frac{i}{g} = x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} , \quad (1.1)$$

$$e^{ip} = \frac{x^+}{x^-}, \qquad \varepsilon = ig\left(\frac{1}{x^+} - \frac{1}{x^-} - x^+ + x^-\right) = \sqrt{1 + 16 g^2 \sin^2 \frac{p}{2}} . \quad (1.2)$$

We define g as

$$g \equiv \frac{\sqrt{\lambda}}{4\pi} = \frac{\sqrt{g_{YM}^2 N}}{4\pi} . \quad (1.3)$$

We consider an open string ending on a D5 brane that wraps $AdS_4 \times S^2$, or $AdS_2 \times S^4$. There is a whole family of BPS branes of this kind that arises by adding flux for the $U(1)$ gauge field on the brane worldvolume on the S^2 or AdS_2. In fact, in the limit of large electric flux on the $AdS_2 \times S^4$ brane we get a boundary condition like the Wilson loop one. In fact the $AdS_2 \times S^4$ branes can be interpreted as Wilson loops in the k-fold antisymmetric representation of $U(N)$. In all these cases one can choose the BMN vacuum in such a way that we preserve the $\tilde{s}\tilde{u}(2|2)_D$ of the spin chain. Therefore, we would get the same matrix structure for the reflection matrix, again assuming that there are no boundary degrees of freedom.
We have the crossing equation

$$R_0(p) R_0(\bar{p}) = \sigma(p, -\bar{p})^2 , \quad (1.4)$$

where the bar indicates the action of the crossing transformation. Here $\sigma(p_1, p_2)$ is the bulk dressing phase. We are going to $\bar{p}$ along the same contour in momentum space that we choose in the formulation of the bulk crossing equation. In addition, we also should impose the unitarity condition

$$R_0(p) R_0(-p) = 1 . \quad (1.5)$$

We now write the ansatz

$$R_0(p) = \frac{1}{\sigma_B(p)\sigma(p,-p)} \left[\frac{1 + \frac{1}{(x^-)^2}}{1 + \frac{1}{(x^+)^2}} \right] . \quad (1.6)$$

Here σ is the bulk dressing phase. This would be our naïve choice for a phase factor. The explicit factors of $x^{\pm}$ have been chosen only to simplify the final formula. We have an unknown factor $\sigma_B(p)$. Now (1.4) becomes

$$\sigma_B(p)\sigma_B(\bar{p}) = \frac{x^- + \frac{1}{x^-}}{x^+ + \frac{1}{x^+}}. \quad (1.7)$$

We can now solve this equation and obtain:

$$\sigma_B = e^{i\chi(x^+) - i\chi(x^-)}, \quad (1.8) \qquad i\chi(x) = i\Phi(x) = \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{x-z} \log\left\{ \frac{\sinh\left[2\pi g\left(z + \frac{1}{z}\right)\right]}{2\pi g\left(z + \frac{1}{z}\right)} \right\}, \quad |x| > 1 \quad (1.9)$$

This expression is valid when $|x| > 1$. The value for χ in other regions is given by analytic continuation. We have also introduced the function $\Phi(x)$ which is given by the integral for all values of x. When $|x| < 1$ these two functions differ by

$$i\chi(x) = i\Phi(x) + \log\left\{ \frac{\sinh\left[2\pi g\left(x + \frac{1}{x}\right)\right]}{2\pi g\left(x + \frac{1}{x}\right)} \right\}, \qquad |x| < 1 \quad (1.10)$$

The ambiguities in the choice of branch cuts for the logarithm cancel out when we compute σ_B in (1.8). Note that $\chi(x) = \chi(-x)$. This is a particular solution of the boundary crossing equation. Instead of (1.2) we define $q = i\varepsilon$ and $E_m = ip$, and use the same formulas as in (1.2). Here q is the mirror momentum and E_m is the mirror energy. In order for these to be real we will need to pick a solution of (1.1) with $|x^+| > 1$ and $|x^-| < 1$. From the expression for q, we can write

$$z^{[\pm a]} = \frac{1}{4g}\left(\sqrt{1 + \frac{16g^2}{a^2 + q^2}} \pm 1\right)(q + ia), \quad (1.11) \qquad E_m = 2\,\mathrm{arcsin}\,h\frac{\sqrt{q^2 + a^2}}{4g}. \quad (1.12)$$

Here $z^{\pm}$ just denote the values of $x^{\pm}$ in the mirror region.
When we have a boundary, this time/space flip turns the boundary into a boundary state. Then a suitable analytic continuation of the boundary reflection matrix characterizes the boundary state. The boundary state creates a superposition of many particles. The total mirror momentum should be zero since it is translational invariant. So, schematically the state has the form

$$|B\rangle = |0\rangle + \int_0^{\infty} \frac{dq}{2\pi} K^{A\dot{A},B\dot{B}}(q) a^{\dagger}_{-qA\dot{A}} a^{\dagger}_{qB\dot{B}} |0\rangle + \ldots \quad (1.13)$$

with

$$K^{A\dot{A},B\dot{B}}(q) = \left[R^{-1}(z^+, z^-)\right]^{A\dot{A}}_{D\dot{D}} \mathcal{C}^{D\dot{D},B\dot{B}}. \quad (1.14)$$

The formula (1.14) can be obtained by performing a $\pi/2$ rotation of the boundary condition. Due to the independence of reflection events from a boundary, we can exponentiate (1.14) to get the full boundary state. Similarly, we can form a future boundary state. This is a boundary state that annihilates the particles. It is given by

$$\langle B| = \langle 0| + \langle 0| \int_0^\infty \frac{dq}{2\pi} \overline{K}_{A\dot{A},B\dot{B}}(q) a_q^{A\dot{A}} a_{-q}^{B\dot{B}} + \ldots \quad (1.15)$$

with

$$\overline{K}_{A\dot{A},B\dot{B}}(q) = \left[R^{-1}\left(-\frac{1}{z^-}, -\frac{1}{z^+} \right) \right]_{B\dot{B}}^{D\dot{D}} \mathcal{C}_{D\dot{D}A\dot{A}} . \quad (1.16)$$

Thence, we can rewrite the eq. (1.15) also as follows:

$$\langle B| = \langle 0| + \langle 0| \int_0^\infty \frac{dq}{2\pi} \left[R^{-1}\left(-\frac{1}{z^-}, -\frac{1}{z^+} \right) \right]_{B\dot{B}}^{D\dot{D}} \mathcal{C}_{D\dot{D}A\dot{A}} a_q^{A\dot{A}} a_{-q}^{B\dot{B}} + \ldots \quad (1.16b)$$

In the relativistic case (1.16) would be $\overline{K}(\theta) = \dfrac{1}{R\left(-i\frac{\pi}{2} - \theta \right)}$.

When L is very large the leading L-dependent contribution comes from the exchange of this pair of particles and we can write the corresponding contribution as

$$\delta\xi = -\int_0^\infty \frac{dq}{2\pi} e^{-2LE_m(q)} t(q), \qquad t(q) = Tr\left[K(q) \overline{K}(q) \right], \quad (1.17)$$

thence,

$$\delta\xi = -\int_0^\infty \frac{dq}{2\pi} e^{-2LE_m(q)}\, Tr\left[K(q) \overline{K}(q) \right]. \quad (1.17b)$$

This formula is correct whenever the integral is finite.
In our case, the phase factor σ_B has a pole at $q = 0$. The physical interpretation of this pole at $q = 0$ is that the boundary state is sourcing single particles states in the mirror theory. Obviously such source has to contain only zero momentum particles.
A careful analysis leads to the formula

$$\xi \approx -\int_0^\infty \frac{dq}{2\pi} \log\left\{ 1 + e^{-2LE_m(q)} Tr\left[K(q) \overline{K}(q) \right] \right\} \approx -\frac{1}{2} e^{-LE_m(0)} \sqrt{q^2 Tr\left[K(q) \overline{K}(q) \right]_{q=0}} . \quad (1.18)$$

In the last equality we extracted the leading term in the integral, which comes only from the coefficient of the pole. Notice that the L dependence is precisely what we expect from the exchange of a single particle.
To write down the full Luscher formula we need to compute $t(q)$ also for the bound states of the mirror theory. We find the following formula:

$$\Delta\xi \approx -\frac{(\cos\phi-\cos\theta)}{\sin\phi}\sum_{a=1}^{\infty}(-1)^a\left(\frac{-1+\sqrt{1+16g^2/a^2}}{1+\sqrt{1+16g^2/a^2}}\right)^{1+L}\sin(a\phi)\frac{a}{\sqrt{1+16g^2/a^2}}F(a,g). \quad (1.19)$$

From the following relation

$$\xi = \frac{\phi^2}{2}\sum_{a=1}^{\infty}\frac{C_a}{\sqrt{1+16g^2/a^2}}, \quad (1.20)$$

we obtain the expression for energy up to 3-loop order

$$\xi = -\phi^2\left[g^2 - g^4\frac{2\pi^2}{3} + g^6\frac{2\pi^4}{3} + O(g^8)\right] = -\phi^2\left[\frac{\lambda}{16\pi^2} - \frac{\lambda^2}{384\pi^2} + \frac{\lambda^3}{6144\pi^2} + \mathcal{O}(\lambda^4)\right], \quad (1.21)$$

in perfect agreement with the expansion of the following expression:

$$B = \frac{1}{4\pi^4}\frac{\sqrt{\tilde{\lambda}}\,I_2\left(\sqrt{\tilde{\lambda}}\right)}{I_1\left(\sqrt{\tilde{\lambda}}\right)} + \mathcal{O}(1/N^2). \quad (1.22)$$

Now we consider an open string operator of the form $B_l Z^L B_r(\theta,\phi)$ and compute the leading correction to the energy for large L, this correction goes as $e^{-(\text{constant})L}$. We will compute the correction for $\phi = 0, \theta \neq 0$ at leading order in the strong coupling expansion. In this case we have a string that moves on $AdS_2 \times S^3$. We fix the solution on the AdS_2 part. This AdS_2 solution is completely characterized by the extent of the spatial worldsheet coordinate σ, which we take to run between $[-s/2, s/2]$. In particular the spacetime energy Δ of the solution is fixed, once s is fixed. As we vary the parameters we will see that L will change, θ will change, and so will $\Delta - L$.
So we now concentrate on the solution on the S^3, which we parametrize as

$$x_1 + ix_2 = e^{i\gamma\tau}\sqrt{1-\rho^2(\sigma)}, \qquad x_3 + ix_4 = \rho(\sigma)e^{i\varphi(\sigma)}. \quad (1.23)$$

Inserting this in the Euler Lagrange equations for the string and imposing the Virasoro constraints, $T_\pm = 1$ one finds two integrals of motion, ℓ and γ. They are given by

$$\ell = \rho^2\varphi' \quad (1.24)$$

and

$$\frac{\rho^2(\rho')^2}{1-\rho^2} = -\ell^2 - (\gamma^2-1)\rho^2 + \gamma^2\rho^4. \quad (1.25)$$

The boundary conditions are $\rho'(0)=0$, $\rho(s/2)=1$. Let us define ρ_0 to be the value of ρ at $\sigma = 0$ where the derivative vanishes. It is a root of

$$0 = -\ell^2 - (\gamma^2-1)\rho_0^2 + \gamma^2\rho_0^4. \quad (1.26)$$

By using (1.25) we can write the following expressions

$$\frac{s}{2}=\int_{\rho_0}^{1} d\rho \frac{\rho}{\sqrt{1-\rho^2}\sqrt{D}}\,, \quad (1.27) \qquad \frac{\theta}{2}=\int_{\rho_0}^{1} d\rho \frac{\ell}{\rho\sqrt{1-\rho^2}\sqrt{D}}\,, \quad (1.28)$$

$$\frac{L}{2}=2g\int_0^{s/2} d\sigma\gamma|x_1+ix_2|^2 = 2g\gamma\int_{\rho_0}^{1} d\rho \frac{\rho\sqrt{1-\rho^2}}{\sqrt{D}}\,, \quad (1.29)$$

$$D=-\ell^2-(\gamma^2-1)\rho^2+\gamma^2\rho^4=(\rho^2-\rho_0^2)[\gamma^2(\rho^2+\rho_0^2)-(\gamma^2-1)]. \quad (1.30)$$

This happens when $\rho_0 \to 0$ and $\gamma \to 1$ and $\ell \to 0$. We need to scale them as

$$\gamma = 1+\varepsilon/2\,, \qquad \ell=\varepsilon\frac{\hat{\ell}}{2}\,, \qquad \rho=\sqrt{\varepsilon}v\,, \quad (1.31)$$

where v is a new rescaled variable and $\hat{\ell}$ is fixed as $\varepsilon \to 0$. Now, to leading order in ε we find that (1.26) becomes

$$0=-\frac{\hat{\ell}^2}{4}-v_0^2+v_0^4\,, \qquad \text{or} \qquad v_0^2=\frac{1+\sqrt{1+\hat{\ell}^2}}{2}\,. \quad (1.32)$$

The integral for θ, (1.28), becomes negligibly small away from $\rho \approx \rho_0$ since there a factor of ℓ multiplying. So it receives all its contribution from the small ρ region, namely the finite v region, see (1.31). We can write

$$\frac{\theta}{2}=\frac{\hat{\ell}}{2}\int_{v_0}^{\infty}\frac{1}{v\sqrt{\tilde{D}}}\,, \qquad \Rightarrow \qquad \hat{\ell}=\tan\theta \quad (1.33) \qquad \tilde{D}=(v^2-v_0^2)(v^2+v_0^2-1). \quad (1.34)$$

We can similarly compute the integral for s,

$$\frac{s}{2}=\int_{\rho_0}^{1}\frac{\rho}{\sqrt{1-\rho^2}}\left(\frac{1}{\sqrt{D}}-\frac{1}{\rho^2}\right)+\int_{\rho_0}^{1}\frac{1}{\rho\sqrt{1-\rho^2}} \quad (1.35)$$

$$=\int_{v_0}^{\infty} v\left(\frac{1}{\sqrt{\tilde{D}}}-\frac{1}{v^2}\right)+\log 2-\log\rho_0 \quad (1.36)$$

$$\frac{s}{2}=\log 4-\frac{1}{2}\log\left(\frac{\varepsilon}{\cos\theta}\right), \qquad \Rightarrow \quad \frac{\varepsilon}{\cos\theta}=16e^{-s}\,, \quad (1.37)$$

where we used $\rho_0=\sqrt{\varepsilon}v_0$ and the result (1.33), and the definition of $\tilde{D}$ in (1.34). Furthermore, we can rewrite the eqs (1.35-1.37) also as follows:

$$\frac{s}{2}=\int_{\rho_0}^{1}\frac{\rho}{\sqrt{1-\rho^2}}\left(\frac{1}{\sqrt{D}}-\frac{1}{\rho^2}\right)+\int_{\rho_0}^{1}\frac{1}{\rho\sqrt{1-\rho^2}}=\int_{v_0}^{\infty} v\left(\frac{1}{\sqrt{\tilde{D}}}-\frac{1}{v^2}\right)+\log 2-\log\rho_0=$$

$$=\log 4-\frac{1}{2}\log\left(\frac{\varepsilon}{\cos\theta}\right)=\log 4-\frac{1}{2}\log\left(16e^{-s}\right). \quad (1.37\text{b})$$

Here we have split the integral in two terms, the first receives contributions only from the small ρ region and the second, which can be done explicitly with no need to take the small ρ_0 limit. We now want to compute L. We will compute instead

$$\frac{L}{4g}-\frac{s}{2}=\frac{\varepsilon}{2}\int_{\rho_0}^{1}d\rho\rho\sqrt{1-\rho^2}\left(\frac{1}{\sqrt{D}}-\frac{1}{\rho^2}\right)-\int_{\rho_0}^{1}d\rho\frac{\rho^3}{\sqrt{1-\rho^2}}\left(\frac{1}{\sqrt{D}}-\frac{1}{\rho^2}+\varepsilon\frac{(1-\rho^2)}{2\rho^4}\right)-\int_{\rho_0}^{1}d\rho\frac{\rho}{\sqrt{1-\rho^2}} \tag{1.38}$$

$$\frac{L}{4g}-\frac{s}{2}=\frac{\varepsilon}{2}\int_{v_0}^{\infty}dvv\left(\frac{1}{\sqrt{\tilde{D}}}-\frac{1}{v^2}\right)-\varepsilon\int_{v_0}^{\infty}dvv^3\left(\frac{1}{\sqrt{\tilde{D}}}-\frac{1}{v^2}+\frac{1}{2v^4}\right)-1+\frac{\rho_0^2}{2} \tag{1.39}$$

$$\frac{L}{4g}-\frac{s}{2}=\varepsilon\left(\frac{1}{4}-\frac{v_0^2}{2}\right)-1+\varepsilon\frac{v_0^2}{2}=-1+\frac{\varepsilon}{4}=-1+\cos\theta 4e^{-s} \tag{1.40}$$

$$L-2gs=-4g+16g\cos\theta e^{-\frac{L}{2g}-2} \tag{1.41}$$

$$\frac{L}{4g}-\frac{s}{2}=\frac{\varepsilon}{2}\int_{\rho_0}^{1}d\rho\rho\sqrt{1-\rho^2}\left(\frac{1}{\sqrt{D}}-\frac{1}{\rho^2}\right)-\int_{\rho_0}^{1}d\rho\frac{\rho^3}{\sqrt{1-\rho^2}}\left(\frac{1}{\sqrt{D}}-\frac{1}{\rho^2}+\varepsilon\frac{(1-\rho^2)}{2\rho^4}\right)-\int_{\rho_0}^{1}d\rho\frac{\rho}{\sqrt{1-\rho^2}}=$$
$$=\frac{\varepsilon}{2}\int_{v_0}^{\infty}dvv\left(\frac{1}{\sqrt{\tilde{D}}}-\frac{1}{v^2}\right)-\varepsilon\int_{v_0}^{\infty}dvv^3\left(\frac{1}{\sqrt{\tilde{D}}}-\frac{1}{v^2}+\frac{1}{2v^4}\right)-1+\frac{\rho_0^2}{2} \tag{1.42}$$

$$\frac{L}{4g}-\frac{s}{2}=\frac{\varepsilon}{2}\int_{\rho_0}^{1}d\rho\rho\sqrt{1-\rho^2}\left(\frac{1}{\sqrt{D}}-\frac{1}{\rho^2}\right)-\int_{\rho_0}^{1}d\rho\frac{\rho^3}{\sqrt{1-\rho^2}}\left(\frac{1}{\sqrt{D}}-\frac{1}{\rho^2}+\varepsilon\frac{(1-\rho^2)}{2\rho^4}\right)-\int_{\rho_0}^{1}d\rho\frac{\rho}{\sqrt{1-\rho^2}}=$$
$$=\frac{\varepsilon}{2}\int_{v_0}^{\infty}dvv\left(\frac{1}{\sqrt{\tilde{D}}}-\frac{1}{v^2}\right)-\varepsilon\int_{v_0}^{\infty}dvv^3\left(\frac{1}{\sqrt{\tilde{D}}}-\frac{1}{v^2}+\frac{1}{2v^4}\right)-1+\frac{\rho_0^2}{2}=$$
$$=\varepsilon\left(\frac{1}{4}-\frac{v_0^2}{2}\right)-1+\varepsilon\frac{v_0^2}{2}=-1+\frac{\varepsilon}{4}=-1+\cos\theta 4e^{-s}. \tag{1.43}$$

We know that for $\theta=0$ the result should vanish due to the BPS condition. Thus we find that

$$\Delta-L=g(1-\cos\theta)\frac{16}{e^2}e^{-\frac{L}{2g}}. \tag{1.44}$$

If we changed the angle in the AdS part, then instead of 1 in (1.44) we would get some function of ϕ. However, since we know that for $\theta=\phi$ we should get zero due to the BPS condition, we conclude that for generic angles we get

$$\Delta-L=g(\cos\theta-\cos\theta)\frac{16}{e^2}e^{-\frac{L}{2g}}=0. \tag{1.45}$$

In order to compare this to the expected answer from the Luscher type correction we need to evaluate the function F in the following expression

$$F(a,g)^2 \equiv e^{i\left(\Phi\left(z^{[+a]}\right)-\Phi\left(z^{[-a]}\right)+\Phi\left(1/z^{[-a]}\right)-\Phi\left(1/z^{[+a]}\right)\right)}\Big|_{q=0},$$

at strong coupling. This involves evaluating the function Φ in (1.8), (1.10) at $z^{[\pm a]}$ at $q=0$. When $q=0$, we have that

$$-1/z^{[-a]}(0) = z^{[+a]}(0) = i\left(\sqrt{1+a^2/\left(16g^2\right)} - a/(4g)\right) = i\left(1+\frac{a}{4g}+\dots\right) \quad (1.46)$$

which is very close to i, where the strong coupling expansion is tricky. We need to compute

$$\log F = i\Phi(y) - i\Phi(1/y) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} dt \frac{\left(y^4-1\right)}{\left(1+y^2\right)^2 - 4y^2\sin^2 t}\log\left[\frac{\sinh 4\pi g\sin t}{4\pi g\sin t}\right] \quad (1.47)$$

with $y = z^{[a]}(0)$. Then the y dependent factor can be well approximated by

$$\frac{\left(y^4-1\right)}{\left(1+y^2\right)^2 - 4y^2\sin^2 t}\Big|_{y=x^{[a]}(0)} \approx \frac{a}{4g}\left[\frac{1}{\sin^2 t + \frac{a^2}{16g^2}}\right]. \quad (1.48)$$

We now insert this into the integral (1.47), and split the integral into two pieces

$$\log F = r_1 + r_2 \quad (1.49)$$

$$r_1 = \frac{a}{2\pi g}\int_0^{\frac{\pi}{2}} dt\left[\frac{1}{\sin^2 t + \frac{a^2}{16g^2}}\right] 4\pi g\sin t = 2a\log\left[\frac{8g}{a}\right] + o(1/g) \quad (1.50)$$

$$r_2 = \frac{a}{2\pi g}\int_0^{\frac{\pi}{2}} dt\left[\frac{1}{\sin^2 t + \frac{a^2}{16g^2}}\right]\log\left[\frac{1-e^{-8\pi g\sin t}}{8\pi g\sin t}\right] = \quad (1.51)$$

$$r_2 = \int_0^{\infty} dt\frac{4a}{v^2+4a^2\pi^2}\log\left[\frac{1-e^{-v}}{v}\right] = 2\left[a\log a - a - \log\Gamma(a+1)\right] \quad (1.52)$$

where we have defined $t = v/(8\pi g)$ in the integral for r_2 and taken the $g\to\infty$ limit. Furthermore, we note that:

$$\log F = i\Phi(y) - i\Phi(1/y) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} dt \frac{(y^4-1)}{(1+y^2)^2 - 4y^2\sin^2 t}\log\left[\frac{\sinh 4\pi g\sin t}{4\pi g\sin t}\right] =$$

$$= \frac{a}{2\pi g}\int_0^{\frac{\pi}{2}} dt\left[\frac{1}{\sin^2 t + \frac{a^2}{16g^2}}\right]4\pi g\sin t + \int_0^{\infty} dt \frac{4a}{v^2+4a^2\pi^2}\log\left[\frac{1-e^{-v}}{v}\right] = 2a\log\left[\frac{8g}{a}\right] + o(1/g) +$$

$$+ 2\left[a\log a - a - \log\Gamma(a+1)\right]. \quad (1.52b)$$

Summarizing, we get that the leading strong coupling approximation is

$$F(a,g) = \frac{2^{6a} g^{2a}}{e^{2a}(a!)^2}. \quad (1.53)$$

We have contributions from the explicit functions in the following equation

$$C_a = (-1)^a a^2 F(a,g)\frac{z_0^{[-a]}}{z_0^{[+a]}} e^{\Delta_{conv}}, \quad (1.54)$$

which can be expanded to any order independently of the Y-system solution. These give

$$a^4\left[\frac{z_0^{[-a]}}{z_0^{[+a]}}\right]^2 F(a,g)^2 = a^4\left(\frac{a-\sqrt{a^2+16g^2}}{a+\sqrt{a^2+16g^2}}\right)^2 e^{i\left(\Phi\left(z_0^{[+a]}\right)-\Phi\left(z_0^{[-a]}\right)+\Phi\left(1/z_0^{[-a]}\right)-\Phi\left(1/z_0^{[+a]}\right)\right)} =$$

$$= 16g^4\left[1 + \left(\pi^2 - \frac{6}{a^2}\right)\frac{8g^2}{3} + \left(7\pi^4 - \frac{150\pi^2}{a^2} + \frac{630}{a^4}\right)\frac{16g^4}{45} + O(g^6)\right]. \quad (1.55)$$

We can rewrite the fermionic convolutions in the TBA equation

$$\Delta_{conv} = \left\{\mathcal{R}_{a1}^{10}\,\hat{*}\log\left(\frac{\Psi}{1/2}\right) - \mathcal{B}_{a1}^{(10)}\,\hat{*}\log\left(\frac{\Phi}{1/2}\right) + \left[\mathcal{R}_{ab}^{(10)} + \mathcal{B}_{a,b-2}^{(10)}\right]*\log\left(\frac{1+\mathcal{Y}_b}{1+\frac{1}{b^2-1}}\right)\right\}\Big|_{u=0} \quad (1.56)$$

as

$$2\mathcal{R}_{a1}^{(10)}\,\hat{*}\log\Psi - 2\mathcal{B}_{a1}^{(10)}\,\hat{*}\log\Phi = K_a\,\hat{*}\log\frac{\Psi}{\Phi} + K_{a,y}\,\hat{*}\log\Psi\Phi. \quad (1.57)$$

It is important to recall that we need only the $u \to 0$ limit of this. For the order g^4 of the first term in (1.57) we get

$$\frac{16\pi^3 g^4}{3}K_a(0) = \frac{32\pi^2 g^4}{3a}. \quad (1.58)$$

The second term in (1.57) is

$$K_{a,y}\hat{*}\log\Psi\Phi=\int_{-2g}^{2g}dvK_{a,y}(0,v)\Big[\log(-4)+2g^2\left(\Psi^{(1)}(v)+\Phi^{(1)}(v)\right)-\left(\Psi^{(1)}(v)+\Phi^{(1)}(v)\right)^2g^4+$$
$$+2\left(\Psi^{(2)}(v)+\Phi^{(2)}(v)\right)g^4+\ldots\Big],\quad (1.59)$$

where its g^4 order is

$$-\frac{56\pi^4g^4}{45}\int_{-1}^{1}d\tilde{v}\,\frac{2\tilde{v}^2}{\pi\sqrt{1-\tilde{v}^2}}-\frac{9\pi^4g^4}{4}+2\left(\frac{4\pi^2}{3}+\frac{32\pi^4}{45}\right)g^4=-\frac{12\pi^4g^4}{45}+\frac{8\pi^2g^4}{3}.\quad (1.60)$$

Thus, the total g^4 order of (1.57) is

$$\frac{32\pi^2g^4}{3a}+\frac{8\pi^2g^4}{3}-\frac{12\pi^4g^4}{45}.\quad (1.61)$$

For the remaining convolution in (1.56), $2\left[\mathcal{R}_{ab}^{(10)}+\mathcal{B}_{a,b-2}^{(10)}\right]*\log(1+\mathcal{Y}_b)$ we use

$$\mathcal{R}_{ab}^{(10)}(0,v)+\mathcal{B}_{a,b-2}^{(10)}(0,v)=K_{b-1}(v)+\sum_{j=-\frac{a-1}{2}}^{\frac{a-3}{2}}K_{b+2j}+\mathrm{O}(g^2).\quad (1.62)$$

We note that the eq. (1.57) can be rewritten also as follows:

$$2\mathcal{R}_{a1}^{(10)}\hat{*}\log\Psi-2\mathcal{B}_{a1}^{(10)}\hat{*}\log\Phi=K_a\hat{*}\log\frac{\Psi}{\Phi}+$$
$$+\int_{-2g}^{2g}dvK_{a,y}(0,v)\Big[\log(-4)+2g^2\left(\Psi^{(1)}(v)+\Phi^{(1)}(v)\right)-\left(\Psi^{(1)}(v)+\Phi^{(1)}(v)\right)^2g^4+2\left(\Psi^{(2)}(v)+\Phi^{(2)}(v)\right)g^4+\ldots\Big],$$
$$(1.62b)$$

We go to Fourier space, where the term order g^4 is

$$2\left[\tilde{K}_{b-1}+\sum_{j=-\frac{a-1}{2}}^{\frac{a-3}{2}}\tilde{K}_{b+2j}\right]\frac{\tilde{\mathcal{Y}}_b^{(2)}}{b^2}=-\frac{8\pi^3}{3}e^{-|w|}-\frac{16\pi^3}{3}\frac{a-1}{a}e^{-\frac{a|w|}{2}}.\quad (1.63)$$

We Fourier transform back and evaluate for $u\to 0$ and we get

$$2\left[K_{b-1}+\sum_{j=-\frac{a-1}{2}}^{\frac{a-3}{2}}K_{b+2j}\right]*\left.\frac{\mathcal{Y}_b^{(2)}}{b^2}\right|_{u=0}=\frac{32\pi^2}{3a^2}-\frac{32\pi^2}{3a}-\frac{8\pi^2}{3}.\quad (1.64)$$

Thus, for Δ_{conv} up the 3-loop order we have

$$\Delta_{conv} = \frac{2\pi^2 g^2}{3} + \left[-\frac{6\pi^4}{45} + \frac{16\pi^2}{3a^2} \right] g^4 + O\left(g^6\right). \quad (1.65)$$

Which, together with (1.55), gives rise to

$$C_a = 4(-1)^a g^2 + 8(-1)^a \left[\pi^2 - \frac{4}{a^2} \right] g^4 + 16(-1)^a \left[\frac{\pi^4}{3} - \frac{4\pi^2}{a^2} + \frac{20}{a^4} \right] g^6 + O\left(g^8\right). \quad (1.66)$$

2. On some equations concerning a two-parameter family of Wilson loop operators in N = 4 supersymmetric Yang-Mills theory which interpolates smoothly between the 1/2 BPS line or circle, principally some equations concerning the one-loop determinants

With regard the calculation of the 2-loop graphs for the Wilson loop with a cusp in the case of non-zero θ, the resulting expression can be written as a sum of the contribution of ladder graphs and the interacting graphs

$$V^{(2)}(\phi,\theta) = V^{(2)}_{lad}(\phi,\theta) + V^{(2)}_{\text{int}}(\phi,\theta)$$

$$V^{(2)}_{lad}(\phi,\theta) = -\frac{(\cos\theta - \cos\phi)^2}{\sin^2\phi} \int_0^\infty \frac{dz}{z} \log\left(\frac{1+ze^{i\phi}}{1+ze^{-i\phi}} \right) \log\left(\frac{z+e^{i\phi}}{z+e^{-i\phi}} \right)$$

$$V^{(2)}_{\text{int}}(\phi,\theta) = 4(\cos\theta - \cos\phi) \int_0^1 dz Y\left(z^2, z^2 + 2z\cos\phi + 1, 1\right). \quad (2.1)$$

The integrand in the last expression is the "scalar triangle graph" – the Feynman diagram arising at one-loop order from the cubic interaction between three scalars separated by distances given by the arguments

$$Y\left(x_{12}^2, x_{23}^2, x_{13}^2\right) = \frac{1}{\pi^2} \int d^4w \frac{1}{|x_1 - w|^2 |x_2 - w|^2 |x_3 - w|^2}, \qquad x_{ij}^2 = \left| x_i - x_j \right|^2. \quad (2.2)$$

This integral is known in closed form. For $x_{12}^2, x_{23}^2 < x_{13}^2$ it is equal to

$$Y\left(x_{12}^2, x_{23}^2, x_{13}^2\right) = \frac{1}{x_{13}^2 A} \left[\frac{\pi^2}{3} - 2Li_2\left(\frac{1+s-t-A}{2} \right) - 2Li_2\left(\frac{1-s+t-A}{2} \right) + \right.$$
$$\left. -\ln s \ln t + 2\ln\left(\frac{1+s-t-A}{2} \right) \ln\left(\frac{1-s+t-A}{2} \right) \right]$$
$$s = \frac{x_{12}^2}{x_{13}^2}, \quad t = \frac{x_{23}^2}{x_{13}^2}, \quad A = \sqrt{(1-s-t)^2 - 4st}. \quad (2.3)$$

Thence, we have that:

$$Y\left(x_{12}^2, x_{23}^2, x_{13}^2\right) = \frac{1}{\pi^2} \int d^4w \frac{1}{|x_1 - w|^2 |x_2 - w|^2 |x_3 - w|^2} =$$

$$= \frac{1}{x_{13}^2 A}\left[\frac{\pi^2}{3} - 2Li_2\left(\frac{1+s-t-A}{2}\right) - 2Li_2\left(\frac{1-s+t-A}{2}\right) + \right.$$
$$\left. -\ln s \ln t + 2\ln\left(\frac{1+s-t-A}{2}\right)\ln\left(\frac{1-s+t-A}{2}\right)\right]. \quad (2.3b)$$

This expression is valid for $s,t<1$ with the principle branch of the logarithms and dilogarithms. If x_{12}^2 is the largest, then the result is the same function divided by s and the replacement $s \to 1/s$ and $t \to t/s$. Likewise when x_{23}^3 is the largest. In our case, if we take $\phi > 2\pi/3$, then for $z \le 1$ the first two arguments of Y in (2.1) are less than unity and in that regime Y evaluates to

$$Y = -\frac{i}{z\sin\phi}\left(\frac{\pi^2}{6} - Li_2\left(-ze^{i\phi}\right) - Li_2\left(1+ze^{-i\phi}\right) - \log(z)\log\left(1+ze^{i\phi}\right) + \log\left(-e^{i\phi}\right)\log\left(1+ze^{-i\phi}\right)\right). \quad (2.4)$$

The integration then gives

$$\int_0^1 dzY = \frac{(\pi-\phi)(\pi+\phi)\phi}{3\sin\phi}. \quad (2.5)$$

With the prefactor we find the final expression (valid by analytical continuation for all $0 \le \phi < \pi$) is

$$V_{\text{int}}^{(2)}(\phi,\theta) = \frac{4}{3}\frac{\cos\theta - \cos\phi}{\sin\phi}(\pi-\phi)(\pi+\phi)\phi. \quad (2.6)$$

The first integral in (2.1) can also be done analytically. Again one should take care in choosing branch cuts for the logarithms, where the principle branch is for small ϕ. The result is

$$V_{lad}^{(2)}(\phi,\theta) = -4\frac{(\cos\theta-\cos\phi)^2}{\sin^2\phi}\left[Li_3\left(e^{2i\phi}\right) - \zeta(3) - i\phi\left(Li_2\left(e^{2i\phi}\right) + \frac{\pi^2}{6}\right) + \frac{i}{3}\phi^3\right]. \quad (2.7)$$

Thence, form (2.1), we have the following expression:

$$V_{lad}^{(2)}(\phi,\theta) = -\frac{(\cos\theta-\cos\phi)^2}{\sin^2\phi}\int_0^\infty \frac{dz}{z}\log\left(\frac{1+ze^{i\phi}}{1+ze^{-i\phi}}\right)\log\left(\frac{z+e^{i\phi}}{z+e^{-i\phi}}\right) =$$
$$= -4\frac{(\cos\theta-\cos\phi)^2}{\sin^2\phi}\left[Li_3\left(e^{2i\phi}\right) - \zeta(3) - i\phi\left(Li_2\left(e^{2i\phi}\right) + \frac{\pi^2}{6}\right) + \frac{i}{3}\phi^3\right]. \quad (2.7b)$$

The finite expression for the regularized 1-loop effective action is:

$$\Gamma_{reg} = -\frac{\mathcal{T}}{2}\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ln\frac{\varepsilon^2\omega^2 \det^8 \mathrm{O}_F^\varepsilon}{\det^5 \mathrm{O}_0^\varepsilon \det^2 \mathrm{O}_1^\varepsilon \det \mathrm{O}_2^\varepsilon}. \quad (2.8)$$

The small ϕ expansion is realized sending $k \to 0$ (equivalently $p \to \infty$) in the expressions or the following determinants

$$\det \mathrm{O}_0^{\varepsilon} \cong \frac{\sinh(2K\omega)}{\omega}, \quad (2.9)$$

$$\det \mathrm{O}_F^{\varepsilon} \cong \frac{8K_2\sqrt{\omega_3^2+k_2^2}\,\sinh(K_2 Z(\alpha_F))}{\varepsilon\pi(1-k^2)(k_1+1)^2\sqrt{(\omega_3^2+1)(\omega_3^2+k_2^2+1)}}\,\frac{\vartheta_2(0,q_2)\vartheta_4\left(\frac{\pi\alpha_F}{2K_2},q_2\right)}{\vartheta'_1(0,q_2)\vartheta_3\left(\frac{\pi\alpha_F}{2K_2},q_2\right)}. \quad (2.10)$$

An efficient way to proceed, considering for example the determinant for the operator O_1, is to transform as follows

$$\alpha_1 = \beta_1 + K_1 + iK'_1 \quad (2.11)$$

which allows to identify the imaginary part of the argument of the hyperbolic function

$$2K_1 Z(\alpha_1) = 2K_1 Z(\beta_1) - 2K_1\frac{sn(\beta_1|k_1^2)dn(\beta_1|k_1^2)}{cn(\beta_1|k_1^2)} - i\pi. \quad (2.12)$$

In applying this approach to the fermionic determinant, one notices that a shift analogue to (2.11) changes the $\sinh$ in $\cosh$. One can then first compute the $k\to 0$ expansion of $\partial Z(\alpha_i|k^2)/\partial\omega$ where the dependence of Z on ω is via α, and then perform an indefinite integration over ω. From examining the expansion of the determinants at small k we find the form

$$\det \mathrm{O}_i = \sum_{l=0}^{\infty} D_i^{(l)} k^{2l}, \qquad i = 0,1,2,F, \quad (2.13)$$

where each $D_i^{(l)}$ is a rational function in ω times $\sinh(\pi\omega)$ and $\cosh(\pi\omega)$.
The zero-th order contribution to the regularized effective action (2.8) in this limit reads then

$$\frac{\Gamma_{reg}^{(0)}}{\mathcal{J}} = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} d\omega \ln\left[\frac{\left(\frac{4\cosh(\pi\omega)}{4\omega^2+1}\right)^8}{\left(\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}\right)^2\left(\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}\right)\left(\frac{\sinh(\pi\omega)}{\omega}\right)^5}\right] =$$

$$= -\frac{1}{4\pi}\int_{-\infty}^{+\infty} d\omega \ln\left[\frac{2^{16}\omega^{10}(\omega^2+1)^3\coth^8(\pi\omega)}{(4\omega^2+1)^8}\right] = 0. \quad (2.14)$$

At order k^2 the result is

$$\frac{\Gamma_{reg}^{(2)}}{\mathcal{J}} = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} d\omega \left[8\frac{\frac{\pi(4\omega^2-3)\omega\sinh(\pi\omega)}{(4\omega^2+1)^2}+\frac{32\omega^2\cosh(\pi\omega)}{(4\omega^2+1)^3}}{\frac{4\cosh(\pi\omega)}{4\omega^2+1}} - 2\frac{\frac{\pi(\omega^2-2)\cosh(\pi\omega)}{4\omega^2(\omega^2+1)}+\frac{(4\omega^2+1)\sinh(\pi\omega)}{2\omega^3(\omega^2+1)^2}}{\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}} + \right.$$

$$-\frac{\frac{2\omega\sinh(\pi\omega)}{(\omega^2+1)^3}+\frac{\pi(\omega^2-3)\cosh(\pi\omega)}{4(\omega^2+1)^2}}{\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}}-5\frac{\frac{\pi}{4}\cosh(\pi\omega)}{\frac{\sinh(\pi\omega)}{\omega}}\Bigg]=\frac{3}{8}. \qquad (2.15)$$

At order k^4 one finds

$$\frac{\Gamma_{reg}^{(4)}}{\mathcal{J}}=-\frac{1}{4\pi}\int_{-\infty}^{+\infty}d\omega\Bigg[\Bigg[8\Bigg(\frac{\frac{3\pi\omega(192\omega^6+272\omega^4-220\omega^2-5)\sinh(\pi\omega)}{16(4\omega^2+1)^4}+\left(\frac{\pi^2(3-4\omega^2)^2\omega^2}{8(4\omega^2+1)^3}+\frac{384\omega^4}{(4\omega^2+1)^5}\right)\cosh(\pi\omega)}{\frac{4\cosh(\pi\omega)}{4\omega^2+1}}+$$

$$-\frac{\left(\frac{\pi(4\omega^2-3)\omega\sinh(\pi\omega)}{(4\omega^2+1)^2}+\frac{32\omega^2\cosh(\pi\omega)}{(4\omega^2+1)^3}\right)^2}{2\left(\frac{4\cosh(\pi\omega)}{4\omega^2+1}\right)^2}\Bigg)-2\Bigg(\frac{\frac{\pi(9\omega^6+37\omega^4-72\omega^2-24)\cosh(\pi\omega)}{64\omega^4(\omega^2+1)}+\left(\frac{28\omega^4+12\omega^2+3}{8\omega^5(\omega^2+1)^3}+\right.}{\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}}$$

$$\frac{\left.+\frac{\pi^2(\omega^2-2)^2}{32\omega^3(\omega^2+1)}\right)\sinh(\pi\omega)}{}-\frac{\left(\frac{\pi(\omega^2-2)\cosh(\pi\omega)}{4\omega^2(\omega^2+1)}+\frac{(4\omega^2+1)\sinh(\pi\omega)}{2\omega^3(\omega^2+1)^2}\right)^2}{2\left(\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}\right)^2}\Bigg)+$$

$$-\Bigg(\frac{\left(\frac{\pi^2(\omega^2-3)^2\omega}{32(\omega^2+1)^3}+\frac{6\omega^3}{(\omega^2+1)^5}\right)\sinh(\pi\omega)+\frac{3\pi(3\omega^6+17\omega^4-55\omega^2-5)\cosh(\pi\omega)}{64(\omega^2+1)^4}}{\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}}+$$

$$-\frac{\left(\frac{2\omega\sinh(\pi\omega)}{(\omega^2+1)^3}+\frac{\pi(\omega^2-3)\cosh(\pi\omega)}{4(\omega^2+1)^2}\right)^2}{2\left(\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}\right)^2}\Bigg)-5\left(\frac{\frac{\pi^2\omega}{32}\sinh(\pi\omega)+\frac{9\pi}{64}\cosh(\pi\omega)}{\frac{\sinh(\pi\omega)}{\omega}}-\frac{\left(\frac{\pi}{4}\cosh(\pi\omega)\right)^2}{2\left(\frac{\sinh(\pi\omega)}{\omega}\right)^2}\right)\Bigg]\Bigg]=\frac{29}{128}-\frac{3\zeta(3)}{16}$$

(2.16)

3. On some results and equations of the mathematician Ramanujan concerning some definite integrals and an infinite product and on some equations concerning the development of derivatives of order n (n positive integer) of various trigonometric functions and divergent series.

On some results and equations of the mathematician Ramanujan concerning some definite integrals and an infinite product

Consider the integral

$$\int_0^\infty \frac{\cos 2mx dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\{1+x^2/(a+2)^2\}...}, \quad (3.1)$$

where m and a are positive.
It can be easily proved that

$$\left\{1-\left(\frac{t}{a}\right)^2\right\}\left\{1-\left(\frac{t}{a+1}\right)^2\right\}\left\{1-\left(\frac{t}{a+2}\right)^2\right\}...\left\{1-\left(\frac{t}{a+n-1}\right)^2\right\} = \frac{\Gamma(a+n-t)\Gamma(a+n+t)\{\Gamma(a)\}^2}{\Gamma(a-t)\Gamma(a+t)\{\Gamma(a+n)\}^2}, \quad (3.2)$$

where n is any positive integer. Hence, by splitting

$$\frac{1}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}...\{1+x^2/(a+n-1)^2\}} \quad (3.3)$$

into partial fractions, we see that it is equal to

$$\frac{2\Gamma(2a)\{\Gamma(a+n)\}^2}{\{\Gamma(a)\}^2\Gamma(n)\Gamma(2a+n)}\left\{\frac{a}{a^2+x^2}-\frac{2a}{1!}\frac{n-1}{n+2a}\frac{a+1}{(a+1)^2+x^2}+\frac{2a(2a+1)}{2!}\frac{(n-1)(n-2)}{(n+2a)(n+2a+1)}\frac{a+2}{(a+2)^2+x^2}-...\right\} \quad (3.4)$$

Multiplying both sides by $\cos 2mx$ and integrating from 0 to ∞ with respect to x, we have

$$\int_0^\infty \frac{\cos 2mx dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}...\{1+x^2/(a+n-1)^2\}} = \frac{\pi\Gamma(2a)\{\Gamma(a+n)\}^2}{\{\Gamma(a)\}^2\Gamma(n)\Gamma(2a+n)}\left\{e^{-2am}-\frac{2a}{1!}\frac{n-1}{n+2a}e^{-2(a+1)m}+...\right\} \quad (3.5)$$

The limit of the right-hand side, as $n \to \infty$, is

$$\frac{\pi\Gamma(2a)}{\{\Gamma(a)\}^2}\left\{e^{-2am}-\frac{2a}{1!}e^{-2(a+1)m}+\frac{2a(2a+1)}{2!}e^{-2(a+2)m}-...\right\} = \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)}\operatorname{sec}h^{2a}m. \quad (3.6)$$

Hence

$$\int_0^\infty \frac{\cos 2mx dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}...} = \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)}\operatorname{sec}h^{2a}m. \ldots\ldots \quad (3.7)$$

Since

$$\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}... = \frac{\{\Gamma(a)\}^2}{\Gamma(a+ix)\Gamma(a-ix)}, \quad (3.8)$$

the formula (3.7) is equivalent to

$$\int_0^{\infty} |\Gamma(a+ix)|^2 \cos 2mx dx = \frac{1}{2}\sqrt{\pi}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\sec h^{2a} m. \ldots\ldots (3.9)$$

Let

$$\int_a^b f(x)F(nx)dx = \psi(n), \quad (3.10) \qquad \text{and} \qquad \int_\alpha^\beta \phi(x)F(nx)dx = \chi(n). \quad (3.11)$$

If we suppose the functions $f, \phi,$ and F to be such that the order of integration is indifferent, we have

$$\int_a^b f(x)\chi(nx)dx = \int_\alpha^\beta dy \int_a^b f(x)\phi(y)F(nxy)dx = \int_\alpha^\beta \phi(y)\chi(ny)dy. \qquad (3.12)$$

We have, for example, the formulae

$$\int_0^{\infty} \frac{\cos 2nx}{\cosh \pi x} dx = \frac{1}{2\cosh n}, \quad (3.13) \qquad \int_0^{\infty} \frac{\cos 2nx dx}{1+2\cosh \frac{2}{3}\pi x} = \frac{\sqrt{3}}{2(1+2\cosh 2n)}, \quad (3.14)$$

$$\int_0^{\infty} e^{-x^2} \cos 2nx dx = \frac{1}{2}\sqrt{\pi} e^{-n^2}. \quad (3.15)$$

By applying the general result (3.12) to the integrals (3.13) and (3.14), we obtain

$$\sqrt{3}\int_0^{\infty} \frac{dx}{\cosh \pi x(1+2\cosh 2nx)} = \int_0^{\infty} \frac{dx}{\cosh nx\left(1+2\cosh \frac{2}{3}\pi x\right)}; \quad (3.16)$$

or, in other words, if $\alpha\beta = \frac{3}{4}\pi^2$, then

$$\sqrt{\alpha}\int_0^{\infty} \frac{dx}{\cosh \alpha x(1+2\cosh \pi x)} = \sqrt{\beta}\int_0^{\infty} \frac{dx}{\cosh \beta x(1+2\cosh \pi x)}. \quad (3.17)$$

In the same way, from (3.14) and (3.15), we obtain

$$\sqrt{\alpha}\int_0^{\infty} \frac{e^{-x^2} dx}{1+2\cosh \alpha x} = \sqrt{\beta}\int_0^{\infty} \frac{e^{-x^2} dx}{1+2\cosh \beta x}, \quad (3.18)$$

with the condition $\alpha\beta = \frac{4}{3}\pi$; and, from (3.13) and (3.15),

$$\sqrt{\alpha}\int_0^{\infty} \frac{e^{-x^2}}{\cosh \alpha x} dx = \sqrt{\beta}\int_0^{\infty} \frac{e^{-x^2}}{\cosh \beta x} dx, \quad (3.19)$$

with the condition $\alpha\beta = \pi$. (Formulae equivalent to (3.18) and (3.19) were given by Hardy).

Suppose now that $a, b,$ and n are positive, and

$$\int_0^\infty \phi(a,x) \begin{matrix}\cos \\ \sin\end{matrix} nx dx = \psi(a,n). \quad (3.20)$$

Then, if the conditions of Fourier's double integral theorem are satisfied, we have

$$\int_0^\infty \psi(b,x) \begin{matrix}\cos \\ \sin\end{matrix} nx dx = \frac{1}{2}\pi\phi(b,n). \quad (3.21)$$

Applying the formula (3.12) to (3.20) and (3.21), we obtain

$$\frac{1}{2}\pi\int_0^\infty \phi(a,x)\phi(b,nx)dx = \int_0^\infty \psi(b,x)\psi(a,nx)dx. \quad (3.22)$$

Thus, when $a = b$, we have the formula

$$\frac{1}{2}\pi\int_0^\infty \phi(x)\phi(nx)dx = \int_0^\infty \psi(x)\psi(nx)dx, \quad (3.23)$$

where

$$\psi(t) = \int_0^\infty \phi(x) \begin{matrix}\cos \\ \sin\end{matrix} tx dx; \quad (3.24)$$

and, in particular, if $n = 1$, then

$$\frac{1}{2}\pi\int_0^\infty \{\phi(x)\}^2 dx = \int_0^\infty \{\psi(x)\}^2 dx. \quad (3.25)$$

If

$$\phi(a,x) = \frac{1}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\ldots} \quad (a > 0), \quad (3.26)$$

then, by (3.7),

$$\psi(a,x) = \frac{1}{2}\sqrt{\pi}\frac{\Gamma\left(a+\frac{1}{2}\right)}{\Gamma(a)}\sec h^{2a}\frac{1}{2}x. \quad (3.27)$$

Hence, by (3.22),

$$\int_0^\infty \phi(a,x)\phi(b,x)dx = \frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)}{2\Gamma(a)\Gamma(b)}\int_0^\infty \sec h^{2a+2b}\frac{1}{2}x dx; \quad (3.28)$$

and so

$$\int_0^\infty \frac{dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\ldots\{1+x^2/b^2\}\{1+x^2/(b+1)^2\}\ldots} = \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{\Gamma(a)\Gamma(b)\Gamma\left(a+b+\frac{1}{2}\right)}, \tag{3.29}$$

a and b being positive: or

$$\int_0^\infty |\Gamma(a+ix)\Gamma(b+ix)|^2\,dx = \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{\Gamma\left(a+b+\frac{1}{2}\right)}. \tag{3.30}$$

We note that $\left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4}\cdot\frac{1}{\pi^2} = \frac{1}{4\pi}$, and $\left(\frac{\sqrt{\pi}}{2}\right)^2\cdot\frac{8}{\pi^2} = \frac{\pi}{4}\cdot\frac{8}{\pi^2} = \frac{2}{\pi}$.

As particular cases of the above result, we have, when $b=1$,

$$\int_0^\infty \frac{x}{\sinh \pi x}\,\frac{dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\ldots} = \frac{a}{2(1+2a)}; \tag{3.31}$$

when $b=2$,

$$\int_0^\infty \frac{x^3}{\sinh \pi x}\,\frac{dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\ldots} = \frac{a^2}{2(1+2a)(3+2a)}; \tag{3.32}$$

and so on. Since $\Pi\{1+x^2/(a+n)^2\}$ can be expressed in finite terms by means of hyperbolic functions when $2a$ is an integer, we can deduce a large number of special formulae from the preceding results.
Suppose now that $\alpha=\beta$ in the following expression:

$$\phi(\alpha,\beta)\phi(\beta,\alpha) = \frac{\{\Gamma(1+\alpha)\Gamma(1+\beta)\}^3}{\Gamma(1+\alpha+2\beta)\Gamma(1+\beta+2\alpha)}\left\{\frac{\cosh \pi(\alpha+\beta)\sqrt{3}-\cos \pi(\alpha-\beta)}{2\pi^2(\alpha^2+\alpha\beta+\beta^2)}\right\}. \tag{3.33}$$

We obtain:

$$\left\{1+\left(\frac{2\alpha}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{2+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha}{3+\alpha}\right)^3\right\}\ldots = \frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(1+3\alpha)}\,\frac{\sinh \pi\alpha\sqrt{3}}{\pi\alpha\sqrt{3}}. \tag{3.34}$$

Similarly, putting $\beta=\alpha+1$ in (3.33), we obtain:

$$\left\{1+\left(\frac{2\alpha+1}{1+\alpha}\right)^3\right\}\left\{1+\left(\frac{2\alpha+1}{2+\alpha}\right)^3\right\}\ldots = \frac{\{\Gamma(1+\alpha)\}^3}{\Gamma(2+3\alpha)}\,\frac{\cosh \pi\left(\frac{1}{2}+\alpha\right)\sqrt{3}}{\pi}. \tag{3.35}$$

Again, since

$$\left\{1+\left(\frac{\alpha}{n}\right)^3\right\}\left\{1+3\left(\frac{\alpha}{2n+\alpha}\right)^2\right\}=\frac{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\alpha^2}{n^2}+\frac{\alpha^4}{n^4}\right)}{\left(1+\frac{\alpha}{2n}\right)^2}, \quad (3.36)$$

it is easy to see that

$$\left[\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)...\right]\left[\left\{1+3\left(\frac{\alpha}{2+\alpha}\right)^2\right\}\left\{1+3\left(\frac{\alpha}{4+\alpha}\right)^2\right\}...\right]=\frac{\Gamma\left(\frac{1}{2}\alpha\right)}{\Gamma\left\{\frac{1}{2}(1+\alpha)\right\}}\left(\frac{\cosh\pi\alpha\sqrt{3}-\cos\pi\alpha}{2^{\alpha+2}\pi\alpha\sqrt{\pi}}\right). \quad (3.36)$$

It is known that, if the real part of α is positive, then

$$\log\Gamma(\alpha)=\left(\alpha-\frac{1}{2}\right)\log\alpha-\alpha+\frac{1}{2}\log 2\pi+2\int_0^\infty\frac{\tan^{-1}(x/\alpha)}{e^{2\pi x}-1}dx. \quad (3.37)$$

From this we can show that, if the real part of α is positive, then

$$\frac{1}{2}\log 2\pi\alpha+\frac{\pi\alpha}{\sqrt{3}}+\log\left\{\left(1+\frac{\alpha^3}{1^3}\right)\left(1+\frac{\alpha^3}{2^3}\right)\left(1+\frac{\alpha^3}{3^3}\right)...\right\}=\log\left(\frac{\cosh\pi\alpha\sqrt{3}-\cos\pi\alpha}{\pi\alpha}\right)+2\int_0^\infty\frac{\tan^{-1}(x/\alpha)^3}{e^{2\pi x}-1}dx \quad (3.38)$$

From this and the previous section it follows that

$$\int_0^\infty\frac{\tan^{-1}x^3}{e^{2\pi nx}-1}dx, \quad (3.39)$$

can be expressed in finite terms if n is a positive integer. Thus, for example,

$$\int_0^\infty\frac{\tan^{-1}x^3}{e^{2\pi x}-1}dx=\frac{1}{4}\log 2\pi-\frac{\pi}{4\sqrt{3}}-\frac{1}{2}\log\left(1+e^{-\pi\sqrt{3}}\right); \quad (3.40)$$

$$\int_0^\infty\frac{\tan^{-1}x^3}{e^{4\pi x}-1}dx=\frac{1}{8}\log 12\pi-\frac{\pi}{4\sqrt{3}}-\frac{1}{4}\log\left(1-e^{-2\pi\sqrt{3}}\right); \quad (3.41)$$

and so on.
It is also easy to see that

$$\frac{1^2}{1^3+n^3}-\frac{2^2}{2^3+n^3}+\frac{3^2}{3^3+n^3}-\frac{4^2}{4^3+n^3}+...=\frac{1}{3}\left(\frac{1}{1+n}-\frac{1}{2+n}+\frac{1}{3+n}-\frac{1}{4+n}+...\right)+$$
$$+\frac{4}{3}\left\{\frac{2-n}{(2-n)^2+3n^2}-\frac{4-n}{(4-n)^2+3n^2}+\frac{6-n}{(6-n)^2+3n^2}-...\right\}. \quad (3.42)$$

Since

$$\frac{\pi}{4\cosh\frac{1}{2}\pi x}=\frac{1}{1^2+x^2}-\frac{3}{3^2+x^2}+\frac{5}{5^2+x^2}-\dots, \quad (3.43)$$

it is clear that the left-hand side of (3.42) can be expressed in finite terms if n is any odd integer. For example,

$$\frac{1^2}{1^3+1}-\frac{2^2}{2^3+1}+\frac{3^2}{3^3+1}-\frac{4^2}{4^3+1}+\dots=\frac{1}{3}\left(1-\log 2+\pi\sec h\frac{1}{2}\pi\sqrt{3}\right). \quad (3.44)$$

The corresponding integral in this case is

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\frac{dx}{n^6+x^6}=\frac{2}{\pi}\int_0^\infty\left\{\frac{1}{2x^2}+\sum_{\nu=1}^{\nu=\infty}\frac{(-1)^\nu}{\nu^2+x^2}\right\}\frac{x^6dx}{n^6+x^6}=\frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}+\dots\right)+$$

$$-\frac{4}{3}\left\{\frac{n+2}{(n+2)^2+3n^2}-\frac{n+4}{(n+4)^2+3n^2}+\frac{n+6}{(n+6)^2+3n^2}-\dots\right\}, \quad (3.45)$$

and so the integral on the left-hand side of (3.45) can be expressed in finite terms if n is any odd integer. For example,

$$\int_0^\infty \frac{x^5}{\sinh \pi x}\frac{dx}{1+x^6}=\frac{1}{3}\left(\log 2-1+\pi\sec h\frac{1}{2}\pi\sqrt{3}\right). \quad (3.46)$$

3.1 *On some equations concerning the development of derivatives of order n (n positive integer) of various trigonometric functions and divergent series.*

We have the following expression:

$$\int_0^t \frac{t^{x-1/2}-t^{-x-1/2}}{t-1}dt=\left(t=e^{-u}\right)=\int_0^\infty\frac{e^{-u(x-1/2)}-e^{-u(-x-1/2)}}{e^{-u}-1}e^{-u}du=\int_0^\infty\frac{\left(e^{ux}-e^{-ux}\right)e^{u/2}}{e^u-1}du=$$

$$=\int_0^\infty\frac{\sinh(ux)}{\sinh(u/2)}du=\pi\tan(\pi x), \qquad |x|<\frac{1}{2}. \quad (3.47)$$

Deriving the eq. (3.47) $(2n)$ times, with respect to x, we obtain:

$$\int_0^\infty\frac{u^{2n}\sinh(ux)}{\sinh(u/2)}du=\pi[\tan(\pi x)]^{(2n)}=\pi^{2n+1}\sum_{h=0}^{n}b_h[\tan(\pi x)]^{2h+1}=\frac{2\pi}{i}\sum_{k\ge1}(-1)^k(-2\pi i)^{2n}k^{2n}e^{-2\pi ixk}. \quad (3.48)$$

Operating on the eq. (3.48), we obtain:

$$\int_0^\infty\frac{u^{2n}\sinh(ux)}{\sinh(u/2)}du=\pi[\tan(\pi x)]^{2n}=\pi^{2n+1}\sum_{h=0}^{n}b_h[\tan(\pi x)]^{2h+1}=\frac{(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\ge1}(-1)^k k^{2n}e^{-2\pi ixk}. \quad (3.49)$$

Putting, in the eq. (3.49), $\tan(\pi x)=t$, whence $\pi x=\arctan(t)$, we obtain:

$$\int_0^\infty \frac{u^{2n}\sinh\left[\frac{u}{\pi}\arctan(t)\right]}{\sinh(u/2)}du = \pi^{2n+1}\sum_{h=0}^{n} b_h t^{2h+1} . \quad (3.50)$$

Deriving the precedent eq. (3.50) with respect to (t), and putting after, $t = 0$, we have:

$$\lim_{t\to 0} D_t \int_0^\infty \frac{u^{2n}\sinh\left[\frac{u}{\pi}\arctan(t)\right]}{\sinh(u/2)}du = \frac{1}{\pi}\int_0^\infty \frac{u^{2n+1}}{\sinh(u/2)}du = \pi^{2n+1}b_0 , \quad (3.51)$$

whence

$$\frac{2}{\pi}\int_0^\infty u^{2n+1}e^{-u/2}\sum_{k\geq 0}e^{-uk}du = \frac{2}{\pi}\sum_{k\geq 0}\frac{(2n+1)!}{\left(\frac{1}{2}+k\right)^{2n+2}} = \frac{2^{2n+3}(2n+1)!}{\pi}\sum_{k\geq 0}\frac{1}{(1+2k)^{2n+2}} . \quad (3.52)$$

We have the following expression:

$$\sum_{k\geq 0}\frac{1}{(1+2k)^{2n}} = \sum_{k\geq 0}\frac{1}{(1+2k)^{2n}} + \sum_{k\geq 1}\frac{1}{(2k)^{2n}} - \sum_{k\geq 1}\frac{1}{(2k)^{2n}} = \sum_{k\geq 1}\frac{1}{(k)^{2n}} - \sum_{k\geq 1}\frac{1}{(2k)^{2n}} = \left(1-2^{-2n}\right)\zeta(2n); \quad (3.53)$$

and we remember that:

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}\left|B_{2n}\right|, \quad (3.54)$$

where B_{2n} is the Bernoulli's number of index $2n$, while $\zeta(s)$ is the Riemann zeta function, defined by $\zeta(s) = \sum_{k\geq 1}\frac{1}{k^s}, \mathrm{Re}(s) > 1$.

Applying the eq. (3.53), we obtain:

$$\sum_{k\geq 0}\frac{1}{(1+2k)^{2n+2}} = \left[1-2^{-(2n+2)}\right]\zeta(2n+2), \quad (3.55)$$

and thence:

$$\pi^{2n+1}b_0 = \frac{2^{2n+3}(2n+1)!}{\pi}\left[1-2^{-(2n+2)}\right]\zeta(2n+2) = \frac{2(2n+1)!}{\pi}\left(2^{2n+2}-1\right)\zeta(2n+2); \quad (3.56)$$

but:

$$\zeta(2n+2) = \frac{2^{2n+1}\pi^{2n+2}}{(2n+2)!}\left|B_{2n+2}\right|, \quad (3.57)$$

and thence:

$$b_0 = \frac{2^{2n+1}}{n+1}\left(2^{2n+2}-1\right)\left|B_{2n+2}\right|. \quad (3.58)$$

With regard this equation, the values of b_0 for n variable from 1 to 6 are: 2, 16, 272, 7936, 353792 and 22368256. We note that 272 is divisible for 16 and 7936, 353792 and 22368256 are all divisible for 64. It's interesting to observe that 64 is 8^2 and we know that 8 is the number of the physical vibrations of the superstrings that can be expressed by the following Ramanujan equation that has 8 "modes" corresponding to the vibrations above mentioned, i.e.,:

$$8 = \frac{1}{3} \cdot \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} . \quad (3.59)$$

If a series, $\sum_{n\geq 0} A_n \frac{x^n}{n!} = f(x)$, is divergent, for $x > a$, (a, constant), multiplying, both sides of the previous relation, for e^{-x}, and integrating, with respect to x, between the limits zero and infinity, we obtain another divergent series, defined by:

$$\int_0^\infty e^{-x} f(x) dx = \sum_{n\geq 0} \frac{A_n}{n!} \int_0^\infty x^n e^{-x} dx . \quad (3.60)$$

Now, we have the following relation:

$$\frac{x}{e^x - 1} = \sum_{k\geq 0} B_k \frac{x^k}{k!} . \quad (3.61)$$

Applying to this relation, the integration of eq. (3.60), we have that:

$$\int_0^\infty \frac{xe^{-x} dx}{e^x - 1} = \sum_{k\geq 0} B_k \frac{1}{k!} \int_0^\infty x^k e^{-x} dx = \sum_{k\geq 0} B_k = 1 - \frac{1}{2} + \sum_{k\geq 1} B_{2k} . \quad (3.62)$$

Now we compute the integral that is to the left-hand side of the (3.62). We obtain:

$$\int_0^\infty \frac{xe^{-x} dx}{e^x - 1} = \int_0^\infty \frac{xe^{-x} e^{-x} dx}{1 - e^{-x}} = \sum_{k\geq 0} \int_0^\infty xe^{-x(k+2)} dx = \sum_{k\geq 0} \frac{1}{(k+2)^2} = \frac{\pi^2}{6} - 1 . \quad (3.63)$$

Thence, from the (3.62), we obtain:

$$\sum_{k\geq 1} B_{2k} = \frac{\pi^2}{6} - \frac{3}{2} . \quad (3.64)$$

The left-hand side of the (3.64) is just a divergent series, that is represented from the value of the right-hand side of this expression.
Substituting in the (3.61), $-x$ to x, we have:

$$\frac{-x}{e^{-x} - 1} = \sum_{k\geq 0} B_k \frac{(-x)^k}{k!} . \quad (3.65)$$

For $|x| \geq 2\pi$, the precedent relation is divergent, thence, applying to the same relation, the integration of eq. (3.60), we have:

$$\int_0^\infty e^{-x} \frac{-x}{e^{-x}-1} dx = \int_0^\infty e^{-x} \frac{x}{1-e^{-x}} dx = \sum_{k\geq 0} \int_0^\infty x e^{-x(1+k)} dx = \sum_{k\geq 0} \frac{1}{(1+k)^2} = \frac{\pi^2}{6}; \quad (3.66)$$

$$\sum_{k\geq 0} B_k \frac{(-1)^k}{k!} \int_0^\infty x^k e^{-x} dx = \sum_{k\geq 0} B_k (-1)^k = 1 + \frac{1}{2} + \sum_{k\geq 1} B_{2k}. \quad (3.67)$$

Equalling the results of the two last relations, we obtain:

$$\sum_{k\geq 1} B_{2k} = \frac{\pi^2}{6} - \frac{3}{2}, \quad (3.68)$$

that is identical to the (3.64).
Now, we have the following integral formula:

$$\int_0^\infty (\ln x)^{2n} \frac{dx}{1+x^2} = \left(\frac{\pi}{2}\right)^{2n+1} |E_{2n}|. \quad (3.69)$$

Because $E_{4k} > 0$, and $E_{4k+2} < 0$, we have:

$$E_{2n} = (-1)^n \left(\frac{2}{\pi}\right)^{2n+1} \int_0^\infty (\ln x)^{2n} \frac{dx}{1+x^2}. \quad (3.70)$$

Applying, to the eq. (3.70) the following geometrical series:

$$\frac{1}{c+t} = \frac{1}{c} - \frac{t}{c^2} + \frac{t^2}{c^3} - \frac{t^3}{c^4} + ..., \quad c > 0, \ t > 0 \quad (3.70b)$$

we obtain:

$$\sum_{k\geq 0} E_{2k} = \frac{2}{\pi} \int_0^\infty \frac{1}{1 + \frac{4}{\pi^2}(\ln x)^2} \frac{dx}{1+x^2} = 2\pi \int_{-\infty}^\infty \frac{1}{\pi^2 + 4y^2} \frac{e^y}{1+e^{2y}} dy \quad (\text{for } x = e^y)$$

$$= \pi \int_{-\infty}^\infty \frac{1}{\pi^2 + 4y^2} \frac{1}{\cosh y} dy = \int_{-\infty}^\infty \frac{1}{1+4z^2} \frac{1}{\cosh(\pi z)} dz. \ (\text{for } y = \pi z) \quad (3.71)$$

Now, we consider the series

$$\sum_{k\geq 0} \frac{(-1)^k}{k+1} \binom{2k}{k} x^k. \quad (3.72)$$

The series (3.72) is a convergent series for $|x| < \frac{1}{4}$, and is divergent for $|x| \geq \frac{1}{4}$. The absolute values of the coefficients of the powers of the series (3.72) are the famous Catalan's numbers, and the primes values, for k = 0, 1, 2, 3,…, are the following: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796,… From the (3.72), we obtain:

$$\sum_{k\geq 0}\frac{(-1)^k}{k+1}\binom{2k}{k}x^k=\sum_{k\geq 0}\frac{(-1)^k}{k+1}\frac{\Gamma(2k+1)}{\Gamma(k+1)\Gamma(k+1)}x^k=\sum_{k\geq 0}\frac{(-1)^k}{k+1}\frac{2^{2k}}{\sqrt{\pi}}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+1)}{\Gamma(k+1)\Gamma(k+1)}x^k=$$

$$=\frac{1}{\sqrt{\pi}}\frac{1}{\Gamma\left(\frac{3}{2}\right)}\sum_{k\geq 0}\frac{(-4)^k\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(k+2)}x^k=\frac{2}{\pi}\sum_{k\geq 0}(-4)^k x^k\int_0^1 t^{k+\frac{1}{2}-1}(1-t)^{\frac{3}{2}-1}dt\,. \quad (3.73)$$

Applying the (3.70b) to the (3.73), we obtain:

$$\sum_{k\geq 0}\frac{(-1)^k}{k+1}\binom{2k}{k}x^k=\frac{2}{\pi}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{3}{2}-1}\frac{1}{1+4xt}dt=\frac{2}{\pi}\int_0^\infty\frac{y^{\frac{1}{2}-1}}{1+y}\frac{dy}{1+y+4xy}\quad \left(\text{for } t=\frac{y}{1+y}\right)=$$

$$=\frac{2}{\pi}\int_0^\infty y^{\frac{1}{2}-1}\left(\frac{1}{1+y}-\frac{1+4x}{1+y+4xy}\right)\frac{dy}{-4x}=\frac{2}{\pi}\frac{1}{-4x}\pi\left(\sin\frac{\pi}{2}\right)^{-1}+\frac{2}{\pi}\frac{1+4x}{-4x}\int_0^\infty\frac{y^{\frac{1}{2}-1}dy}{1+y(1+4x)}=$$

$$=\frac{1}{-2x}+\frac{2}{\pi}\frac{1+4x}{4x}\frac{1}{\sqrt{1+4x}}\pi\left(\sin\frac{\pi}{2}\right)^{-1}=\frac{\sqrt{1+4x}-1}{2x},\quad (3.74)\qquad \text{i.e.:}$$

$$\sum_{k\geq 0}\frac{(-1)^k}{k+1}\binom{2k}{k}x^k=\frac{\sqrt{1+4x}-1}{2x}\,. \quad (3.75)$$

Now we consider the series:

$$\sum_{k\geq 0}(-1)^k(\ln x)^{2k}=\frac{1}{1+(\ln x)^2}\,. \quad (3.76)$$

The right-hand side of (3.76) has been obtained applying the (3.70b). Multiplying for $\frac{x^{q-1}}{1+x}$, $0<\mathrm{Re}(q)<1$, both sides of (3.76) and integrating with respect to x, between the limits 0 and ∞, we obtain another divergent series, defined by:

$$\sum_{k\geq 0}(-1)^k\int_0^\infty(\ln x)^{2k}\frac{x^{q-1}}{1+x}dx=\int_0^\infty\frac{x^{q-1}}{1+x}\frac{dx}{1+(\ln x)^2}\,. \quad (3.77)$$

Operating, we have:

$$\sum_{k\geq 0}(-1)^k\int_0^\infty(\ln x)^{2k}\frac{x^{q-1}}{1+x}dx=\sum_{k\geq 0}(-1)^k D_q^{(2k)}\int_0^\infty\frac{x^{q-1}}{1+x}dx=\sum_{k\geq 0}(-1)^k D_q^{(2k)}\left(\frac{\pi}{\sin\pi q}\right);\quad (3.78)$$

$$\int_0^\infty\frac{x^{q-1}}{1+x}\frac{dx}{1+(\ln x)^2}=\int_{-\infty}^\infty\frac{e^{qz}}{1+e^z}\frac{dz}{1+z^2}\,.\quad (\text{for } x=e^z)\quad (3.79)$$

From the compute of the integral on the right-hand side of (3.79), we obtain:

$$\int_{-\infty}^{\infty}\frac{e^{qz}}{1+e^{z}}\frac{dz}{1+z^{2}}=-2\pi\sum_{k\geq 0}\frac{\sin[\pi q(2h+1)]}{\pi^{2}(2h+1)^{2}-1}+\frac{\pi}{2}\frac{\cos\left(\frac{1}{2}-q\right)}{\cos\frac{1}{2}}+2i\pi\sum_{k\geq 0}\frac{\cos[\pi q(2h+1)]}{\pi^{2}(2h+1)^{2}-1}-\frac{i\pi}{2}\frac{\sin\left(\frac{1}{2}-q\right)}{\cos\frac{1}{2}}. \quad (3.80)$$

3.1 Mathematical connections

We note that eqs. (1.18) and (1.21) can be connected with the eqs. (3.30) and (3.59) if we consider $\left(\frac{\sqrt{\pi}}{2}\right)^{2}=\frac{\pi}{4}\cdot\frac{2}{\pi^{2}}=\frac{1}{2\pi}$. Indeed, we obtain:

$$\xi\approx-\int_{0}^{\infty}\frac{dq}{2\pi}\log\left\{1+e^{-2LE_{m}(q)}Tr\left[K(q)\overline{K}(q)\right]\right\}\approx-\frac{1}{2}e^{-LE_{m}(0)}\sqrt{q^{2}Tr\left[K(q)\overline{K}(q)\right]_{q=0}}=$$

$$=\xi=-\phi^{2}\left[\frac{\lambda}{16\pi^{2}}-\frac{\lambda^{2}}{384\pi^{2}}+\frac{\lambda^{3}}{6144\pi^{2}}+\mathcal{O}(\lambda^{4})\right]\Rightarrow$$

$$\Rightarrow\int_{0}^{\infty}\left|\Gamma(a+ix)\Gamma(b+ix)\right|^{2}dx=\frac{1}{2}\sqrt{\pi}\,\frac{\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{\Gamma\left(a+b+\frac{1}{2}\right)}\Rightarrow$$

$$\Rightarrow 8=\frac{1}{3}\cdot\frac{4\left[anti\log\frac{\int_{0}^{\infty}\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^{2}w'}dx}{e^{-\frac{\pi^{2}}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^{2}w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (3.81)$$

We note that the eq. (1.52b) can be connected with the eqs. (3.45) and (3.73). Indeed, we obtain the following mathematical connections:

$$\log F=i\Phi(y)-i\Phi(1/y)=\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}dt\frac{(y^{4}-1)}{(1+y^{2})^{2}-4y^{2}\sin^{2}t}\log\left[\frac{\sinh 4\pi g\sin t}{4\pi g\sin t}\right]=$$

$$=\frac{a}{2\pi g}\int_{0}^{\frac{\pi}{2}}dt\left[\frac{1}{\sin^{2}t+\frac{a^{2}}{16g^{2}}}\right]4\pi g\sin t+\int_{0}^{\infty}dt\frac{4a}{v^{2}+4a^{2}\pi^{2}}\log\left[\frac{1-e^{-v}}{v}\right]=2a\log\left[\frac{8g}{a}\right]+o(1/g)+$$

$$+2[a\log a-a-\log\Gamma(a+1)]\Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{x^5}{\sinh \pi x}\frac{dx}{n^6+x^6} = \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{2x^2}+\sum_{\nu=1}^{\nu=\infty}\frac{(-1)^\nu}{\nu^2+x^2}\right\}\frac{x^6 dx}{n^6+x^6} = \frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}+...\right)+$$

$$-\frac{4}{3}\left\{\frac{n+2}{(n+2)^2+3n^2}-\frac{n+4}{(n+4)^2+3n^2}+\frac{n+6}{(n+6)^2+3n^2}-...\right\}\Rightarrow$$

$$\Rightarrow \sum_{k\geq 0}\frac{(-1)^k}{k+1}\binom{2k}{k}x^k = \sum_{k\geq 0}\frac{(-1)^k}{k+1}\frac{\Gamma(2k+1)}{\Gamma(k+1)\Gamma(k+1)}x^k = \sum_{k\geq 0}\frac{(-1)^k}{k+1}\frac{2^{2k}}{\sqrt{\pi}}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+1)}{\Gamma(k+1)\Gamma(k+1)}x^k =$$

$$= \frac{1}{\sqrt{\pi}}\frac{1}{\Gamma\left(\frac{3}{2}\right)}\sum_{k\geq 0}\frac{(-4)^k\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(k+2)}x^k = \frac{2}{\pi}\sum_{k\geq 0}(-4)^k x^k\int_0^1 t^{k+\frac{1}{2}-1}(1-t)^{\frac{3}{2}-1}dt\,. \quad (3.82)$$

Now, we note that we can to connect the eq. (2.7b) with the (3.66) and we obtain the following mathematical connection:

$$V_{lad}^{(2)}(\phi,\theta) = -\frac{(\cos\theta-\cos\phi)^2}{\sin^2\phi}\int_0^\infty \frac{dz}{z}\log\left(\frac{1+ze^{i\phi}}{1+ze^{-i\phi}}\right)\log\left(\frac{z+e^{i\phi}}{z+e^{-i\phi}}\right) =$$

$$= -4\frac{(\cos\theta-\cos\phi)^2}{\sin^2\phi}\left[Li_3\left(e^{2i\phi}\right)-\zeta(3)-i\phi\left(Li_2\left(e^{2i\phi}\right)+\frac{\pi^2}{6}\right)+\frac{i}{3}\phi^3\right]\Rightarrow$$

$$\Rightarrow \int_0^\infty e^{-x}\frac{-x}{e^{-x}-1}dx = \int_0^\infty e^{-x}\frac{x}{1-e^{-x}}dx = \sum_{k\geq 0}\int_0^\infty xe^{-x(1+k)}dx = \sum_{k\geq 0}\frac{1}{(1+k)^2} = \frac{\pi^2}{6}\,. \quad (3.83)$$

We note that the eq. (2.16), can be connected with the eq. (3.30), if we consider $\left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4}\cdot\frac{1}{\pi^2} = \frac{1}{4\pi}$ and with the eq. (3.59). Indeed, we obtain the following mathematical connections:

$$\frac{\Gamma_{reg}^{(4)}}{\mathcal{J}} = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}d\omega\left[8\left(\frac{\frac{3\pi\omega(192\omega^6+272\omega^4-220\omega^2-5)\sinh(\pi\omega)}{16(4\omega^2+1)^4}+\left(\frac{\pi^2(3-4\omega^2)^2\omega^2}{8(4\omega^2+1)^3}+\frac{384\omega^4}{(4\omega^2+1)^5}\right)\cosh(\pi\omega)}{\frac{4\cosh(\pi\omega)}{4\omega^2+1}}+\right.\right.$$

$$\left.-\frac{\left(\frac{\pi(4\omega^2-3)\omega\sinh(\pi\omega)}{(4\omega^2+1)^2}+\frac{32\omega^2\cosh(\pi\omega)}{(4\omega^2+1)^3}\right)^2}{2\left(\frac{4\cosh(\pi\omega)}{4\omega^2+1}\right)^2}\right)-2\left(\frac{\frac{\pi(9\omega^6+37\omega^4-72\omega^2-24)\cosh(\pi\omega)}{64\omega^4(\omega^2+1)}+\left(\frac{28\omega^4+12\omega^2+3}{8\omega^5(\omega^2+1)^3}+\right.}{\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}}\right.$$

$$+\frac{\frac{\pi^2(\omega^2-2)^2}{32\omega^3(\omega^2+1)}\Big)\sinh(\pi\omega)}{-}-\frac{\left(\frac{\pi(\omega^2-2)\cosh(\pi\omega)}{4\omega^2(\omega^2+1)}+\frac{(4\omega^2+1)\sinh(\pi\omega)}{2\omega^3(\omega^2+1)^2}\right)^2}{2\left(\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}\right)^2}\Bigg)+$$

$$-\left(\frac{\left(\frac{\pi^2(\omega^2-3)^2\omega}{32(\omega^2+1)^3}+\frac{6\omega^3}{(\omega^2+1)^5}\right)\sinh(\pi\omega)+\frac{3\pi(3\omega^6+17\omega^4-55\omega^2-5)\cosh(\pi\omega)}{64(\omega^2+1)^4}}{\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}}+\right.$$

$$-\frac{\left(\frac{2\omega\sinh(\pi\omega)}{(\omega^2+1)^3}+\frac{\pi(\omega^2-3)\cosh(\pi\omega)}{4(\omega^2+1)^2}\right)^2}{2\left(\frac{\sinh(\pi\omega)}{\omega(\omega^2+1)}\right)^2}\Bigg)-5\left(\frac{\frac{\pi^2\omega}{32}\sinh(\pi\omega)+\frac{9\pi}{64}\cosh(\pi\omega)}{\frac{\sinh(\pi\omega)}{\omega}}-\frac{\left(\frac{\pi}{4}\cosh(\pi\omega)\right)^2}{2\left(\frac{\sinh(\pi\omega)}{\omega}\right)^2}\right)\Bigg]=\frac{29}{128}-\frac{3\zeta(3)}{16}\Rightarrow$$

$$\Rightarrow\int_0^\infty|\Gamma(a+ix)\Gamma(b+ix)|^2dx=\frac{1}{2}\sqrt{\pi}\frac{\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{\Gamma\left(a+b+\frac{1}{2}\right)}\Rightarrow$$

$$\Rightarrow 8=\frac{1}{3}\cdot\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (3.84)$$

4. On some equations concerning the relationship between Yang-Mills theory and gravity and, consequently, the complete four-loop four-point amplitude of N = 4 super-Yang-Mills theory including the nonplanar contributions regarding the gauge theory and the gravity amplitudes.

At tree-level, the Kawai-Lewellen-Tye (KLT) relations given a complete description of the relationship between closed string amplitudes and open string amplitudes. For example, the open string amplitude for gluons is

$$A_n\sim\int\frac{dx_1...dx_n}{v_{abc}}\prod_{1\le i<j\le n}|x_i-x_j|^{k_i\cdot k_j}\exp\left[\sum_{i<j}\left(\frac{\varepsilon_i\cdot\varepsilon_j}{(x_i-x_j)^2}+\frac{k_i\cdot\varepsilon_j-k_j\cdot\varepsilon_i}{(x_i-x_j)}\right)\right]\Bigg|_{multi-linear}, \quad (4.1)$$

where

$$\mathcal{V}_{abc} = \frac{dx_a dx_b dx_c}{\left|(x_a - x_b)(x_b - x_c)(x_c - x_a)\right|} . \quad (4.2)$$

The corresponding n-graviton tree amplitude in string theory is

$$M_n \sim \int \frac{d^2 z_1 ... d^2 z_n}{\Delta_{abc}} \prod_{1 \le i < j \le n} (z_i - z_j)^{k_i \cdot k_j} \exp\left[\sum_{i<j}\left(\frac{\varepsilon_i \cdot \varepsilon_j}{(z_i - z_j)^2} + \frac{k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i}{(z_i - z_j)}\right)\right] \times \prod_{1 \le i < j \le n} (\bar{z}_i - \bar{z}_j)^{k_i \cdot k_j} \cdot$$

$$\cdot \exp\left[\sum_{i<j}\left(\frac{\bar{\varepsilon}_i \cdot \bar{\varepsilon}_j}{(\bar{z}_i - \bar{z}_j)^2} + \frac{k_i \cdot \bar{\varepsilon}_j - k_j \cdot \bar{\varepsilon}_i}{(\bar{z}_i - \bar{z}_j)}\right)\right]\Bigg|_{multi-linear} , \quad (4.3)$$

where

$$\Delta_{abc} = \frac{d^2 z_a d^2 z_b d^2 z_c}{|z_a - z_b|^2 |z_b - z_c|^2 |z_c - z_a|^2} , \quad (4.4)$$

z_a, z_b, z_c are any three of the z_i, and 'multi-linear' means linear in each ε_i and each $\bar{\varepsilon}_i$. In this expression we have taken the graviton polarization vector to be a product of gluon polarization vectors

$$\varepsilon_i^{\mu\nu} = \varepsilon_i^{\mu} \bar{\varepsilon}_i^{\nu} . \quad (4.5)$$

The KLT relation for four-point amplitude is:

$$M_4^{tree}(1,2,3,4) = -i s_{12} A_4^{tree}(1,2,3,4) A_4^{tree}(1,2,4,3), \quad (4.6)$$

with M and A represented from the expressions (4.3) and (4.1).
Applying the relation (4.6) yields the four-graviton amplitude

$$M_4^{tree}(1,2,3,4) = \frac{16 i K^2}{stu} . \quad (4.7)$$

A relation we will use also is the following:

$$stu M_4^{tree}(1,2,3,4) = -i\left(st\left[A_4^{tree}(1,2,3,4)\right]\right)^2 . \quad (4.8)$$

The one-loop $N = 8$ four-graviton amplitude is,

$$\mathcal{M}_4^{N=8,1-loop}(1,2,3,4) = -i\left(\frac{\kappa}{2}\right) s_{12} s_{23} s_{13} M_4^{tree}(1,2,3,4)\left(\mathcal{I}_4^{1-loop}(s_{12}, s_{23}) + \mathcal{I}_4^{1-loop}(s_{12}, s_{13}) + \mathcal{I}_4^{1-loop}(s_{23}, s_{13})\right), \quad (4.9)$$

with the integral functions defined as:

$$\mathcal{I}_4^{1-loop}(s_{12}, s_{23}) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p - k_1)^2 (p - k_1 - k_2)^2 (p + k_4)^2} . \quad (4.10)$$

These amplitudes were first obtained by Green, Schwarz and Brink in the field-theory limit of superstring theory.
The key relation for evaluating the $N=4$ two-particle cuts exactly to all loop orders is,

$$\sum_{S_1,S_2\in\{N=4\}} A_4^{tree}\left(-\ell_1^{s_1},1,2,\ell_2^{s_2}\right)\times A_4^{tree}\left(-\ell_2^{s_2},3,4,\ell_1^{s_1}\right)=-istA_4^{tree}(1,2,3,4)\frac{1}{(\ell_1-k_1)^2}\frac{1}{(\ell_2-k_3)^2}\,. \quad (4.11)$$

Using the KLT relation (4.6) we can use eq. (4.11) to obtain the equivalent relation for $N=8$ supergravity,

$$\begin{aligned}
\sum_{N=8states} M_4^{tree}(-\ell_1,1,2,\ell_2)\times M_4^{tree}(-\ell_2,3,4,\ell_1) &= -s^2\left(\sum_{N=4states} A_4^{tree}(-\ell_1,1,2,\ell_2)\times A_4^{tree}(-\ell_2,3,4,\ell_1)\right)\times\\
&\times\left(\sum_{N=4states} A_4^{tree}(\ell_2,1,2,-\ell_1)\times A_4^{tree}(\ell_1,3,4,-\ell_2)\right)=\\
&= s^2(st)^2\left[A_4^{tree}(1,2,3,4)\right]^2\frac{1}{(\ell_1-k_1)^2(\ell_2-k_3)^2(\ell_2+k_1)^2(\ell_1+k_3)^2}=\\
&= is^2stuM_4^{tree}(1,2,3,4)\frac{1}{(\ell_1-k_1)^2(\ell_2-k_3)^2(\ell_1-k_2)^2(\ell_2-k_4)^2}\,. \quad (4.12)
\end{aligned}$$

We perform a partial-fraction decomposition of the denominators,

$$-\frac{s}{(\ell_1-k_1)^2(\ell_1-k_2)^2}=\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\,;\quad -\frac{s}{(\ell_2-k_3)^2(\ell_2-k_4)^2}=\frac{1}{(\ell_2-k_3)^2}+\frac{1}{(\ell_2-k_4)^2}\,, \quad (4.13)$$

to obtain the $N=8$ basic two-particle on-shell sewing relation,

$$\begin{aligned}
&\sum_{N=8states} M_4^{tree}(-\ell_1,1,2,\ell_2)\times M_4^{tree}(-\ell_2,3,4,\ell_1)=\\
&= istuM_4^{tree}(1,2,3,4)\left[\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\right]\left[\frac{1}{(\ell_2-k_3)^2}+\frac{1}{(\ell_2-k_4)^2}\right]. \quad (4.14)
\end{aligned}$$

We may recycle the sewing relation (4.14) to obtain two-particle cuts of higher-loop amplitudes. Consider the two-particle s-cut with a tree amplitude on the left and a one-loop amplitude on the right,

$$M_4^{2-loop}(1,2,3,4)\Big|_{s-cut}=\int\frac{d^D\ell_1}{(2\pi)^D}\sum_{N=8states}\frac{i}{\ell_1^2}M_4^{tree}(-\ell_1,1,2,\ell_2)\frac{i}{\ell_2^2}M_4^{1-loop}(-\ell_2,3,4,\ell_1)\Big|_{\ell_1^2=\ell_2^2=0}\,. \quad (4.15)$$

Inserting eq. (4.9) for M_4^{1-loop} and applying the sewing relation (4.14), we have

$$\begin{aligned}
M_4^{2-loop}(1,2,3,4)\Big|_{s-cut} &= -stuM_4^{tree}\int\frac{d^D\ell_1}{(2\pi)^D}s(\ell_2-k_3)^2(\ell_2-k_4)^2\times\\
&\times\left[\frac{1}{(\ell_1-k_1)^2}+\frac{1}{(\ell_1-k_2)^2}\right]\frac{i}{\ell_1^2}\left[\frac{s}{(\ell_2-k_3)^2(\ell_2-k_4)^2}\right]\frac{i}{\ell_2^2}\times
\end{aligned}$$

$$\times\left[\mathcal{J}_4^{1-loop}\left(s,(\ell_2-k_3)^2\right)+\mathcal{J}_4^{1-loop}\left((\ell_2-k_3)^2,(\ell_2-k_4)^2\right)+\mathcal{J}_4^{1-loop}\left((\ell_2-k_4)^2,s\right)\right]\Big|_{\ell_1^2=\ell_2^2=0}. \quad (4.16)$$

The unwanted propagators cancel and our final result is remarkably simple

$$M_4^{2-loop}(1,2,3,4)\Big|_{s-cut}=stuM_4^{tree}s^2\left(\mathcal{J}_4^{2-loop,P}(s,t)+\mathcal{J}_4^{2-loop,P}(s,u)+\mathcal{J}_4^{2-loop,NP}(s,t)+\mathcal{J}_4^{2-loop,NP}(s,u)\right)\Big|_{s-cut}, \quad (4.17)$$

where the scalar integrals $\mathcal{J}_4^{2-loop,P}$ and $\mathcal{J}_4^{2-loop,NP}$ are defined in the following equations:

$$\mathcal{J}_4^{2-loop,P}(s_{12},s_{23})=\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+q)^2q^2(q-k_4)^2(q-k_3-k_4)^2},$$
$$\mathcal{J}_4^{2-loop,NP}(s_{12},s_{23})=\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{1}{p^2(p-k_2)^2(p+q)^2(p+q+k_1)^2q^2(q-k_3)^2(q-k_3-k_4)^2}. \quad (4.18)$$

A straightforward Feynman parameterization of the integrals (4.18) gives

$$\mathcal{J}_4^{2-loop,X}(s,t)=\frac{\Gamma(7-D)}{(4\pi)^D}\int_0^1 d^7a\,\delta\left(1-\sum_{i=1}^7 a_i\right)\left(-Q_X(s,t,a_i)\right)^{D-7}\left(\Delta_X(a_i)\right)^{7-3D/2}, \quad X=P,NP. \quad (4.19)$$

In extracting the $1/\varepsilon$ pole term for $D=n-2\varepsilon$ it is legitimate to replace $\mathcal{J}_4^{2-loop,X}(s,t)$ with

$$\mathcal{J}_4^{2-loop,X}(s,t)\Big|_{pole}=\frac{\Gamma(7-D)}{(4\pi)^D}\int_0^1 d^7a\,\delta\left(1-\sum_{i=1}^7 a_i\right)\left(-Q_X(s,t,a_i)\right)^{n-7}\left(\Delta_X(a_i)\right)^{7-n-D/2}. \quad (4.20)$$

For the case $D=7-2\varepsilon$, where we can set $\varepsilon=0$ from the beginning, we get:

$$\mathcal{J}_4^{2-loop,P,D=7-2\varepsilon}\Big|_{pole}=\frac{1}{2\varepsilon(4\pi)^7}\frac{1}{4}\int_0^1 dy\,y^2(1-y)^2\int_0^1 dx\frac{x^{3/2}}{[1-x(1-y(1-y))]^{7/2}}=\frac{1}{2\varepsilon(4\pi)^7}\frac{\pi}{10}. \quad (4.21)$$

For $D=9-2\varepsilon$, setting $\varepsilon=0$ would lead to

$$\mathcal{J}_4^{2-loop,P,D=9-2\varepsilon}\Big|_{pole}=\frac{1}{4\varepsilon(4\pi)^9}\frac{1}{498960}\int_0^1\frac{dy}{[y(1-y)]^{3/2}}\Big[3s^2\left(16y^2(1-y)^2-77y(1-y)+132\right)+$$
$$+8sty(1-y)(2y(1-y)+11)+80t^2y^2(1-y)^2\Big]. \quad (4.22)$$

We find that the (x,y)-integral for the s^2 term can be conveniently written as the sum of two terms,

$$\mathcal{J}_4^{2-loop,P,D=9-2\varepsilon}\Big|_{pole,s^2term}=\frac{s^2}{4\varepsilon(4\pi)^9}\int_0^1 dy\int_0^1 dx\left[C_1(x,y)+C_2(x,y)\right], \quad (4.23)$$

where

$$C_1(x,y)=\frac{[y(1-y)]^4}{480}\frac{x^{5/2+\varepsilon}\left(-x^2y(1-y)+(1-x)(2-3x)\right)}{[1-x(1-y(1-y))]^{13/2-\varepsilon}},$$

$$C_2(x,y)=\frac{[y(1-y)]^2}{360}\frac{x^{5/2+\varepsilon}}{[1-x(1-y(1-y))]^{9/2-\varepsilon}}. \quad (4.24)$$

We can rewrite the eq. (4.23) also as follows:

$$\mathcal{J}_4^{2-loop,P,D=9-2\varepsilon}\Big|_{pole,s^2term}=\frac{s^2}{4\varepsilon(4\pi)^9}\int_0^1 dy\int_0^1 dx\ \frac{[y(1-y)]^4}{480}\frac{x^{5/2+\varepsilon}\left(-x^2y(1-y)+(1-x)(2-3x)\right)}{[1-x(1-y(1-y))]^{13/2-\varepsilon}}+$$
$$+\frac{[y(1-y)]^2}{360}\frac{x^{5/2+\varepsilon}}{[1-x(1-y(1-y))]^{9/2-\varepsilon}}. \quad (4.24b)$$

The integral of C_1 converges for $\varepsilon=0$, and $\int_0^1 dxdyC_1(x,y)=-5\pi/11088$. The integral over C_2 requires analytic continuation in ε. Somewhat more generally, we need

$$I(p,q,\alpha)\equiv\int_0^1 dy[y(1-y)]^p\int_0^1 dx\frac{x^{\alpha-q-2+\varepsilon}(1-x)^q}{[1-x(1-y(1-y))]^{\alpha-\varepsilon}}=\frac{\Gamma(\alpha-q-1+\varepsilon)\Gamma(q+1)}{\Gamma(\alpha+\varepsilon)}\times$$
$$\times\int_0^1 dy[y(1-y)]^p\,{}_2F_1(\alpha-\varepsilon,\alpha-q-1+\varepsilon;\alpha+\varepsilon;1-y(1-y))=\frac{\Gamma(\alpha-q-1+\varepsilon)\Gamma(q+1)}{\Gamma(\alpha+\varepsilon)}\int_0^1 dy[y(1-y)]^p\times$$
$$\times\left\{\frac{\Gamma(\alpha+\varepsilon)\Gamma(-\alpha+q+1+\varepsilon)}{\Gamma(2\varepsilon)\Gamma(q+1)}{}_2F_1(\alpha-\varepsilon,\alpha-q-1+\varepsilon;\alpha-q-\varepsilon;y(1-y))+[y(1-y)]^{-\alpha+q+1+\varepsilon}\cdot\right.$$
$$\left.\cdot\frac{\Gamma(\alpha+\varepsilon)\Gamma(\alpha-q-1-\varepsilon)}{\Gamma(\alpha-\varepsilon)\Gamma(\alpha-q-1+\varepsilon)}{}_2F_1(2\varepsilon,q+1;-\alpha+q+2+\varepsilon;y(1-y))\right\}, \quad (4.25)$$

where p and q are positive integers and α is a positive half-integer. In the limit $\varepsilon\to 0$ the factor of $1/\Gamma(2\varepsilon)$ in eq. (4.25) causes the term containing it to vanish, and the surviving hypergeometric function can be set to 1. Performing the remaining y-integral gives

$$I(p,q,\alpha)=\frac{\Gamma(\alpha-q-1-\varepsilon)\Gamma(q+1)}{\Gamma(\alpha-\varepsilon)}\int_0^1 dy[y(1-y)]^{-\alpha+p+q+1+\varepsilon}=\frac{\Gamma(\alpha-q-1)\Gamma(q+1)\Gamma^2(-\alpha+p+q+2)}{\Gamma(\alpha)\Gamma(2(-\alpha+p+q+2))}. \quad (4.26)$$

Thus $I(p,q,\alpha)=0$ (after analytic continuation in ε) unless $\alpha<p+q+2$. In the present case, $p=2,q=0,$ and $\alpha=9/2$, so the integral of C_2 vanishes.
The final result for the planar double-box pole at $D=9-2\varepsilon$ and $D=11-2\varepsilon$ is then

$$\mathcal{J}_4^{2-loop,P,D=9-2\varepsilon}\Big|_{pole}=\frac{1}{4\varepsilon(4\pi)^9}\frac{\pi}{99792}\left(-45s^2+18st+2t^2\right), \quad (4.27)$$
$$\mathcal{J}_4^{2-loop,P,D=11-2\varepsilon}\Big|_{pole}=\frac{1}{48\varepsilon(4\pi)^{11}}\frac{\pi}{196911000}\left(2100s^4-880s^3t+215s^2t^2+30st^3+12t^4\right). \quad (4.28)$$

The non-planar double-box integrals are handled analogously, with the results:

$$\mathcal{J}_4^{2-loop,NP,D=7-2\varepsilon}\Big|_{pole}=\frac{1}{2\varepsilon(4\pi)^7}\frac{\pi}{15}, \quad (4.29)$$

$$\mathcal{J}_4^{2-loop,NP,D=9-2\varepsilon}\Big|_{pole} = \frac{1}{4\varepsilon(4\pi)^9}\frac{-\pi}{83160}\left(75s^2+2tu\right), \quad (4.30)$$

$$\mathcal{J}_4^{2-loop,NP,D=11-2\varepsilon}\Big|_{pole} = \frac{1}{48\varepsilon(4\pi)^{11}}\frac{\pi}{1654052400}\left(40383s^4-1138s^2tu+144t^2u^2\right). \quad (4.31).$$

Thence, from the eq.(4.24b), we obtain the following expression:

$$\mathcal{J}_4^{2-loop,P,D=9-2\varepsilon}\Big|_{pole,s^2 term} = \frac{s^2}{4\varepsilon(4\pi)^9}\int_0^1 dy\int_0^1 dx\ \frac{[y(1-y)]^4}{480}\frac{x^{5/2+\varepsilon}\left(-x^2y(1-y)+(1-x)(2-3x)\right)}{[1-x(1-y(1-y))]^{13/2-\varepsilon}} +$$
$$+\frac{[y(1-y)]^2}{360}\frac{x^{5/2+\varepsilon}}{[1-x(1-y(1-y))]^{9/2-\varepsilon}} \Rightarrow$$
$$\Rightarrow \mathcal{J}_4^{2-loop,P,D=9-2\varepsilon}\Big|_{pole} = \frac{1}{4\varepsilon(4\pi)^9}\frac{\pi}{99792}\left(-45s^2+18st+2t^2\right). \quad (4.32)$$

We note that this equation can be related to the number 24 (480 = 24 * 20; 360 = 24 * 15; 99792 = 4158 * 24) that is connected to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}.$$

Indeed, we obtain the following mathematical connection:

$$\mathcal{J}_4^{2-loop,P,D=9-2\varepsilon}\Big|_{pole,s^2 term} = \frac{s^2}{4\varepsilon(4\pi)^9}\int_0^1 dy\int_0^1 dx\ \frac{[y(1-y)]^4}{480}\frac{x^{5/2+\varepsilon}\left(-x^2y(1-y)+(1-x)(2-3x)\right)}{[1-x(1-y(1-y))]^{13/2-\varepsilon}} +$$
$$+\frac{[y(1-y)]^2}{360}\frac{x^{5/2+\varepsilon}}{[1-x(1-y(1-y))]^{9/2-\varepsilon}} \Rightarrow$$
$$\Rightarrow \mathcal{J}_4^{2-loop,P,D=9-2\varepsilon}\Big|_{pole} = \frac{1}{4\varepsilon(4\pi)^9}\frac{\pi}{99792}\left(-45s^2+18st+2t^2\right) \Rightarrow$$
$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.32b)$$

The two-loop MSYM (maximally supersymmetric N = 4 Yang-Mills theory) four-point amplitudes for the planar contribution, is:

$$M_4^{(2)}(\varepsilon)=\frac{1}{4}st\left(sI_4^{(2)}(s,t)+tI_4^{(2)}(t,s)\right), \quad (4.33)$$

The two-loop scalar integral $I_4^{(2)}$ is defined in the following equation:

$$I_4^{(2)}(s,t)=\left(-ie^{\varepsilon\gamma}\pi^{-d/2}\right)^2\int\frac{d^dpd^dq}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+q)^2q^2(q-k_4)^2(q-k_3-k_4)^2}. \quad (4.34)$$

The three-loop planar amplitude gives the explicit form of the integrand,

$$M_4^{(3)}(\varepsilon)=-\frac{1}{8}st\left(s^2I_4^{(3)a}(s,t)+2sI_4^{(3)b}(t,s)+t^2I_4^{(3)a}(t,s)+2tI_4^{(3)b}(s,t)\right). \quad (4.35)$$

The two three-loop integrals $I_4^{(3)a}, I_4^{(3)b}$, appearing in the four-point amplitude (4.35), are:

$$I_4^{(3)a}(s,t)=\left(-ie^{\varepsilon\gamma}\pi^{-d/2}\right)^3\int\frac{d^dpd^drd^dq}{p^2(p-k_1)^2(p-k_1-k_2)^2}\times$$
$$\times\frac{1}{(p+r)^2r^2(q-r)^2(r-k_3-k_4)^2q^2(q-k_4)^2(q-k_3-k_4)^2}, \quad (4.36)$$

and

$$I_4^{(3)b}(s,t)=\left(-ie^{\varepsilon\gamma}\pi^{-d/2}\right)^3\int\frac{d^dpd^drd^dq(p+r)^2}{p^2q^2r^2(p-k_1)^2(p+r-k_1)^2}\times$$
$$\times\frac{1}{(p+r-k_1-k_2)^2(p+r+k_4)^2(q-k_4)^2(r+p+q)^2(p+q)^2}, \quad (4.37)$$

where dimensional regularization with $d=4-2\varepsilon$ is implied.
Mellin-Barnes (MB) integrations are introduced in order to replace a sum of terms raised to some power by their products raised to certain powers, at the cost of having extra integrations:

$$\frac{1}{(X+Y)^\lambda}=\frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}dz\Gamma(\lambda+z)\Gamma(-z)\frac{Y^z}{X^{\lambda+z}}, \quad (4.38)$$

where $-\mathrm{Re}\,\lambda<\beta<0$.
An eightfold MB representation can be derived with the eleventh index corresponding to the numerator $\left[(p+r)^2\right]^{-a_{11}}$. For our integral with the powers $a_1=...=a_{10}=1$ and $a_{11}=-1$, this gives

$$I_4^{(3)b}(s,t)=-\frac{e^{3\varepsilon\gamma}}{\Gamma(-2\varepsilon)(-s)^{1+3\varepsilon}t^2}\times\frac{1}{(2\pi i)^8}\int_{-i\infty}^{+i\infty}...\int_{-i\infty}^{+i\infty}dwdz_1\left(\prod_{j=2}^{7}dz_j\Gamma(-z_j)\right)\left(\frac{t}{s}\right)^w\Gamma(1+3\varepsilon+w)\times$$

$$\times \frac{\Gamma(-3\varepsilon - w)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-\varepsilon-z_1-z_3)\Gamma(1+z_1+z_4)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-z_6)\Gamma(1-2\varepsilon+z_1+z_2+z_3)} \times$$

$$\times \frac{\Gamma(-1-\varepsilon-z_1-z_2-z_4)\Gamma(2+\varepsilon+z_1+z_2+z_3+z_4)}{\Gamma(-1-4\varepsilon-z_5)\Gamma(1-z_4-z_7)\Gamma(2+2\varepsilon+z_4+z_5+z_6+z_7)} \times \Gamma(-\varepsilon+z_1+z_3-z_5)\Gamma(2-w+z_5)$$

$$\Gamma(-1+w-z_5-z_6)\times\Gamma(z_5+z_7-z_1)\Gamma(1+z_5+z_6)\Gamma(-1+w-z_4-z_5-z_7)\times\Gamma(-\varepsilon-z_1+z_2-z_5-z_6-z_7)$$

$$\Gamma(1-\varepsilon-w+z_4+z_5+z_6+z_7)\times\Gamma(1+\varepsilon-z_1-z_2-z_3+z_5+z_6+z_7). \qquad (4.39)$$

There is a factor of $\Gamma(-2\varepsilon)$ (i.e. the two gamma functions) in the denominator, so that the integral is effectively sevenfold.

The starting point is to take minus the residue at $z_5 = -2+\varepsilon+x_2-z_1-z_2-z_3-z_4$ and shift the integration contour correspondingly. The value of the residue is then symmetrized by $x_1 \leftrightarrow x_2$. This sum leads, in the limit $x_1, x_2 \to 0$, to the following fourfold MB integral:

$$I_4^{(3)b,c-c-c,res}(s,t) = -\frac{e^{3\varepsilon\gamma}}{(-s)^{1+3\varepsilon}t^2}\frac{\Gamma(-3\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma(-2\varepsilon)}$$

$$\times \frac{1}{(2\pi i)^4}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty}\prod_{j=1}^{4} dz_j \frac{\Gamma(1+z_1)\Gamma(-1-\varepsilon-z_1-z_2)\Gamma(-z_2)\Gamma(-\varepsilon-z_3)\Gamma(-z_3)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-2\varepsilon+z_1+z_2+z_3)}$$

$$\times \frac{\Gamma(1+z_1+z_2+z_3)\Gamma(1+\varepsilon+z_2+z_3)\Gamma(-1-\varepsilon+z_2-z_4)\Gamma(-1-z_1-z_4)\Gamma(1+z_4)}{\Gamma(-2-4\varepsilon-z_1-z_4)\Gamma(-1+2\varepsilon-z_1-z_4)\Gamma(3+z_1+z_4)}$$

$$\times\Gamma(-1+\varepsilon-z_1-z_2-z_3-z_4)\Gamma(2+z_1+z_4)\Gamma(-2-\varepsilon-z_1-z_4)\Gamma(2-\varepsilon+z_1+z_3+z_4)$$

$$\times[2\gamma+L+\psi(-3\varepsilon)+\psi(-2\varepsilon)-\psi(1+3\varepsilon)-\psi(1+z_1+z_2+z_3)+\psi(-1-z_1-z_4)-\psi(-2-4\varepsilon-z_1-z_4)$$

$$+\psi(-1+2\varepsilon-z_1-z_4)+\psi(-1-\varepsilon+z_2-z_4)-\psi(-1+\varepsilon-z_1-z_2-z_3-z_4)+\psi(2-\varepsilon+z_1+z_3+z_4)], \qquad (4.40)$$

where $L = \ln(s/t)$.

In the integral over the shifted contour in z_5, one can set $x_1 = x_2 = 0$ to obtain the following fivefold integral:

$$I_4^{(3)b,c-c-c,\mathrm{int}}(s,t) = -\frac{2e^{3\varepsilon\gamma}}{(-s)^{1+3\varepsilon}(-t)^2}\frac{\Gamma(1+3\varepsilon)\Gamma(-3\varepsilon)}{\Gamma(-2\varepsilon)}$$

$$\times \frac{1}{(2\pi i)^5}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty}\prod_{j=1}^{5} dz_j \frac{\Gamma(1+z_1)\Gamma(1+\varepsilon+z_2+z_3)\Gamma(-z_2)\Gamma(-\varepsilon-z_3)\Gamma(-z_3)}{\Gamma(1-z_2)\Gamma(1-z_3)}$$

$$\times \frac{\Gamma(-1-\varepsilon-z_1-z_2)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-z_1-z_4)}{\Gamma(1-2\varepsilon+z_1+z_2+z_3)\Gamma(-2-4\varepsilon-z_1-z_4)}$$

$$\times \frac{\Gamma(1+z_4)\Gamma(\varepsilon-z_2-z_3-z_5)\Gamma^*(-2+\varepsilon-z_1-z_2-z_3-z_4-z_5)}{\Gamma(1+\varepsilon-z_2-z_3-z_5)\Gamma(1+\varepsilon+z_2+z_3+z_5)}$$

$$\times\Gamma(1+z_5)\Gamma(-1-\varepsilon+z_2-z_4)\Gamma(-2\varepsilon+z_2+z_3+z_5)\times\Gamma(2-\varepsilon+z_1+z_2+z_3+z_4+z_5)\Gamma(-z_2-z_5), \qquad (4.41)$$

where the asterisk on one of the gamma functions implies that the first pole is considered to be of the opposite nature. With regard the evaluation of (4.40) and (4.41), in an expansion in ε, after the resolution of the singularities in ε, one obtains 60 contributions where an expansion of the integrand in ε becomes possible. Eventually, one reproduces the following leading asymptotic behaviour:

$$I_4^{(3)b}(s,t) = -\frac{1}{(-s)^{1+3\varepsilon}t^2} \times \left\{ \frac{16}{9}\frac{1}{\varepsilon^6} + \frac{13}{6}L\frac{1}{\varepsilon^5} + \left[\frac{1}{2}L^2 - \frac{19}{12}\pi^2\right]\frac{1}{\varepsilon^4} + \left[-\frac{1}{6}L^3 - \frac{67}{72}\pi^2 L - \frac{241}{18}\zeta_3\right]\frac{1}{\varepsilon^3} + \right.$$
$$+\left[\frac{1}{24}L^4 + \frac{13}{24}\pi^2 L^2 - \frac{67}{6}\zeta_3 L - \frac{19}{6480}\pi^4\right]\frac{1}{\varepsilon^2} + \left[-\frac{1}{120}L^5 - \frac{13}{72}\pi^2 L^3 - \frac{5}{2}\zeta_3 L^2 - \frac{6523}{8640}\pi^4 L + \frac{1385}{216}\pi^2\zeta_3 + \right.$$
$$\left. - \frac{1129}{10}\zeta_5\right]\frac{1}{\varepsilon} + \frac{1}{720}L^6 + \frac{13}{288}\pi^2 L^4 + \frac{5}{6}\zeta_3 L^3 + \frac{331}{960}\pi^4 L^2 + \left(\frac{317}{72}\pi^2\zeta_3 - \frac{1203}{10}\zeta_5\right)L - \frac{180631}{3265920}\pi^6 +$$
$$\left. - \frac{163}{6}\zeta_3^2 + O\left(\frac{s}{t}\right)\right\}. \quad (4.42)$$

Thence, the eq. (4.39) can be rewritten also as follows:

$$I_4^{(3)b}(s,t) = -\frac{e^{3\varepsilon\gamma}}{\Gamma(-2\varepsilon)(-s)^{1+3\varepsilon}t^2} \times \frac{1}{(2\pi i)^8}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty} dw dz_1 \left(\prod_{j=2}^{7} dz_j \Gamma(-z_j)\right)\left(\frac{t}{s}\right)^w \Gamma(1+3\varepsilon+w) \times$$
$$\times \frac{\Gamma(-3\varepsilon-w)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-\varepsilon-z_1-z_3)\Gamma(1+z_1+z_4)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-z_6)\Gamma(1-2\varepsilon+z_1+z_2+z_3)} \times$$
$$\times \frac{\Gamma(-1-\varepsilon-z_1-z_2-z_4)\Gamma(2+\varepsilon+z_1+z_2+z_3+z_4)}{\Gamma(-1-4\varepsilon-z_5)\Gamma(1-z_4-z_7)\Gamma(2+2\varepsilon+z_4+z_5+z_6+z_7)} \times \Gamma(-\varepsilon+z_1+z_3-z_5)\Gamma(2-w+z_5)$$
$$\Gamma(-1+w-z_5-z_6) \times \Gamma(z_5+z_7-z_1)\Gamma(1+z_5+z_6)\Gamma(-1+w-z_4-z_5-z_7) \times \Gamma(-\varepsilon-z_1+z_2-z_5-z_6-z_7)$$
$$\Gamma(1-\varepsilon-w+z_4+z_5+z_6+z_7) \times \Gamma(1+\varepsilon-z_1-z_2-z_3+z_5+z_6+z_7) \Rightarrow$$
$$\Rightarrow -\frac{1}{(-s)^{1+3\varepsilon}t^2} \times \left\{ \frac{16}{9}\frac{1}{\varepsilon^6} + \frac{13}{6}L\frac{1}{\varepsilon^5} + \left[\frac{1}{2}L^2 - \frac{19}{12}\pi^2\right]\frac{1}{\varepsilon^4} + \left[-\frac{1}{6}L^3 - \frac{67}{72}\pi^2 L - \frac{241}{18}\zeta_3\right]\frac{1}{\varepsilon^3} + \right.$$
$$+\left[\frac{1}{24}L^4 + \frac{13}{24}\pi^2 L^2 - \frac{67}{6}\zeta_3 L - \frac{19}{6480}\pi^4\right]\frac{1}{\varepsilon^2} + \left[-\frac{1}{120}L^5 - \frac{13}{72}\pi^2 L^3 - \frac{5}{2}\zeta_3 L^2 - \frac{6523}{8640}\pi^4 L + \frac{1385}{216}\pi^2\zeta_3 + \right.$$
$$\left. - \frac{1129}{10}\zeta_5\right]\frac{1}{\varepsilon} + \frac{1}{720}L^6 + \frac{13}{288}\pi^2 L^4 + \frac{5}{6}\zeta_3 L^3 + \frac{331}{960}\pi^4 L^2 + \left(\frac{317}{72}\pi^2\zeta_3 - \frac{1203}{10}\zeta_5\right)L - \frac{180631}{3265920}\pi^6 +$$
$$\left. - \frac{163}{6}\zeta_3^2 + O\left(\frac{s}{t}\right)\right\}. \quad (4.42b)$$

Also this expression can be related to the number 24 (120 = 24 * 5; 72 = 24 * 3; 8640 = 360 * 24; 216 = 24 * 9; 720 = 24 * 30; 288 = 24 * 12; 960 = 24 * 40; 3265920 = 24 * 136080) that is connected to the physical vibrations of the bosonic strings. Indeed, we obtain:

$$I_4^{(3)b}(s,t) = -\frac{e^{3\varepsilon\gamma}}{\Gamma(-2\varepsilon)(-s)^{1+3\varepsilon}t^2} \times \frac{1}{(2\pi i)^8}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty} dw dz_1 \left(\prod_{j=2}^{7} dz_j \Gamma(-z_j)\right)\left(\frac{t}{s}\right)^w \Gamma(1+3\varepsilon+w) \times$$
$$\times \frac{\Gamma(-3\varepsilon-w)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-\varepsilon-z_1-z_3)\Gamma(1+z_1+z_4)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-z_6)\Gamma(1-2\varepsilon+z_1+z_2+z_3)} \times$$
$$\times \frac{\Gamma(-1-\varepsilon-z_1-z_2-z_4)\Gamma(2+\varepsilon+z_1+z_2+z_3+z_4)}{\Gamma(-1-4\varepsilon-z_5)\Gamma(1-z_4-z_7)\Gamma(2+2\varepsilon+z_4+z_5+z_6+z_7)} \times \Gamma(-\varepsilon+z_1+z_3-z_5)\Gamma(2-w+z_5)$$
$$\Gamma(-1+w-z_5-z_6) \times \Gamma(z_5+z_7-z_1)\Gamma(1+z_5+z_6)\Gamma(-1+w-z_4-z_5-z_7) \times \Gamma(-\varepsilon-z_1+z_2-z_5-z_6-z_7)$$
$$\Gamma(1-\varepsilon-w+z_4+z_5+z_6+z_7) \times \Gamma(1+\varepsilon-z_1-z_2-z_3+z_5+z_6+z_7) \Rightarrow$$

$$\Rightarrow -\frac{1}{(-s)^{1+3\varepsilon}t^2}\times\left\{\frac{16}{9}\frac{1}{\varepsilon^6}+\frac{13}{6}L\frac{1}{\varepsilon^5}+\left[\frac{1}{2}L^2-\frac{19}{12}\pi^2\right]\frac{1}{\varepsilon^4}+\left[-\frac{1}{6}L^3-\frac{67}{72}\pi^2 L-\frac{241}{18}\zeta_3\right]\frac{1}{\varepsilon^3}+\right.$$

$$+\left[\frac{1}{24}L^4+\frac{13}{24}\pi^2L^2-\frac{67}{6}\zeta_3 L-\frac{19}{6480}\pi^4\right]\frac{1}{\varepsilon^2}+\left[-\frac{1}{120}L^5-\frac{13}{72}\pi^2L^3-\frac{5}{2}\zeta_3L^2-\frac{6523}{8640}\pi^4L+\frac{1385}{216}\pi^2\zeta_3+\right.$$

$$\left.-\frac{1129}{10}\zeta_5\right]\frac{1}{\varepsilon}+\frac{1}{720}L^6+\frac{13}{288}\pi^2L^4+\frac{5}{6}\zeta_3L^3+\frac{331}{960}\pi^4L^2+\left(\frac{317}{72}\pi^2\zeta_3-\frac{1203}{10}\zeta_5\right)L-\frac{180631}{3265920}\pi^6+$$

$$\left.-\frac{163}{6}\zeta_3^2+\mathrm{O}\left(\frac{s}{t}\right)\right\}\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.42c)$$

The iterative structure of the four-point MSYM amplitude found at two loops is

$$M_4^{(2)}(\varepsilon)=\frac{1}{2}\left(M_4^{(1)}(\varepsilon)\right)^2+f^{(2)}(\varepsilon)M_4^{(1)}(2\varepsilon)+C^{(2)}+\mathrm{O}(\varepsilon), \quad (4.43)$$

where

$$f^{(2)}(\varepsilon)=-\left(\zeta_2+\zeta_3\varepsilon+\zeta_4\varepsilon^2+\ldots\right), \quad (4.44)$$

and the constant $C^{(2)}$ is given by

$$C^{(2)}=-\frac{1}{2}\zeta_2^2. \quad (4.45)$$

The iterative relation for the three-loop four-point amplitude is the following:

$$M_4^{(3)}(\varepsilon)=-\frac{1}{3}\left[M_4^{(1)}(\varepsilon)\right]^3+M_4^{(1)}(\varepsilon)M_4^{(2)}(\varepsilon)+f^{(3)}(\varepsilon)M_4^{(1)}(3\varepsilon)+C^{(3)}+\mathrm{O}(\varepsilon), \quad (4.46)$$

where

$$f^{(3)}(\varepsilon)=\frac{11}{2}\zeta_4+\varepsilon(6\zeta_5+5\zeta_2\zeta_3)+\varepsilon^2\left(c_1\zeta_6+c_2\zeta_3^2\right), \quad (4.47)$$

and the constant $C^{(3)}$ is given by

$$C^{(3)}=\left(\frac{341}{216}+\frac{2}{9}c_1\right)\zeta_6+\left(-\frac{17}{9}+\frac{2}{9}c_2\right)\zeta_3^2. \quad (4.48)$$

The constants c_1 and c_2 are expected to be rational numbers.
A general n-point scattering amplitude can be factorized into the following form,

$$\mathcal{M}_n = J\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), \varepsilon\right) \times S\left(k_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu), \varepsilon\right) \times h_n\left(k_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu), \varepsilon\right). \quad (4.49)$$

For a general theory, the Sudakov form factor at scale Q^2 can be written as

$$\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), \varepsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2} \frac{d\xi^2}{\xi^2}\left[\mathcal{K}^{[g]}(\alpha_s(\mu), \varepsilon) + \mathcal{G}^{[g]}\left(-1, \bar{\alpha}_s\left(\frac{\mu^2}{\xi^2}, \alpha_s(\mu), \varepsilon\right), \varepsilon\right) + \right.\right.$$
$$\left.\left. + \frac{1}{2}\int_{\xi^2}^{\mu^2} \frac{d\tilde{\mu}^2}{\tilde{\mu}^2} \gamma_K^{[g]}\left(\bar{\alpha}_s\left(\frac{\mu^2}{\tilde{\mu}^2}, \alpha_s(\mu), \varepsilon\right)\right)\right]\right\}, \quad (4.50)$$

where $\gamma_K^{[g]}$ denotes the soft or (Wilson line) cusp anomalous dimension, which will produce a $1/\varepsilon^2$ pole after integration. The function $\mathcal{K}^{[g]}$ is a series of counter terms (pure poles in ε), while $\mathcal{G}^{[g]}$ includes non-singular dependence on ε before integration, and produces a $1/\varepsilon$ pole after integration.
The integral over $\mathcal{G}$ is very simple,

$$\int_0^{-Q^2} \frac{d\xi^2}{\xi^2}\mathcal{G}^{[g]} = -\sum_{l=1}^{\infty} \frac{a^l}{l\varepsilon}\left(\frac{\mu^2}{-Q^2}\right)^{l\varepsilon} \hat{\mathcal{G}}_0^{(l)}. \quad (4.51)$$

The first integral over γ_K gives,

$$\int_{\xi^2}^{\mu^2} \frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\gamma_K^{[g]} = \sum_{l=1}^{\infty} \frac{a^l}{l\varepsilon}\left[\left(\frac{\mu^2}{\xi^2}\right)^{l\varepsilon} - 1\right]\hat{\gamma}_K^{(l)}. \quad (4.52)$$

Adding the $\mathcal{K}^{[g]}$ term to 1/2 of eq. (4.52), using the following equation

$$\mathcal{K}^{[g]}(\alpha_s, \varepsilon) = \sum_{l=1}^{\infty} \frac{1}{2l\varepsilon} a^l \hat{\gamma}_K^{(l)}, \quad (4.53)$$

we see that the “– 1” is cancelled. Then the integral over ξ is properly regulated, and evaluates to

$$-\frac{1}{2}\sum_{l=1}^{\infty} \frac{a^l}{(l\varepsilon)^2}\left(\frac{\mu^2}{-Q^2}\right)^{l\varepsilon} \hat{\gamma}_K^{(l)}. \quad (4.54)$$

Combining this result with eq. (4.51) gives:

$$\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), \varepsilon\right) = \exp\left[-\frac{1}{4}\sum_{l=1}^{\infty} a^l\left(\frac{\mu^2}{-Q^2}\right)^{l\varepsilon}\left(\frac{\hat{\gamma}_K^{(l)}}{(l\varepsilon)^2} + \frac{2\hat{\mathcal{G}}_0^{(l)}}{l\varepsilon}\right)\right]. \quad (4.55)$$

Thence, the eq. (4.50) can be rewritten also as follows:

$$\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2},\alpha_s(\mu),\varepsilon\right)=\exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\left[\mathcal{K}^{[g]}(\alpha_s(\mu),\varepsilon)+\mathcal{G}^{[g]}\left(-1,\overline{\alpha}_s\left(\frac{\mu^2}{\xi^2},\alpha_s(\mu),\varepsilon\right),\varepsilon\right)+\right.\right.$$

$$\left.\left.+\frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\tilde{\mu}^2}{\tilde{\mu}^2}\gamma_K^{[g]}\left(\overline{\alpha}_s\left(\frac{\mu^2}{\tilde{\mu}^2},\alpha_s(\mu),\varepsilon\right)\right)\right]\right\}\Rightarrow\exp\left[-\frac{1}{4}\sum_{l=1}^{\infty}a^l\left(\frac{\mu^2}{-Q^2}\right)^{l\varepsilon}\left(\frac{\hat{\gamma}_K^{(l)}}{(l\varepsilon)^2}+\frac{2\hat{\mathcal{G}}_0^{(l)}}{l\varepsilon}\right)\right].\quad(4.55b)$$

For the complete amplitude for a general gauge group G, including all non-planar contributions, the parent-graph decomposition,

$$\mathrm{A}_n^{(L)}=g^{2L+n-2}\sum_{i\in parent}a_iC_iI_i\,,\quad(4.56)$$

is more convenient than the color-trace representation.

The L-loop four-point amplitude is a Feynman integral with the following general structure and normalization,

$$I_i=(-i)^L\int\left(\prod_{j=1}^{L}\frac{d^D\ell_j}{(2\pi)^D}\right)\frac{N_i(\ell_j,k_m)}{\prod_{n=1}^{3L+1}l_n^2}\,,\quad(4.57)$$

where $k_m, m=1,2,3,$ are the three independent external momenta, ℓ_j are the L independent loop momenta, and l_n are the momenta of the $(3L+1)$ propagators which are linear combinations of the ℓ_j and the k_m.

At one loop, the structure of the $\mathcal{N}=4$ sYM four-point amplitude is especially simple. We modify eq. (4.56) slightly by extracting an overall prefactor, and write the result as,

$$\mathrm{A}_4^{(1)}=-\frac{1}{8}g^4\mathcal{K}\sum_{S_4}C_{1234}^{box}I^{box}(s_{12},s_{23}),\quad(4.58)$$

where g is the gauge coupling.

At two loops, the full $\mathcal{N}=4$ sYM amplitude is given by a similar permutation sum as for the one-loop case (4.58),

$$\mathrm{A}_4^{(2)}=-\frac{1}{4}g^6\mathcal{K}\sum_{S_4}\left[C_{1234}^{(P)}I^{(P)}(s_{12},s_{23})+C_{1234}^{(NP)}I^{(NP)}(s_{12},s_{23})\right].\quad(4.59)$$

The UV divergence of the color-dressed amplitude depends only on the three integrals V_1, V_2 and V_8:

$$\mathrm{A}_4^{(4)}(1,2,3,4)\Big|_{pole}^{SU(N_c)}=-6g^{10}\mathcal{K}N_c^2\left(N_c^2V_1+12(V_1+2V_2+V_8)\right)\times$$
$$\times\left(s_{12}(Tr_{1324}+Tr_{1423})+s_{23}(Tr_{1243}+Tr_{1342})+s_{13}(Tr_{1234}+Tr_{1432})\right).\quad(4.60)$$

Thus, we find that double-trace terms are absent from the divergence in the critical dimension D_c, as for the case at three loops.

For a general gauge group the leading UV divergence at four loops has a similarly simple structure, proportional to the tree-level color tensor:

$$\mathrm{A}_4^{(4)}(1,2,3,4)\Big|_{pole}^{G}=g^{10}\mathcal{K}\mathcal{V}^{(4)}\left(s_{12}\tilde{f}^{a_2a_3b}\tilde{f}^{ba_4a_1}+s_{23}\tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4}\right),\quad(4.61)$$

where

$$\mathcal{V}^{(4)} = 3\left(C_{V_1}V_1 + 2C_{V_2}V_2 + C_{V_8}V_8\right). \quad (4.62)$$

The coefficients C_{V_1}, C_{V_2} and C_{V_8} are the group invariants associated with the corresponding vacuum diagrams. As at three loops, the four-loop group invariants are not independent; rather, they satisfy the following relations:

$$C_{V_1} - C_{V_2} = \frac{C_A^4}{8}, \quad (4.63) \qquad \frac{1}{3}C_{V_1} + \frac{2}{3}C_{V_2} = \frac{d_A^{abcd}d_A^{abcd}}{N_A}, \quad (4.64) \qquad C_{V_8} = C_{V_2}. \quad (4.65)$$

As with the three-loop case, it is possible to rearrange the UV-divergent contributions at four loops into one-particle-reducible parent graphs. The divergent part of the amplitude then has the simple form

$$\mathrm{A}_4^{(4)}(1,2,3,4) = -g^{10}\mathcal{K}\sum_{S_4}\left[\frac{1}{16}C_{1234}^{V_1}I_{V_1} + \frac{1}{8}C_{1234}^{V_2}I_{V_2} + \frac{1}{16}C_{1234}^{V_8}I_{V_8}\right] + \text{subleading}. \quad (4.66)$$

The one-loop bubble integral is simple to evaluate. For an arbitrary dimension D and powers n_1 and n_2 of the two propagators, it is given by

$$I^{bubble}(n_1,n_2) \equiv -i\int\frac{d^D p}{(2\pi)^D}\frac{1}{\left((p+k)^2\right)^{n_1}\left(p^2\right)^{n_2}} = \frac{(-1)^{n_1+n_2}}{(4\pi)^{D/2}}G(n_1,n_2)\left(-k^2\right)^{-(n_1+n_2-D/2)}, \quad (4.67)$$

where

$$G(n_1,n_2) = \frac{\Gamma(-D/2+n_1+n_2)\Gamma(D/2-n_1)\Gamma(D/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(D-n_1-n_2)}. \quad (4.68)$$

Thence, we can rewrite the eq. (4.67) also as follows:

$$I^{bubble}(n_1,n_2) \equiv -i\int\frac{d^D p}{(2\pi)^D}\frac{1}{\left((p+k)^2\right)^{n_1}\left(p^2\right)^{n_2}} = \frac{(-1)^{n_1+n_2}}{(4\pi)^{D/2}}\frac{\Gamma(-D/2+n_1+n_2)\Gamma(D/2-n_1)\Gamma(D/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(D-n_1-n_2)}\left(-k^2\right)^{-(n_1+n_2-D/2)}. \quad (4.68b)$$

In $D = 11/2 - 2\varepsilon$ dimensions, we have $(n_1,n_2) = \left(\frac{3}{4}+3\varepsilon,2\right)$ and $(n_1,n_2) = \left(\frac{7}{4}+3\varepsilon,1\right)$. Inserting these values into eq. (4.67) we find that both integrals have the same UV pole,

$$G\left(\frac{3}{4}+3\varepsilon,2\right) = \frac{4}{21}\frac{1}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\varepsilon} + \mathrm{O}(1), \quad (4.69) \qquad G\left(\frac{7}{4}+3\varepsilon,1\right) = \frac{4}{21}\frac{1}{\Gamma\left(\frac{3}{4}\right)}\frac{1}{\varepsilon} + \mathrm{O}(1), \quad (4.70)$$

The finite three-loop two-point integrals can be reduced to a set of master integrals using the method of integration by parts and we obtain, for example, the following relations for the $1/\varepsilon$ pole terms:

$$V_1 = \left[\frac{6272}{25}\Gamma^5\left(\frac{3}{4}\right) - \frac{256}{5}\Gamma^4\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right) + 8\frac{\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} G^{(2)}\left(\frac{9}{4},\frac{11}{2}\right) \right] \frac{G\left(\frac{3}{4}+3\varepsilon,2\right)}{(4\pi)^{11}} \qquad (4.71)$$

$$= \left[\frac{12992}{25}\Gamma^5\left(\frac{3}{4}\right) - \frac{496}{5}\Gamma^4\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right) + \frac{1}{2}\frac{\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} G^{(2)}\left(\frac{9}{4},\frac{11}{2}\right) \right] \frac{G\left(\frac{7}{4}+3\varepsilon,1\right)}{(4\pi)^{11}} \qquad (4.72)$$

$$= \left[\frac{12352}{25}\Gamma^5\left(\frac{3}{4}\right) - \frac{288}{5}\Gamma^4\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right) - 5\frac{\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(G^{(2)}\left(\frac{9}{4},\frac{11}{2}\right) - 6G^{(2)}\left(\frac{5}{4},\frac{11}{2}\right)\right) \right]$$

$$\times \frac{G\left(\frac{7}{4}+3\varepsilon,1\right)}{(4\pi)^{11}}. \qquad (4.73)$$

Thence, we can rewrite the eq. (4.68b), considering one of these solutions (for example the 4.71), also as follows:

$$I^{bubble}(n_1,n_2) \equiv -i\int \frac{d^D p}{(2\pi)^D} \frac{1}{\left((p+k)^2\right)^{n_1}\left(p^2\right)^{n_2}} = \frac{(-1)^{n_1+n_2}}{(4\pi)^{D/2}} \frac{\Gamma(-D/2+n_1+n_2)\Gamma(D/2-n_1)\Gamma(D/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(D-n_1-n_2)}$$

$$\left(-k^2\right)^{-(n_1+n_2-D/2)} \Rightarrow$$

$$\Rightarrow \left[\frac{6272}{25}\Gamma^5\left(\frac{3}{4}\right) - \frac{256}{5}\Gamma^4\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right) + 8\frac{\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} G^{(2)}\left(\frac{9}{4},\frac{11}{2}\right) \right] \frac{G\left(\frac{3}{4}+3\varepsilon,2\right)}{(4\pi)^{11}}. \qquad (4.73b)$$

We note that this equation can be connected with the following Ramanujan equation that has 8 “modes” corresponding to the vibrations of the superstrings:

$$8 = \frac{1}{3} \cdot \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)} \right]}.$$

Indeed, 256 and 6272 are multiples of 8. Thence, we obtain:

$$I^{bubble}(n_1,n_2) \equiv -i\int \frac{d^D p}{(2\pi)^D} \frac{1}{\left((p+k)^2\right)^{n_1}\left(p^2\right)^{n_2}} = \frac{(-1)^{n_1+n_2}}{(4\pi)^{D/2}} \frac{\Gamma(-D/2+n_1+n_2)\Gamma(D/2-n_1)\Gamma(D/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(D-n_1-n_2)}$$

$$\left(-k^2\right)^{-(n_1+n_2-D/2)} \Rightarrow$$

$$\Rightarrow\left[\frac{6272}{25}\Gamma^5\left(\frac{3}{4}\right)-\frac{256}{5}\Gamma^4\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)+8\frac{\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)}G^{(2)}\left(\frac{9}{4},\frac{11}{2}\right)\right]\frac{G\left(\frac{3}{4}+3\varepsilon,2\right)}{(4\pi)^{11}}\Rightarrow$$

$$\Rightarrow 8=\frac{1}{3}\cdot\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.73c)$$

Equating the three forms for V_1 at order $1/\varepsilon$ yields

$$G^{(2)}\left(\frac{5}{4},\frac{11}{2}\right)=-\frac{64}{25}\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(\frac{1}{2}\right)+\frac{928}{125}\frac{\Gamma^3\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}+O(\varepsilon), \quad (4.74)$$

$$G^{(2)}\left(\frac{9}{4},\frac{11}{2}\right)=-\frac{32}{5}\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(\frac{1}{2}\right)+\frac{896}{25}\frac{\Gamma^3\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}+O(\varepsilon), \quad (4.75)$$

and

$$V_1=\frac{1}{(4\pi)^{11}\varepsilon}\left[\frac{512}{5}\Gamma^4\left(\frac{3}{4}\right)-\frac{2048}{105}\Gamma^3\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)\right]+O(1). \quad (4.76)$$

With regard the analytic evaluation of V_2, after using eqs. (4.74) and (4.75) for $G^{(2)}\left(\frac{5}{4},\frac{11}{2}\right)$ and $G^{(2)}\left(\frac{9}{4},\frac{11}{2}\right)$, they all give the same result,

$$V_2=\frac{1}{(4\pi)^{11}\varepsilon}\left[\frac{4352}{105}\Gamma^4\left(\frac{3}{4}\right)+\frac{832}{105}\Gamma^3\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)\right]+O(1). \quad (4.77)$$

There are four inequivalent ways of factorizing the non-planar vacuum integral V_8 into a product of three-loop and one-loop two-point integrals.
That is, all four ways of factorizing V_8 lead to the same expression,

$$V_8 = \frac{1}{(4\pi)^{11}} \frac{4}{21} \frac{1}{\Gamma\left(\frac{3}{4}\right)} \frac{V_8^{fin}}{\varepsilon} + O(1), \quad (4.78)$$

where

$$V_8^{fin} = -\frac{5248}{125}\Gamma^5\left(\frac{3}{4}\right) + \frac{224}{25}\Gamma^4\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right) + 2NO_m . \quad (4.79)$$

Although it is not needed for the four-loop $\mathcal{N} = 4$ sYM amplitude, a similar factorization and reduction procedure for V_9 gives

$$V_9 = \frac{1}{(4\pi)^{11}} \frac{4}{21} \frac{1}{\Gamma\left(\frac{3}{4}\right)} \frac{V_9^{fin}}{\varepsilon} + O(1), \quad (4.80)$$

where

$$V_9^{fin} = -\frac{15552}{125}\Gamma^5\left(\frac{3}{4}\right) + \frac{576}{25}\Gamma^4\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right) - 2NO_m . \quad (4.81)$$

In general, a massless m-point L-loop gauge-theory amplitude $A_m^{(L)}$ in D space-time dimensions, with all particles in the adjoint representation, may be written as

$$A_m^{(L)} = i^L g^{m-2+2L} \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \quad (4.82)$$

where g is the gauge coupling constant. The sum runs over the complete set Γ of m-point L-loop graphs with only cubic (trivalent) vertices, including all permutations of external legs.
More surprising than the duality itself is a consequent relation between gauge and gravity amplitudes. Once the gauge-theory amplitudes are arranged into a form satisfying the following equation

$$C_i = C_j + C_k \qquad \Rightarrow \qquad n_i = n_j + n_k , \quad (4.83)$$

the numerator factors of the corresponding L-loop gravity amplitudes, $\mathcal{M}_m^{(L)}$, can be obtained simply by multiplying together two copies of gauge-theory numerator factors,

$$\mathcal{M}_m^{(L)} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \quad (4.84)$$

where κ is the gravitational coupling. The $\tilde{n}_i$ represent numerator factors of a second gauge-theory amplitude and the sum runs over the same set of graphs as in eq. (4.82). At least one family of numerators (n_i or $\tilde{n}_i$) must satisfy the duality (4.83). The construction (4.84) is expected to hold in a large class of gravity theories, including all theories that are the low-energy limits of string theories. At tree level, this double-copy property encodes the KLT relations between gravity and

gauge-theory amplitudes. For $\mathcal{N}=8$ supergravity both n_i and $\tilde{n}_i$ are numerators of $\mathcal{N}=4$ sYM theory.
In terms of the 85 distinct graphs, the four-loop sYM amplitude is given by

$$\mathrm{A}_4^{(4)} = g^{10} st A_4^{tree} \sum_{S_4} \sum_{i=1}^{85} \int \left(\prod_{j=5}^{8} \frac{d^D l_j}{(2\pi)^D} \right) \frac{1}{S_i} \frac{N_i(k_j, l_j) C_i}{\prod_{\alpha_i=1}^{13} p_{\alpha_i}^2}, \quad (4.85)$$

where l_5, l_6, l_7, l_8 are the four independent loop momenta and k_1, k_2, k_3 are the three independent external momenta. The p_{α_i} are the momenta of the internal propagators and are linear combinations of the independent loop momenta l_j and the external momenta k_m. As usual, $d^D l_j/(2\pi)^D$ is the D-dimensional integration measure for the j^{th} loop momentum. The numerator factors $N_i(k_j, l_j)$ are polynomial in both internal and external momenta. The full amplitude is obtained by summing over the group S_4 of 24 permutations of the external leg labels.
Using the double-copy relation (4.84), the four-loop four-point $\mathcal{N}=8$ supergravity amplitude is obtained simply by trading the color factor C_i for $\tilde{n}_i = st\tilde{A}_4^{tree} N_i$ in eq. (4.85). Employing the relation $s^2 t^2 A_4^{tree} \tilde{A}_4^{tree} = \mathrm{i}\, stu M_4^{tree}$ and changing the gauge coupling to the gravitational coupling, we have

$$\mathcal{M}_4^{(4)} = -\left(\frac{\kappa}{2}\right)^{10} stu M_4^{tree} \sum_{S_4} \sum_{i=1}^{82} \int \left(\prod_{j=5}^{8} \frac{d^D l_j}{(2\pi)^D} \right) \frac{1}{S_i} \frac{N_i^2(k_j, l_j)}{\prod_{\alpha_i=1}^{13} p_{\alpha_i}^2}, \quad (4.86)$$

where $N_i(k_j, l_j)$ are the gauge-theory numerator factors.
We note that in the eqs. (4.85) and (4.86) there are the numbers 5, 8 and 13 that are all Fibonacci's numbers. Furthermore, also the number of permutations, i.e. 24 is important, because represent the number of the physical vibrations of the bosonic strings.
The UV divergence in the critical dimension $D=11/2$ is given by,

$$\mathrm{A}_4^{(4)}(1,2,3,4)\Big|_{pole}^{SU(N_c)} = -6 g^{10} \mathcal{K} N_c^2 \left(N_c^2 V_1 + 12(V_1 + 2V_2 + V_8) \right)$$
$$\times \left(s(Tr_{1324} + Tr_{1423}) + t(Tr_{1243} + Tr_{1342}) + u(Tr_{1234} + Tr_{1432}) \right). \quad (4.87)$$

It is interesting to note, from eq. (4.87) that the single-trace UV divergence in $D=11/2$ has N_c^4 and N_c^2 components, but the N_c^0 component vanishes.
We note that the eqs. (4.85) and (4.87) can be related. Indeed, we have the following mathematical connection:

$$\mathrm{A}_4^{(4)} = g^{10} st A_4^{tree} \sum_{S_4} \sum_{i=1}^{85} \int \left(\prod_{j=5}^{8} \frac{d^D l_j}{(2\pi)^D} \right) \frac{1}{S_i} \frac{N_i(k_j, l_j) C_i}{\prod_{\alpha_i=1}^{13} p_{\alpha_i}^2} \Rightarrow$$
$$\Rightarrow \mathrm{A}_4^{(4)}(1,2,3,4)\Big|_{pole}^{SU(N_c)} = -6 g^{10} \mathcal{K} N_c^2 \left(N_c^2 V_1 + 12(V_1 + 2V_2 + V_8) \right)$$
$$\times \left(s(Tr_{1324} + Tr_{1423}) + t(Tr_{1243} + Tr_{1342}) + u(Tr_{1234} + Tr_{1432}) \right). \quad (4.87b)$$

The values of the three master integrals appearing in eq. (4.87) are:

$$V_1 = \frac{1}{(4\pi)^{11}\varepsilon}\left[\frac{512}{5}\Gamma^4\left(\frac{3}{4}\right) - \frac{2048}{105}\Gamma^3\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)\right] + O(1),$$

$$V_2 = \frac{1}{(4\pi)^{11}\varepsilon}\left[-\frac{4352}{105}\Gamma^4\left(\frac{3}{4}\right) + \frac{832}{105}\Gamma^3\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)\right] + O(1),$$

$$V_8 = \frac{1}{(4\pi)^{11}\varepsilon}\left[-\frac{20992}{2625}\Gamma^4\left(\frac{3}{4}\right) + \frac{128}{75}\Gamma^3\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right) + \frac{8}{21\Gamma\left(\frac{3}{4}\right)}NO_m\right] + O(1), \quad (4.88)$$

where NO_m denotes a certain three-loop two-point nonplanar integral. While its analytic expression in $D = 11/2$ is not known, it may be evaluated numerically using the Gegen-bauer polynomial x-space technique (GPXT), with the result $NO_m = -6.1983992267...$. It's interesting note that this number is very near at the value of the aurea ratio multiplied for 10, i.e. $0{,}61803398 \times 10 = 6{,}1803398 \cong 6{,}180 \approx 6{,}198$.

With regard the $\mathcal{N} = 4$ sYM color double-trace UV divergence, the UV divergence from graphs 1 through 50 has the following $Tr_{12}Tr_{34}$ component:

$$A_4^{(4)}\Big|_{\substack{Tr_{12}Tr_{34} \\ pole1-50}} = \frac{g^{10}\mathcal{K}}{3(4\pi)^{12}\varepsilon}Tr_{12}Tr_{34}N_c \times \left[\left(s^2+t^2+u^2\right)\left(N_c^2\left(1-4\zeta_3+10\zeta_5\right)+180\zeta_5\right) - 9s^2\left(N_c^2\zeta_3+25\zeta_5\right)\right],$$

(4.89)

where $\mathcal{K}$ is defined in the following equation:

$$\mathcal{K} \equiv stA^{tree}(1,2,3,4). \quad (4.90)$$

Next we evaluate the remaining graphs, 51 through 85. Because the color factors of 1PR graphs do not contain double traces, it follows that, in fact, only the integrals I_{51}, I_{52} and I_{72} contribute to the double-trace terms. The numerator factors of these three integrals are all equal, $N_{51} = N_{52} = N_{72}$. All these three integrals contain an essentially identical subdivergence, from a three-point three-loop subgraph whose external legs carry momentum k_4, l_5 and $l_5 + k_4$. The three-point subgraph reduces to a propagator (two-point) subgraph for the respective cases of graphs 51, 52 and 72. These integrals can be evaluated in $D = 6 - 2\varepsilon$ using IBP identities and gluing relations through the necessary order, $O(\varepsilon^0)$. The results are:

$$P_{51}(l_5^2) = -(-l_5^2)^{-3\varepsilon}\frac{e^{-3\gamma\varepsilon}}{(4\pi)^{9-3\varepsilon}}\left[\frac{1}{6\varepsilon} + \frac{25}{9} + \zeta_3 - \frac{10}{3}\zeta_5 + O(\varepsilon)\right], \quad (4.91)$$

$$P_{52}(l_5^2) = -(-l_5^2)^{-3\varepsilon}\frac{e^{-3\gamma\varepsilon}}{(4\pi)^{9-3\varepsilon}}\left[\frac{1}{6\varepsilon}\left(\zeta_3 - \frac{1}{3}\right) - \frac{25}{27} + \frac{17}{18}\zeta_3 + \frac{1}{4}\zeta_4 + O(\varepsilon)\right], \quad (4.92)$$

$$P_{72}(l_5^2) = -(-l_5^2)^{-3\varepsilon}\frac{e^{-3\gamma\varepsilon}}{(4\pi)^{9-3\varepsilon}}\left[\frac{1}{6\varepsilon}\left(\zeta_3 - \frac{1}{3}\right) - \frac{25}{27} + \frac{17}{18}\zeta_3 + \frac{1}{4}\zeta_4 + O(\varepsilon)\right]. \quad (4.93)$$

In the trace basis, the color factors for graphs 51, 52 and 72 have the following form:

$$C_{51} = N_c^2\left(N_c^2+12\right)\left(Tr_{1234}+Tr_{1432}\right) + 2N_c\left(N_c^2+12\right)\left(Tr_{12}Tr_{34}+Tr_{13}Tr_{24}+Tr_{14}Tr_{23}\right), \quad (4.94)$$

$$C_{52} = C_{72} = 12N_c^2\left(Tr_{1234} + Tr_{1432}\right) + 24N_c\left(Tr_{12}Tr_{34} + Tr_{13}Tr_{24} + Tr_{14}Tr_{23}\right). \quad (4.95)$$

Taking into account the relative symmetry factors, we see that the relevant linear combination of propagator integrals for the N_c^1 part is $P_{N_c^1} \equiv P_{51} + 2P_{52} + P_{72}$, which is given by,

$$P_{N_c^1}\left(l_5^2\right) = -\left(-l_5^2\right)^{-3\varepsilon} \frac{e^{-3\gamma\varepsilon}}{(4\pi)^{9-3\varepsilon}}\left[\frac{\zeta_3}{2\varepsilon} + \frac{23}{6}\zeta_3 + \frac{3}{4}\zeta_4 - \frac{10}{3}\zeta_5 + \mathrm{O}(\varepsilon)\right]. \quad (4.96)$$

Next we need to identify a subtraction that accounts for the three-loop counterterm needed to cancel the pole given in the following equation:

$$\begin{aligned} \mathrm{A}_4^{(3)}(1,2,3,4)\Big|_{pole}^{SU(N_c)} &= 2g^8\mathcal{K}\left(N_c^3V^{(A)} + 12N_c\left(V^{(A)} + 3V^{(B)}\right)\right) \\ &\times\left(s\left(Tr_{1324} + Tr_{1423}\right) + t\left(Tr_{1243} + Tr_{1342}\right) + u\left(Tr_{1234} + Tr_{1432}\right)\right). \quad (4.97) \end{aligned}$$

Now we will choose an $\overline{MS}$ scheme for the three-loop renormalization, where the necessary counterterms are

$$P_{51}^{c.t.} = \frac{e^{-3\gamma\varepsilon}}{(4\pi)^{9-3\varepsilon}}\frac{1}{6\varepsilon}, \quad (4.98) \qquad P_{N_c^1}^{c.t.} = \frac{e^{-3\gamma\varepsilon}}{(4\pi)^{9-3\varepsilon}}\frac{\zeta_3}{2\varepsilon}. \quad (4.99)$$

Notice that the factor of $\left(-l_5^2\right)^{-3\varepsilon}$ in eqs. (4.91) and (4.96) is absent in the counterterm contributions (4.98) and (4.99).

We will discuss in detail only the graph 51 for which we consider the following subtracted integral,

$$I_{51}^{sub} \equiv -i\int \frac{d^{6-2\varepsilon}l_5}{(2\pi)^{6-2\varepsilon}} \frac{N_{51}(l_5)}{l_5^2\left(l_5-k_1\right)^2\left(l_5-k_{12}\right)^2\left(l_5-k_{123}\right)^2}\left[P_{51}\left(l_5^2\right) + P_{51}^{c.t.}\right]. \quad (4.100)$$

The quadratic terms in the numerator factor N_{51} are given by,

$$N_{51}^{quad} = \frac{1}{2}\left[-6\left(t\tau_{15}^2 + u\tau_{25}^2 + s\tau_{35}^2\right) + 5\left(s\tau_{15}\tau_{25} + t\tau_{25}\tau_{35} + u\tau_{15}\tau_{35}\right) - \left(s^2 + t^2 + u^2\right)l_5^2\right]. \quad (4.101)$$

The $1/\varepsilon^2$ and $1/\varepsilon$ terms in I_{51}^{sub} in eq. (4.100) are correctly captured by

$$I_{51}^{sub} = -i\left(\frac{5}{D} - \frac{1}{2}\right)\left(s^2 + t^2 + u^2\right)\int \frac{d^D l_5}{(2\pi)^D} \frac{P_{51}\left(l_5^2\right) + P_{51}^{c.t.}}{\left(l_5-k_1\right)^2\left(l_5-k_{12}\right)^2\left(l_5-k_{123}\right)^2} + \mathrm{O}(\varepsilon). \quad (4.102)$$

To extract the UV pole it is sufficient to simplify it to the form of a massive bubble integral by rearranging the external momenta,

$$I_{51}^{sub} = -i\left(\frac{5}{D} - \frac{1}{2}\right)\left(s^2 + t^2 + u^2\right)\int \frac{d^D l_5}{(2\pi)^D} \frac{P_{51}\left(l_5^2\right) + P_{51}^{c.t.}}{\left(l_5^2\right)^2\left(l_5-k_{12}\right)^2} + \mathrm{O}\left(\varepsilon^0\right). \quad (4.103)$$

The momentum-independent parts of these bubble integrals are given by

$$P^{(1)}(2+3\varepsilon,1;6-2\varepsilon)=-\frac{e^{-\gamma\varepsilon}}{(4\pi)^{3-\varepsilon}}\left[\frac{1}{8\varepsilon}+\frac{7}{16}+\mathrm{O}(\varepsilon)\right], \quad (4.104)$$

$$P^{(1)}(2,1;6-2\varepsilon)=-\frac{e^{-\gamma\varepsilon}}{(4\pi)^{3-\varepsilon}}\left[\frac{1}{2\varepsilon}+1+\mathrm{O}(\varepsilon)\right]. \quad (4.105)$$

Including the overall factors, we get,

$$\begin{aligned} I_{51}^{sub} &= \left(\frac{5}{6-2\varepsilon}-\frac{1}{2}\right)\frac{e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}\left(s^2+t^2+u^2\right)\times\left\{\left[\frac{1}{6\varepsilon}+\frac{25}{9}+\zeta_3-\frac{10}{3}\zeta_5\right]\left(\frac{1}{8\varepsilon}+\frac{7}{16}\right)-\frac{1}{6\varepsilon}\left(\frac{1}{2\varepsilon}+1\right)+\mathrm{O}(\varepsilon^0)\right\} \\ &= \frac{e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}\frac{s^2+t^2+u^2}{24}\left[-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{29}{18}+\zeta_3-\frac{10}{3}\zeta_5\right)\right]+\mathrm{O}(\varepsilon^0). \quad (4.106) \end{aligned}$$

Similarly, the N_c^1 double-trace contribution is obtained using the same formula (4.103) with P_{51} replaced by $P_{N_c^1}$, taken from eqs. (4.96) and (4.99),

$$\begin{aligned} I_{N_c^1}^{sub} &= \left(\frac{5}{6-2\varepsilon}-\frac{1}{2}\right)\frac{e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}\left(s^2+t^2+u^2\right)\times\left\{\left[\frac{\zeta_3}{2\varepsilon}+\frac{23}{6}\zeta_3+\frac{3}{4}\zeta_4-\frac{10}{3}\zeta_5\right]\left(\frac{1}{8\varepsilon}+\frac{7}{16}\right)-\frac{\zeta_3}{2\varepsilon}\left(\frac{1}{2\varepsilon}+1\right)+\mathrm{O}(\varepsilon^0)\right\} \\ &= \frac{e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}\frac{s^2+t^2+u^2}{24}\left[-\frac{3\zeta_3}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{1}{3}\zeta_3+\frac{3}{4}\zeta_4-\frac{10}{3}\zeta_5\right)\right]+\mathrm{O}(\varepsilon^0). \quad (4.107) \end{aligned}$$

We notice that, similarly to the numerator factor N_{51}^{quad} in eq. (4.101), both (4.106) and (4.107) have manifest permutation symmetry. Plugging eqs. (4.106) and (4.107) into the full amplitude, including the double-trace part of the color factors, the sum over all 24 permutations, and the overall prefactor, we obtain,

$$\begin{aligned} \mathrm{A}_4^{(4)}\Big|_{pole51-85}^{sub} &= \frac{g^{10}\mathcal{K}e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}N_c\left(Tr_{12}Tr_{34}+Tr_{13}Tr_{24}+Tr_{14}Tr_{23}\right)\left(s^2+t^2+u^2\right) \\ &\times\left\{-\frac{N_c^2+36\zeta_3}{2\varepsilon^2}+\frac{1}{\varepsilon}\left[N_c^2\left(\frac{29}{18}+\zeta_3-\frac{10}{3}\zeta_5\right)+4\zeta_3+9\zeta_4-40\zeta_5\right]\right\}. \quad (4.108) \end{aligned}$$

We note that the eq. (4.102), can be rewritten also as follows:

$$\begin{aligned} I_{51}^{sub} &= -i\left(\frac{5}{D}-\frac{1}{2}\right)\left(s^2+t^2+u^2\right)\int\frac{d^D l_5}{(2\pi)^D}\frac{P_{51}(l_5^2)+P_{51}^{c.t.}}{(l_5-k_1)^2(l_5-k_{12})^2(l_5-k_{123})^2}+\mathrm{O}(\varepsilon)\Rightarrow \\ &\Rightarrow -i\left(\frac{5}{D}-\frac{1}{2}\right)\left(s^2+t^2+u^2\right)\int\frac{d^D l_5}{(2\pi)^D}\frac{P_{51}(l_5^2)+P_{51}^{c.t.}}{(l_5^2)^2(l_5-k_{12})^2}+\mathrm{O}(\varepsilon^0)\Rightarrow \\ &\Rightarrow\left(\frac{5}{6-2\varepsilon}-\frac{1}{2}\right)\frac{e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}\left(s^2+t^2+u^2\right)\times\left\{\left[\frac{1}{6\varepsilon}+\frac{25}{9}+\zeta_3-\frac{10}{3}\zeta_5\right]\left(\frac{1}{8\varepsilon}+\frac{7}{16}\right)-\frac{1}{6\varepsilon}\left(\frac{1}{2\varepsilon}+1\right)+\mathrm{O}(\varepsilon^0)\right\} \\ &= \frac{e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}\frac{s^2+t^2+u^2}{24}\left[-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{29}{18}+\zeta_3-\frac{10}{3}\zeta_5\right)\right]+\mathrm{O}(\varepsilon^0). \quad (4.108b) \end{aligned}$$

Also this equation (there are the numbers 8 and 24), can be related with the Ramanujan equation that has 8 "modes" corresponding to the vibrations of the superstrings. Thence, we obtain:

$$I_{51}^{sub} = -i\left(\frac{5}{D}-\frac{1}{2}\right)\left(s^2+t^2+u^2\right)\int \frac{d^D l_5}{(2\pi)^D}\frac{P_{51}\left(l_5^2\right)+P_{51}^{c.t.}}{(l_5-k_1)^2(l_5-k_{12})^2(l_5-k_{123})^2}+O(\varepsilon)\Rightarrow$$

$$\Rightarrow -i\left(\frac{5}{D}-\frac{1}{2}\right)\left(s^2+t^2+u^2\right)\int \frac{d^D l_5}{(2\pi)^D}\frac{P_{51}\left(l_5^2\right)+P_{51}^{c.t.}}{\left(l_5^2\right)^2(l_5-k_{12})^2}+O\left(\varepsilon^0\right)\Rightarrow$$

$$\Rightarrow\left(\frac{5}{6-2\varepsilon}-\frac{1}{2}\right)\frac{e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}\left(s^2+t^2+u^2\right)\times\left\{\left[\frac{1}{6\varepsilon}+\frac{25}{9}+\zeta_3-\frac{10}{3}\zeta_5\right]\left(\frac{1}{8\varepsilon}+\frac{7}{16}\right)-\frac{1}{6\varepsilon}\left(\frac{1}{2\varepsilon}+1\right)+O\left(\varepsilon^0\right)\right\}$$

$$=\frac{e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}\frac{s^2+t^2+u^2}{24}\left[-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{29}{18}+\zeta_3-\frac{10}{3}\zeta_5\right)\right]+O\left(\varepsilon^0\right)\Rightarrow$$

$$\Rightarrow 8=\frac{1}{3}\cdot\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.108c)$$

Finally, we add the contribution (4.89) from the graphs 1-50 that have no subdivergences, in order to obtain the total four-loop divergence:

$$A_4^{(4)}\Big|_{pole}^{double-trace}=A_4^{(4)}\Big|_{pole1-50}^{double-trace}+A_4^{(4)}\Big|_{pole51-85}^{sub}=\frac{g^{10}\mathcal{K}e^{-4\gamma\varepsilon}}{(4\pi)^{12-4\varepsilon}}N_c\left\{\left(Tr_{12}Tr_{34}+Tr_{14}Tr_{23}+Tr_{13}Tr_{24}\right)\left(s^2+t^2+u^2\right)\right.$$

$$\times\left[-\frac{N_c^2+36\zeta_3}{2\varepsilon^2}+\frac{1}{\varepsilon}\left(N_c^2\left(\frac{35}{18}-\frac{\zeta_3}{3}\right)+4\zeta_3+9\zeta_4+20\zeta_5\right)\right]-\frac{3}{\varepsilon}\left(N_c^2\zeta_3+25\zeta_5\right)\left(Tr_{12}Tr_{34}s^2+Tr_{14}Tr_{23}t^2+\right.$$

$$\left.\left.+Tr_{13}Tr_{24}u^2\right)\right\}. \quad (4.109)$$

Of course, the double-trace part of the four-loop counterterm must be chosen to cancel these poles,

$$A_{4;c.t.}^{(4)}\Big|^{double-trace}=-A_4^{(4)}\Big|_{pole}^{double-trace}+O(1), \quad (4.110)$$

corresponding to a nonvanishing divergent coefficient for a counterterm of the schematic form, $Tr\left(\mathcal{D}^{4-k}F^2\right)Tr\left(\mathcal{D}^k F^2\right)$. In conclusion, the double-trace terms in the four-point $\mathcal{N}=4$ sYM amplitude to diverge at four-loops, saturating the double-trace finiteness bound of $D_c=4+8/L$.

Appendix A.

The new possible method of factorization of a number (principally applied to the numbers 496 and 6480)

With regard this new possible method of factorization, we can obtain the following conclusion:

We consider, for example, the N=(3^2)x+5;

[Similar conclusions will be valid for the cases : N=(3^2)x+1;N=(3^2)x+3;N=(3^2)x+7. And of course, also for the cases in which the square is different from 3^2=9.]

We take for example the follow: N=185=5*37=9*20+5

We can identify N among the factors on the right of the following column:

1*5 R=4
2*14 R=8
3*23 R=12
...
21*185 R=84
...

In addition, among the factors of right, we'll find the multiples of N in the following points:

...
206*(185*10) R=824=84+(185*4)
...
391*(185*19) R=1564=84+(185*8)
...

To factorize N we must try a multiple, among the factors in the right of this column, which corresponds R equal to a number squared. In this case, the multiple is the following:

...
576*(185*28) R=2304=48^2=84+(185*12)

Where R divided by 4 gives a number squared equal to:

Q'=576=24^2=21+(185*3)

[Q' is equal to the left factor of the column.]

This means that will be enough to find a square equal to: 21+185y ;where y, in this particular column, can take any integer value. Generalizing, we can say that for each N(odd)=9x+5, we need to find:

Q'=[(N-5)/9+1]+Ny

And some of those Q', there are an infinite number, can be calculated as following:

Q'1=[(N-2)/3]^2

Q'2={[(N-2)/3]*2+2}^2
Q'3={[(N-2)/3]*4+2}^2

Q'4={[(N-2)/3]*5+4}^2
Q'5={[(N-2)/3]*7+4}^2

Q'6={[(N-2)/3]*8+6}^2
Q'7={[(N-2)/3]*10+6}^2

etc.

The rule that generates them, should be evident. The point is that these aren't the squares from which it is possible factorize N. The squares that are needed instead are obtained by:

[Q'1^(1/2)-F1z]^2   [Where F1z is a multiple of one of the factors of N]

[Q'2^(1/2)+F1z]^2
[Q'3^(1/2)-F1z]^2

[Q'4^(1/2)+F1z]^2
[Q'5^(1/2)-F1z]^2

etc.

Also in this case, the rule should be evident.

In the case of N=185=5*37, with F1z=37*1, the various Q' that we need are:

(61-37)^2=24^2=21+185*3

[(61*2+2)+37]^2=161^2=21+185*140
[(61*4+2)-37]^2=209^2=21+185*236

[(61*5+4)+37]^2=346^2=21+185*647
[(61*7+4)-37]^2=394^2=21+185*839

etc.

From these squares can be factorized N. But we need to know the factors of N.

Taking always for example N=185, we can say that: being Q'1=61^2=Som.1/121, to obtain the factorization of N is necessary to find a multiple of N = Som. 121/q [where q is the minor term of the summation.]

In this case, given that:

Som 1/121=Q'1=61^2=21+185*20 and;

Som 1/47=(61-37)^2=21+185*3

we have that:

Som 121/49=185*17

In summary, the problem at this point is, given N(odd)=9x+5, calculate:

Som. {[(N-2)/3]*2-1}/ q = multiple of N [where q is the minor term of the summation.]

We don’t know if this is a problem easily solved, regardless of the knowledge of the factors of N, but considering the initial problem for factorize an any N, it seems to us should be a small step forward. The initial problem was, in fact, given N, calculate Q=R+(Ti+2)+(Ti+4)+... + (Ti+x)

According to the rules previously mentioned, we know that, in the case of N(odd)=9x+5,

[(N-2)/3]^2=[(N-5)/9+1]+(Multiple of N), and that,

{[(N-2)/3]-F1z}^2=[(N-5)/9+1]+(Multiple of N), thence,

[(N-2)/3]^2-{[(N-2)/3]-F1z}^2=(Multiple of N)

In the case of N=185=5*37,

1*5
2*14
3*23
...
21*185
we have that:

[(185-2)/3]^2=[(185-5)/9+1]+(20*185),

{[(185-2)/3]-(37*1)}^2=[(185-5)/9+1]+(3*185), and thence we have that,

[(185-2)/3]^2-{[(185-2)/3]-(37*1)}^2=(20-3)*185.

Doing the calculations, we have that:

61^2=21+20*185,

(61-37)^2=21+3*185, and thence we have that,

61^2-(61-37)^2=17*185;

that, considering the squares as sums of odd numbers, can be written also as:

Som1/121=21+20*185

(sum of all the odd numbers from 1 to 121 which is equal to 3721)

Som1/47=21+3*185

(sum of all the odd numbers from 1 to 47 that is 576), thence,

Som1/121-Som1/47=Som49/121=17*185

(In fact: 3721 – 576 = 3145 = 17 * 185)

In conclusion, in the case of N=185, we need to factorize: Som q/121=Multiple of 185, where q(49) is the minor term of the summation .

In the case of 6480, with Q equal to 9, we will proceed to obtain a number not divisible by 9 that we can factorize in the manner indicated above: 6480/9=720; 720/9=80, from which (80*4)=9*35+5. We note also that: 6480 = 24*270 = 24*9*30.

This type of calculations that we have shown, can be applied only to numbers of the form 5 + 9x. A particular form from which we obtain R = 4 in correspondence of the first product and an increase of R of the same value:

1*5 R=4 (5 = 5+9*0)
2*14 R=8 (14 = 5+9*1)
3*23 R=12 (23 = 5+9*2)
...

The basic principle, according to which to reach to the factorization of N is necessary to find a square equal to a multiple of N added to another number, linked to N itself, also applies to other cases. Cases, however, do not exhibit the same conditions as that in which N = 5 + 9x.
So, for factorize the number 496, multiply it so as to obtain a number of desired form.
It will not change the final results.

N'=496*5=2480, indeed: N'=5+9*275

Now, we know that:

[(N'-2)/3]^2=[(N-5)/9+1]+(Multiple of N'), i.e.; [(2480-2)/3]^2=[(2480-5)/9+1]+(Multiple of N'),

826^2=276+2480*275.

That we can also write as:

Som 1/1651=276+2480*275

For factorize N' we have to find the following square:

(826-F1z)^2=276+(Multiple of N')
(826-F2z')^2=276+(Multiple of N')
(826-F3z")^2=276+(Multiple of N')

(826-F4z''')^2=276+(Multiple of N')
(826-F5z'''')^2=276+(Multiple of N')

[The squares are 5 in total, one less than the number of factors of N' (2;2;2;2;31;5).]

The squares in question are:

(826-660)^2=276+2480*11 ;whence:
826^2-(826-660)^2=Som1/1651-Som1/331=Som333/1651=2480*264
(826-620)^2=276+2480*17
826^2-(826-620)^2=Som1/1651-Som1/411=Som413/1651=2480*258
(826-412)^2=276+2480*69
826^2-(826-412)^2=Som1/1651-Som1/827=Som829/1651=2480*206
(826-372)^2=276+2480*83
826^2-(826-372)^2=Som1/1651-Som1/907=Som909/1651=2480*192
(826-40)^2=276+2480*249
826^2-(826-40)^2=Som1/1651-Som1/1571=Som1573/1651=2480*26

Not knowing a priori the values of z, z ', z"; ... , the only way to proceed in order to obtain the various "F" is the following:

F1z=660=2*2*3*5*11; from which, at least in one case: F1=2; z=330; or in another:

F1=5; z=132

F1z'=620=2*2*5*31

F2=2;

z'=310 F2=31; z'=20

F2z"=412=2*2*103

F3=2; z"=206 F3=2;
z"=206

F3z'''=372=2*2*3*31

F4=31; z'''=12 F4=2;
z'''=186

F4z''''=40=2*2*2*5

F5=5; z''''=8
F5=2; z''''=20

[The ways in which the various "F" and relative "z" are ranked are only some of the possible, precisely because of the current difficulty of knowing a priori of the values of the various "z".]

It is therefore a result inevitably partial, but certainly useful.
We remember that 16 is connected to the 496, indeed 496 = 16 * 31 and that 16 = 2 * 8, thence is connected with the Ramanujan's function concerning the modes corresponding to the physical vibrations of the superstrings, i.e.:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4} \right)} \right]} . \quad (1)$$

SCHEME

In this scheme, the # sign indicates multiplication

SX /// DX

... / 1#17 / 1#15 / 1#13 / 1#11 / 1#9 / 1#7 / 1#5 / 1#3 / 1#1 /// 2#2 / 2#4 / 2#6 / 2#8 // 3#3 / 3#5 / 3#7 / 3#9 // 4#4 / 4#6 / 4#8 / 4#10 // 5#5 / 5#7 / 5#9 / 5#11 // 6#6 / ...
... / 2#26 / 2#24 / 2#22 / 2#20/ 2#18/2#16/ 2#14/2#12/2#10///3#11/3#13/3#15/3#17 //4#12/4#14/4#16/ 4#18//5#13/5#15/5#17/5#19//6#14/6#16/6#18/6#20// 7#15/ ...
... /3#35 /3#33 / 3#31 / 3#29/ 3#27 /3#25/3#23/3#21 /3#19///4#20/4#22/4#24/4#26//5#21/ 5#23/5#25/5#27//6#22/ 6#24/6#26/6#28//7#23/7#25/7#27/7#29// 8#24/ ...
... /4#44/ 4#42 / 4#40 / 4#38/ 4#36/ 4#34/4#32/4#30/4#28/// 5#29/5#31/5#33/5#35//6#30/ 6#32/6#34/6#36//7#31/7#33/ 7#35/7#37//8#32/8#34/8#36/8#38//9#33/ ...
... ...

Let us return to the method used to factorize 6480 and us refer, for the moment, to the scheme above. Wanting to place the number in the right part of one of the pairs of factors present in it, the first step is to verify which of the following forms the number belongs.

F1=9x+1
F2=9x+3
F3=9x+5
F4=9x+7
F5=9x+9
F6=9x+11
F7=9x+13
F8=9x+15
F9=9x+17

6480=9*719+9

Now, since 6480 a multiple of 9 can not in any way onward, through multiplication, to the form 9x +5, which would allow me to factorize this with the usual method. Method believe that however should have a variant refers to all other forms, excluding the one to which belongs 6480. In fact, considering that scheme of numbers, in which the term of the right of each pair increases of Q' = 3^2, all numbers multiples of Q' behave in a different way from the others.
[The same is true for numbers multiples of 5^2 in the scheme in which the terms of the right of the various pairs increase in 5^2, for the numbers multiples of 7^2 in the scheme in which the terms of the right of the various pairs increase of 7^2.]
And as the numbers multiples of 9, behave even the numbers in which the term of the right of the pair of factors that compose them is a multiple of 3. So the numbers are divisible by Q'^1/2.

Referring to the SX part of the scheme, we note that:

(1*3=3) R=1
2*12=24 R=1
3*21=63 R=1
...

(1*9=9) R=0
2*18=36 R=0
3*27=81 R=0
...

(1*15=15) R=1
2*24=48 R=1
3*33=99 R=1
...

(1*21=21) R=4
2*30=60 R=4
3*39=117 R=4
...

(1*27=27) R=9
2*36=72 R=9
3*45=135 R=9
...

(1*33=33) R=16
2*42=84 R=16
3*51=153 R=16
...

The term of the right of each pair initial (in parentheses) is a multiple of 3, then, given that the square of which increases is 9, the number obtained from the product of each pair is a multiple of 3. In these cases the value of R is always a square, whose variation is perfectly understandable.
[Similar considerations will also apply to the pairs present in the side DX of the scheme]
The fact that 6480 is a multiple of 9 allows us to factorize without necessarily consider it as a term of the right of a pair of factors but as a product of them to which inevitably corresponds to a value

of R equal to a square.

It should also be said that every number present in this scheme should refer to a pair whose factors are both even or odd.

Then proceed as follows:

[We must also say that the natural way of proceeding would be to continue dividing N to Q, since this represents the first step, with Q equal to 9 in this case, to obtain a number not divisible by 9 to factorize in the usual manner : 6480/9 = 720; 720/9 = 80, from which (80*4) = 9*35+5]

N=6480=9*720

We divide N for 2 so as to obtain a pair whose factors are both even or odd.
6480=2*3240

from which, by subtracting a multiple of 9 by the second term (in this case 9*1) and a multiple equal to 1 (in this case 1*1) from the first term, we obtain:

1*3231

This is one of the initial pairs, where the term on the right is a multiple of 3, present at the side SX of scheme of the numbers.

SX

1*1 ... (1*3) ... (1*9) ... (1*15) ... (1*21)
... (1*27) ... (1*33) ... (1*39) ... ...
(1*3231)
R=1 R=0 R=1
R=4 R=9 R=16
R=25 R=?
Som1/1 Som0 Som1/1 Som1/3
Som1/5 Som1/7 Som1/9 Som1/?

Below each pair of terms we have reported the respective value of R which, we remember, remains unchanged.

Indeed:

1*3 R=1
2*12 R=1
3*21 R=1
...

The next step is to calculate the value of R corresponding to 1*3231. Then we compute the second term of the summation refers to R in this way:

3231=n

[(n-9)/6]*2-1

from which:

R=?=Som1/1073

from which:

R=537^2

Indeed:

1*3231 + R =540^2

[To check that R is correct we must add it to the product corresponding to it and verify that we get a square.]

Thence:

2*3240 + R=543^2

Knowing R we can factorize N:

2*3240=Som(1073+2)/x [with x > of (1073+2).]

we obtain:

F1=1080
F2=6=2*3 [The result is trivial because we know that N is divisible for 3^2.]

At this point, we note that the couple of factors (2 and 3240) that comprise N is even, thence we can proceed with a new factorization dividing by two the second term of the same pair:

2*3240=2*2*1620=4*1620

We obtain:

1*1593

Indeed:

1*1593
2*1602
3*1611
4*1620
....

Where 1 * 1593 is one of the initial couples (where the term on the right is a multiple of 3) that are present to the left (SX) of the schema of numbers.

we compute R=264^2=Som1/527

and factorize N:

4*1620=Som529/x

we obtain:

F1=540
F2=12=4*3 [Also in this case the result is trivial because we know that N is divisible for 3^2.]

Proceed with a new factorization:

4*1620=4*2*810=8*810

we obtain:

1*747

We compute R=123^2 and factorize N:

8*810=Som247/x

we obtain:

F1=270
F2=24=8*3   [as above.]

At this point we have to stop since: 8*810 = 8*2*405 = 16*405 where: 16 and 405 aren't both even or odd.

Then divide by 3 the second term of the couple. [We can do this because we know that N is divisible by 9.]

16*405=16*3*135

from which we can proceed with the factorization of the second term of the couple:

3*135

We obtain:

1*117

we compute R=18^2 and factorize 405:

3*135=Som37/x

we obtain:

F1=45
F2=9=3*3 [as above.]

Similar conclusions are obtained in all the cases in which N is inserted in a pattern of numbers where the term on the right of each pair of factors increases by a number, squared, for which is divisible N.
So having N = 6480 = 2*2*2*2*3*3*3*3*5, the increases to exclude are:

Q'= 4; Q' = 16; Q'= 36; Q' = 9; Q'= 81; etc..

We remember that 6480 appear in the formula (4.42b). Indeed:

$$I_4^{(3)b}(s,t) = -\frac{e^{3\varepsilon\gamma}}{\Gamma(-2\varepsilon)(-s)^{1+3\varepsilon}t^2} \times \frac{1}{(2\pi i)^8}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty} dw dz_1 \left(\prod_{j=2}^{7} dz_j \Gamma(-z_j)\right)\left(\frac{t}{s}\right)^w \Gamma(1+3\varepsilon+w)\times$$
$$\times\frac{\Gamma(-3\varepsilon-w)\Gamma(1+z_1+z_2+z_3)\Gamma(-1-\varepsilon-z_1-z_3)\Gamma(1+z_1+z_4)}{\Gamma(1-z_2)\Gamma(1-z_3)\Gamma(1-z_6)\Gamma(1-2\varepsilon+z_1+z_2+z_3)}\times$$
$$\times\frac{\Gamma(-1-\varepsilon-z_1-z_2-z_4)\Gamma(2+\varepsilon+z_1+z_2+z_3+z_4)}{\Gamma(-1-4\varepsilon-z_5)\Gamma(1-z_4-z_7)\Gamma(2+2\varepsilon+z_4+z_5+z_6+z_7)}\times\Gamma(-\varepsilon+z_1+z_3-z_5)\Gamma(2-w+z_5)$$
$$\Gamma(-1+w-z_5-z_6)\times\Gamma(z_5+z_7-z_1)\Gamma(1+z_5+z_6)\Gamma(-1+w-z_4-z_5-z_7)\times\Gamma(-\varepsilon-z_1+z_2-z_5-z_6-z_7)$$
$$\Gamma(1-\varepsilon-w+z_4+z_5+z_6+z_7)\times\Gamma(1+\varepsilon-z_1-z_2-z_3+z_5+z_6+z_7)\Rightarrow$$
$$\Rightarrow -\frac{1}{(-s)^{1+3\varepsilon}t^2}\times\left\{\frac{16}{9}\frac{1}{\varepsilon^6}+\frac{13}{6}L\frac{1}{\varepsilon^5}+\left[\frac{1}{2}L^2-\frac{19}{12}\pi^2\right]\frac{1}{\varepsilon^4}+\left[-\frac{1}{6}L^3-\frac{67}{72}\pi^2 L-\frac{241}{18}\zeta_3\right]\frac{1}{\varepsilon^3}+\right.$$
$$+\left[\frac{1}{24}L^4+\frac{13}{24}\pi^2L^2-\frac{67}{6}\zeta_3 L-\frac{19}{6480}\pi^4\right]\frac{1}{\varepsilon^2}+\left[-\frac{1}{120}L^5-\frac{13}{72}\pi^2L^3-\frac{5}{2}\zeta_3L^2-\frac{6523}{8640}\pi^4L+\frac{1385}{216}\pi^2\zeta_3+\right.$$
$$\left.-\frac{1129}{10}\zeta_5\right]\frac{1}{\varepsilon}+\frac{1}{720}L^6+\frac{13}{288}\pi^2L^4+\frac{5}{6}\zeta_3L^3+\frac{331}{960}\pi^4L^2+\left(\frac{317}{72}\pi^2\zeta_3-\frac{1203}{10}\zeta_5\right)L-\frac{180631}{3265920}\pi^6+$$
$$\left.-\frac{163}{6}\zeta_3^2+O\left(\frac{s}{t}\right)\right\}. \quad (2)$$

Furthermore, we note that $6480 = 2^4 * 3^4 * 5$ where 2, 3 and 5 are also Fibonacci's numbers and prime numbers and 16 is equal to 8 * 2, where 8 is connected with the following Ramanujan's function:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (3)$$

Now we consider the following equation:

1°) [(72x'+44+2N+18NM-18Nx)/6]^2=[(36Q-9V+36x'+20)/4]*[(4Q-V+36x'+20)/4]+4Q where:

2°) x'={-180+[180^2+4*144*(V-56)]^1/2+}/288

3°) V=4(Nx+R)-4y

and:

4°) x=(288^2*y^2-18432Ny-9216y+32^2*N^2+1280N+256-48^2*R)/(48^2*N)

with N, M, Q and R equal to different known numeric values.

This equation represents a try of generalization of the previous method, for instance the one applied to the number 496. A try that, as the method to which it is addressed, is connected to the schema of numbers that we have obtained.
In this scheme and its variants, (one attached refers to an increase in the first term of each pair equal to 1^2 and of the second term equal to 3^2. It is the first diagram obtained and also the one provided with further demonstrations . But we can to obtain the other, similar, depending from the values of the square in question), are also connected to the equations for the factorization of the numbers of which we know the approximate ratio between the factors.

Scheme of all the non-prime numbers such as: N=5+8x.

```
      __
1t  I__  1°=21                                   =21
    I     2°=21+(24)                             =45
2t  I__  3°=2°+(0)                               =45
    I     4°=21+(24+24)                          =69
3t  I     5°=4°+(16)                             =85
    I__   6°=5°+(16-24)                          =77
    I     7°=21+(24+24+24)                       =93
4t  I     8°=7°+(16+16)                          =125
    I     9°=8°+(16+16-24)                       =133
    I__10°=9°+(16+16-24-24)                      =117
    I   11°=21+(24+24+24+24)                     =117
    I   12°=11°+(16+16+16)                       =165
5t  I   13°=12°+(16+16+16-24)                    =189
    I   14°=13°+(16+16+16-24-24)                 =189
    I__15°=14°+(16+16+16-24-24-24)               =165
    I   16°=21+(24+24+24+24+24)                  =141
    I   17°=16°+(16+16+16+16)                    =205
6t  I   18°=17°+(16+16+16+16-24)                 =245
    I   19°=18°+(16+16+16+16-24-24)              =261
```

```
    I   20°=19°+(16+16+16+16-24-24-24)                 =253
    I__21°=20°+(16+16+16+16-24-24-24-24)              =221
    I   22°=21+(24+24+24+24+24+24)                     =165
    I   23°=22°+(16+16+16+16+16)                        =245
    I   24°=23°+(16+16+16+16+16-24)                    =301
7t  I  25°=24°+(16+16+16+16+16-24-24)                  =333
    I  26°=25°+(16+16+16+16+16-24-24-24)               =341
    I  27°=26°+(16+16+16+16+16-24-24-24-24)            =325
    I_28°=27°+(16+16+16+16+16-24-24-24-24-24)        =285
    I ...
    I ...
    I ...
```

We note that in this scheme there are the number 16 and 24 (where 16 = 2 * 8), numbers connected respectively to the modes corresponding to the physical vibrations of the superstrings and of the bosonic strings, through the following Ramanujan's equations:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} . \quad (4)$$

$$24 = \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} . \quad (5)$$

We know that the factorization of a number can be obtained from the difference of two squares, but also, in the case where N=5+8x, by the following difference:

[Y(M16-12)+M40+45]-12Y^2=N

with M>(Y-1), from which we obtain N divisible by (2Y+5).

The formula is obtained from the schema of above, which in turn is derived from the schema of the initial numbers. Moreover this is not valid for the first number of each section indicated on the left, i.e., for the numbers of the type:

N = 21+24 z.

There will be similar formulas for the cases where: N = 1+8 x, N = 3+8 x and N = 7+8 x.

Now we briefly describe the method that we have used to arrive at the scheme of numbers and we illustrate successfully a part of it entirely similar to the other.

Suppose that we have:

N = 13 with R = 3; Ti = 7 and Tu = 7 +2 = 9

We use “Tu”, which is the term odd next to “Ti”.

Let us the problem of obtaining R starting from the sum of the differences between the various terms odd next to it and the sums in which can be decompose the squares.

In the case of N = 13 and Tu = 9 we consider Q = 36 = Som1/11= (1+11) + (3+9) + (5+7) = 3*12 and we proceed as follows:

(Tu-12) + (Tu +2-12) + (Tu +4-12) = (9-12) + (11-12) + (13-12) = -3-1 +1 = -3

This means that starting from Tu = 9 and adding the two terms odd subsequent to it, we can obtain a square simply by adding 3. In fact:

Q = 36 = (9 +11+13)+3

Where 3 correspond to the value that we want to get.

In particular we can see as for each value of N for which R corresponds to Tu /3 the square that must be considered is always (2R)^2.
Thence we can immediately calculate what is the square which added to N provides us with another square, and we are able to factorize N.

In fact doing the same for the other possible odd values of R, we obtain:

N = 13, R = 3; Tu = 9 from which: Q = 36 and therefore N = 1*13
N = 44, R = 5; Tu = 15 from which: Q = 100 and therefore N = 2*22
N = 93, R = 7; Tu = 21 from which: Q = 196 and therefore N = 3*31
...

Here is as come out the products between factors that increase from time to time of 1 and 9.
This represents the starting point.

The operations can, however, be as follows:

N = 13 = Som1/13-Som1/11 = 7^2-6^2
N = 44 = Som1/23-Som1/19 = 12^2-10^2
N = 93 = Som1/33-Som1/27 = 17^2-14^2
...

The important fact, however, regard the values of “R” and “Tu” for each number that follow a clear pattern. The one from which we can derive the formulas for the factorization.

Generalizing the case we obtain the following diagram that represents a part of the general one.

.../ 31^2-30^2=1*61[R=83;Tu=25] /19^2-18^2=1*37[R=27;Tu=17]/ 7^2-6^2=1*13[R=3;Tu=9]/ 5^2-2^2=3*7[R=15;Tu=13]/ 8^2-2^2=6*10[R=61;Tu=33]/...
.../ 36^2-34^2=2*70[R+2;Tu+6] / 24^2-22^2=2*46[R+2;Tu+6] /12^2-10^2=2*22[R+2;Tu+6]/10^2-6^2=4*16[R+2;Tu+6] /13^2-6^2=7*19[R+2;Tu+6] /...
.../ 41^2-38^2=3*79["] / 29^2-26^2=3*55["] /17^2-14^2=3*31["] /15^2-10^2=5*25["] /18^2-10^2=8*28["] /...

It is noted that the starting values of R and Tu follow a particular consequence, in this scheme as in the other, that are similar, which will form the general one, identical to that seen previously.
[To verify the starting values of the values of “R” and “Tu” just do the test on the products that we encounter as we go down each column.]

The kind of increasing trend of the values of “R” and “Tu” we understand also why those that we get directly from N correspond to the exact, and therefore useful to the factorization, only from a certain point onwards. Precisely, when R becomes less than “Tu”, that is, when the relationship between the two factors of N approaches the square, in this case 9, of which increases the factor of right.

However, as we have said earlier, just increase R and consequently “Tu”, for use effectively the formulas also when the relationship between the factors moves very away from the square.

We believe that there is very much to be explored in this regard.

We return to the formula

[Y(M16-12) +M40+45]-12Y^2 = N

with M>(Y-1), from which we obtain N divisible by (2Y+5).
The formula works as follows:

We consider the pattern of numbers seen previously, the term M in the formula indicates the amount of “16” present in the various summations, while the term Y the quantity of “24”.

Furthermore, the first number of each section is divisible by 3, the second is divisible by 5, the third is divisible by 7, and so on.
Therefore, given that the formula does not allow to calculate the first numbers of each section, those of the type 21+24z, divisible by 3, we obtain N divisible (for 2Y+5).

In fact, putting Y = 0 and M = any number, considering thence the second number of each section, is obtained from the formula: N divisible by 5.

To give an example:

the number 285, the 28th of the scheme, we can calculate it from the formula putting Y = 5 (the

“24” is repeated 5 times) and M = 5 (the “16” is repeated 5 times).

From which:

5*(5*16-12) +5*40 +45-12*5^2 = 285, divisible by: 2Y+5 = 15. In fact:

285 = 15*19

The number 253, the 20th of the schema, we can calculate it placing Y = 3 and M = 4. From which:

3*(4*16-12) +4*40 +45-12* 3^2 = 253, divisible by: 2Y +5 = 11. In fact:

253 = 11*23

In conclusion, the best way to proceed to the factorization of a number, without of course be able to locate it in the diagram, we believe that can be the following:

we consider all the possible values of M, by building the following series:

M = 1;   Y(4)-12Y^2 = N-45-40
M = 2;   Y(20)-12Y^2 = N-45-80
M = 3,   Y(36)-12Y^2 = N-45-120
...

So if, for example, N = 389 = 5 +8*48, we obtain:

M = 1;   Y (4)-12Y^2 = 304
M = 2;   Y (20)-12Y^2 = 264
M = 3,   Y (36)-12Y^2 = 224
M = 4,   Y (52)-12Y^2 = 184
M = 5,   Y (68)-12Y^2 = 144
M = 6,   Y (84)-12Y^2 = 104
...

At this point, we just have to try Y values less than or equal to M to obtain a new series unchanged within which identify N. In this case, however, N is a prime number, thence, none of the possible values of Y may be fine.

With N = 405 = 5 +8*50 instead we get:

M1;…. = 320
M2; ... = 280
M3;… = 240
M4;… = 200
...

Trying the values of Y we get:

M1; (320) Y = 0; = 0   M2; (280) Y = 0; = 0   M3; (240) Y = 0; = 0
M4; (200) Y = 0; = 0   M5; (160) Y = 0; = 0   M6; (120) Y = 0; = 0

Y = 1; = -8 Y = 1; = 8 Y = 1; = 24
Y = 1; = 40 Y = 1; = 56 Y = 1; = 72

Y = 2; = -8 Y = 2; = 24 Y = 2; = 56 Y = 2; = 88 Y = 2; = 120 [YES]

Y = 3; = 0, Y = 3; = 48 Y = 3; = 96 Y = 3; = 144

Y = 4; = 16 Y = 4; = 80 Y = 4; = 144

Y = 5; = 40 Y = 5; = 120 [YES]

Y = 6; = 72

Obtained the correspondence with M6;(120), Y = 2 and Y = 5, we can say that:

2*(6*16-12) +6*40 +45-12*2^2 = 5*(6*16-12) +6*40 +45-12*5^2 = 405

From which:

405/(2Y+5) = 405/9 = 45 and 405/15 = 27

From the series shown above, which remains unchanged as we said, we can also see many types of orders, from which it may be possible to speed up the steps.

Also in this possible method of factorization, it is evident that the various and useful numerical results concerning the numbers 496 and 6480, can be considered as news possible solutions concerning the equations of the string theory (superstrings and bosonic strings). This is the goal, the fundamental result of this interesting Appendix.

Appendix B.

NEW MATHEMATICAL OBSERVATIONS CONCERNING VARIOUS NUMBERS INCLUDED IN SOME EQUATIONS REGARDING THE RELATIONSHIP BETWEEN YANG-MILLS THEORY AND GRAVITY

First provisional observations

First series (from eqs. 4.42):

2, 5, 6, 9, 10, 12, 13, 16, 17, 18, 19, 24, 67, 72, 120, 163, 216, 241,

288, 317, 331, 341, 720, 960, 1129, 1203, 1385, 6480, 6523, 8640,

180631, 3265920

Numbers in green near to the Fibonacci's numbers, that we will cover in the unified Table

Second series (from eqs. 4.71-4.81 and 4.88)

5, 8, 21, 25, 32, 64, 75, 105, 125, 128, 224, 256, 288, 496, 512, 576,

832, 896, 928, 2048, 2625, 4352, 5248, 6272, 12352, 12992, 15552,

20992.

224 =14*16, 14=G2

496 perfect number and double of 248 = E8

5, 8 and 21 Fibonacci's numbers

Blue = multiple of 5 and 25

Red = powers of 2

UNIFIED TABLE FIRST AND SECOND SERIES

In red the Fibonacci's numbers, Lie's numbers or partitions that coincide with numbers of the two series

FIRST SERIES	SECOND SERIES	FIBONACCI	LIE	PARTITIONS	
2		2	3	2	3
5	5	5		5	
6	8	8	7	7	

9		8		
10				
12				11
13		13	13	13 mean between 11 and 15
16				15
17				
18				
19	21	21	21	22
24	25 32	34	31	30 42
67	64	55	57	56
72	75	72 mean between 55 and 89	73	77
				96 mean between 72 and 120
120	105 125 128		133	135
163		144	133	176
216	224	233	241	
241	256	233	241	231
288	288	305 meand 233 and 610	273	297
317		305 mean 233 and 610	307	
331		377	381	
341		377	381	385
	496	493,5 mean between 377 and 610	507	490
	512		507	490
	576		577	627
720			703	627
	832	798,5 mean between 610 e 987		792
	896		871	897 mean between 792 and 1002
960	928	987	931	

1129			1123		
1203			1191	1255	
1385		1597	1333	1255	
	2048		2071	1958	
	2625	2584	2653		
	4352	4182	4231	4565	
	5248		5257	5604	
6480			6481		Diff. 1
6523			6643		
	6762	6766	6807	6842	
8640			8557	8349	
	12352	10498	12433	12310	
	12992		12883	12310	
	15552	14106 mean	15501	16430 mean 14883 and 17977	
180631			180201	173525	
	20992	28212	20881		
3265920			3267057	3238993 not real but valued	

TABLE FACTORIZATIONS

Numbers first series	Factors	Numbers second series	Factors	Observations
2	prime	5	prime	
5	prime	8	2^3	
6	2*3			
9	3^2			

10	2*5			
12	2^2*3			3 Lie’s number
13	prime			
16	2^4			
17	prime			
18	2*3^2	21	3*7	3 and 7 Lie’s numbers
19	prime	25	5^2	
		32	2^5	
24	2^3*3	64	2^6	3 Lie’s number
67	prime	75	3*5^2	3 Lie’s number
72	2^3*3^2			3 Lie’s number
		105	3*5*7	7 Lie’s number
		125	5^3	
		128	2^7	
120	2^3*3*5			3 Lie’s number
163	prime	224	2^5*7	7 Lie’s number
216	2*4*3^3	256	2^8	3 Lie’s number
241	prime	288	2^5*3^2	3 Lie’s number
288	2^5*3^2			3 Lie’s number
317	prime			
331	prime			
341	11*31	496	2^4*31	31 Lie’s number
		512	2^9	
		576	2^6*3^2	3 Lie’s number
720	2^4*3^2*5	832	2^6 *13	13 Lie’s number
		896	2^7*7	7 Lie’s number
		928	2^5*29	
960	2^6*3*5			3 Lie’s number
1129	prime			
1203	3*701			
1385	5*277	2048	2^11	

		2625	3*5^3*7	3 and 7 Lie's numbers
		4352	2^8*17	
		5248	2^7*41	
6480	2^4*3^4*5			3 Lie's number
6523	11*593	6762	2*3*7^2*23	7 Lie's number
8640	2^6*3^3*5	12352	2^6*193	3 Lie's number
		12992	2^6*7*29	7 Lie's number
		15552	2^6*3^5	3 Lie's number
180631	11*16421	20992	2^9*41	
3265920	2^7*3^6*5*7			3 and7 Lie's numbers

We note that 6480 = 2^4 * 3^4 * 5 where 2, 3 and 5 are also Fibonacci's numbers and prime numbers and 16 is equal to 8 * 2, where 8 is connected with the following Ramanujan's function:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4} \right)} \right]} . \quad (1)$$

Final observations

Between almost all the factors of the numbers of the two series, there are always the Lie's numbers smaller, for example 3, 7, 13, and 31, alone or together, especially 3 and 7, and the product 3*7= 21 other Lie's number which however doesn't appear directly between the various factors, but as 3*7 where 3 and 7 are among the factors of a number of the two series, for example 6762=21*322, or 2625=21 *125, with 125 number of the second series; finishing at 3265920=21*155520 , with 155520 = 15552*10 , with 15552 number

of the second series. This fact (higher frequency of Lie's numbers 3, 7,13, 21 and 31 between the prime factors and not the primes, but only the number 21 between them) certainly reflects the symmetries of the exceptional groups of Lie in the natural phenomenon treated in this work. Other frequently factors are powers of 2 and 3. Wer also remember that some powers of 2 are between the numbers of the second series (those maked in red in the first provisional observations):

8 = 2^3,

32 = 2^5

64 = 2^6

128= 2^7

256= 2^8

512 = 2^9

2048= 2^11

that often appear between the factors of some numbers (in addition to being themselves numbers of the second series), for example 2^7 = 128 in the number 3265929 (first series), that, indie, is divisible by 128, since 3265920 = 128* 25515.
About the Lie's groups, we recall that the number of their dimension is a multiple of the Lie's numbers.

G2=14 = 2*7

F4 = 52 = 4*13

E6 = 78 = 6*13

E7= 133 =19*7

E8 = 248 = 8*31

Tables with 2^a*k with k numbers of the series

TABLE 1

2^a	2^e *k numbers of the first series	k numbers of the two series	observations
2^2	2^2*3 =12	3	no
2^3	2^3*3 =24	3	no
2^3	2^3*3^2= 72	9	yes, but of the first series
2^3	2^3*3*5 =120	15	no
2^5	2^5*3^2 =288	9	yes, but of the first series
2^4	2^4*3^2*5 =720	45	no
2^6	2^6*3*5 =960	15	no
2^4	2^4*3^4*5=6480	405	no
2^6	2^6*3^3*5 =8640	135	no
2^7	2^7*3^6*5*7=3265920	25515	no

In this first Table 1, there is only k = 9 as number of the first series, and none of the second series

Table 2 with powers of 2 and numbers of the second series, 2^a*k , with k numbers of the first series

2^a	2^a*k Numbers of the second series	k =numbers of the two series	observations
2^5	2^5*7 = 224	7	no
2^5	2^5*9 =288	9	yes
2^4	2^4*31 =496	31	no
2^6	2^6*13 =832	13	yes
2^7	2^7*7 =896	7	no
2^5	2^5*29 =928	29	no
2^8	2^8*17 =4352	17	yes
2^7	2^7*41= 5248	41	no

2^6	**2^6*193 = 12352**	**193**	**no**
2^6	**2^6*7*29= 12992**	**203**	**no**
2^6	**2^6*3^5= 15552**	**3^5 = 243**	**no**
2^9	**2^9*41= 20992**	**41**	**no**

In this Table 2, there are k = 9, 13 and 17 as numbers of the first series. Therefore there aren't values of k as numbers of the second series. We note that 9, 13 and 17 are part of the arithmetic progression

9 +4 = 13, 13+4 = 17, as tern of the general progression

1+4+4+4... = 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61,

65, 69, 73, 77, 81, 85, 89, 93, 97, 101, 105, 109, 113, 117, 121,

125,129

Of this progression are part the numbers marked in purple, belonging to the initial phase of the first series, and the numbers marked in blue, the largest and belonging to the second series. Only 5 belongs to both series. Thence, also this progression may be important, at least in the initial phase of the two series. Many numbers of the two series differ by 1 from numbers of the said progression, for example all the powers of 2:

5=4+1, 9=8+1, 33=32+1, 65 = 64+1 ; 129 = 128+1, 257=256+1

etc.

Second series

5, 8, 21, **25**, 32, 64, 75, **105**, **125**, 128, 224, 256, 288, 496, 512, 576,

832, 896, 928, 2048, 2625, 4352, 5248, 6272, 12352, 12992, 15552,

20992.

First series

2, **5**, 6, **9**, 10, 12, **13**, 16, **17**, 18, 19, 24, 67, 72, 120, 163, 216, 241, 288, 317, 331, 341, 720, 960, 1129, 1203, 1385, 6480, 6523, 8640,

180631, 3265920

We also recall that 6, 24, 120 and 720, in the first series are also factorial numbers
6=2!
24=3!
120=4!
720=5!

But this connection with the factorials seems to end here, although it may be of some importance.

TABLE SUBSEQUENT RATIOS

NUMBERS FIRST SERIES	SUBSEQUENT RATIOS	NUMBERS SECOND SERIES	SUBSEQUENT RATIOS	OBSERVATIONS
2	-	5	-	
5	5/2= 2,50	8	8/5 =1,6 ≈1,618	
6	6/5= 1,20			
9	… 1,50			
10	1,11≈√√1,618			
12	1,20			
13	1,08			
16	1,23			
17	1,06			
18	1,05	21	2,625≈1,618^2 =2,6179	
19	1,05	25	1,19	
		32	1,28≈√1,618	
24	1,26 ≈√1,618	64	2	
67	2,79	75	1,17≈√√√3,14	
72	1,07			
		105	1,40	
		125	1,19	
		128	1,02	
120	1,66≈ 1,618			
163	1,35	224	1,75≈√3,14	
216	1,32	256	1,14≈√√√3,14	

241	1,11≈√√1,618	288	
288	1,19		
317	1,10≈√√1,618		
331	1,04		
341	1,03	496	1,72≈√3,14
		512	1,03
		576	1,12≈√√1,618
720	2,11	832	1,44
		896	1,07
		928	1,03
960	1,33≈√√3,14		
1129	1,17		
1203	1,06		
1385	1,15≈√√√3,14	2048	2,20
		2625	1,28≈√1,618
		4352	1,65≈ 1,618
		5248	1,20
6480	4,67		
6523	1,006	6762	1,28≈√1,618
8640	1,32≈√√3,14	12352	1,82
		12992	1,05
		15552	1,19
180631	20,90	20992	1,34
3265920	18,08		

Provisional conclusions

The subsequent ratios, at least in the initial phase of the first series, vary between 1 and 2, with a few exceptions (2,50; 2,79; 2,11). The arithmetic mean up to 1,15 of 1385 is

34,66/27 =1,2837 ≈√1,618 = 1,2720

thence there is a relationship with phi, with a good numerical evidence.
The same also for the second series, where there aren't the large ratios (20,90 and 18,08) of the final phase of the first series. Now the arithmetic mean of all the subsequent ratios between the numbers of the second series is:

$$36{,}785 / 28 = 1{,}31375 \approx \sqrt{3{,}14}$$

Here the connection is with 3,14 = π

Ratios small very frequent are also:
1,03 , 1,04, 1,05, 1,06, 1,07, symmetrical to 1,05 as their arithmetic mean. This number 1,05, that equal to about 1,0619 ≈√√√1,618 and to 1,074 = √√√√3,14, and with 1,03 and 1,04 as about square roots of 1,06 and 1,07, because √1,06= 1,029 ≈ 1,03 and √1,074= 1,036 ≈ 1,04. Thence, all the subsequent ratios, from the smallest to the largest, seem connected to 1,618 = Φ and to 3,14 = π

We also note that many numbers taken at random from the two series are divisible by 16. These numbers are 720, 960, 4352, 5248, 6480, 8640, 15552, 20992 e 3265920. Indeed, we have that

720/16 = 45; 960/16 = 60; 4352/16 = 272; 5248/16 = 328;

6480/16 = 405; 8640/16 = 540; 15552/16 = 972;

20992/16 = 1312; 3265920/16 = 204120 .

These numbers are also divisible by 8 and/or 24. Indeed, we have:

720/24 = 30; 960/24 = 40; 4352/8 = 544/8 = 68; 5248/8 = 656/8

= 82; 6480/24 = 270; 8640/24 = 360/24 = 15; 15552/24 = 648;

20992/8 = 2624/8 = 328/8 =41; 3265920/24 = 136080/24 = 5670.

We note that 16 is connected to the 496, indeed 496 = 16 * 31 and 8 and 24 are the numbers connected respectively to the modes corresponding to the physical vibrations of the superstrings and of the bosonic strings, through the following Ramanujan's equations:

$$8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} . \quad (2)$$

$$24 = \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} . \quad (3)$$

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