The Fermat classes and the proof of Beal conjecture

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Abstract : If after 374 years the famous theorem of Fermat-Wiles was demonstrated in 150 pages by A. Wiles [4], The purpose of this article is to give a proofs both for the Fermat last theorem and the Beal conjecture by using the **Fermat class** concept.

Résumé : Si après 374 ans le célèbre théorème de Fermat-Wiles a été démontré en 150 pages par A. Wiles [4], le but de cet article est d'en donner des démonstrations à la fois du dernier théorème de Fermat et de la conjecture de beal en utilisant la notion des classes de Fermat.

Keywords : Fermat, Fermat-Wiles theorem, Fermat's great theorem, Beal conjecture, Diophantine equation

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1 Introduction, notations and definitions

Set out by Pierre de Fermat [3], it was not until more than three centuries ago that Fermat's great theorem was published, validated and established by the British mathematician Andrew Wiles [4] in 1995.
In mathematics, and more precisely in number theory, the last theorem of Fermat [3], or Fermat's great theorem, or since his Fermat-Wiles theorem demonstration [4], is as follows: There are no non-zero integers a, b, and c such that: $a^n + b^n = c^n$, as soon as n is an integer strictly greater than 2 ".
The Beal conjecture [2] is the following conjecture in number theory: If $a^x + b^y = c^z$ where a, b, c, x, y and z are positive integers with x, y, z > 2, then a, b, and c have a common prime factor. Equivalently, There are no solutions to the above equation in positive integers a, b, c, x, y, z with a, b and c being pairwise coprime and all of x, y, z being greater than 2.
The purpose of this article is to give a proofs both for the Fermat last theorem and the Beal conjecture by using the **Fermat class** concept.
Let be two equations $x^a + y^b - z^c = 0$ with $(x, y, z) \in E^3$ and $(a, b, c) \in F^3$, and $X^A + Y^B - Z^C = 0$ with $(X, Y, Z) \in E'^3$ and $(A, B, C) \in F'^3$,in the following $F = F' = \mathbb{N}$ and E and E' are subsets of $\mathbb{R}$.
The two equations $x^a + y^b - z^c = 0$ with $(x, y, z) \in E^3$ and $(a, b, c) \in F^3$; and $X^A + Y^B - Z^C = 0$ with $(X, Y, Z) \in E'^3$ and $(A, B, C) \in F'^3$, are said to be equivalent if the resolution of one is reduced to the resolution of the other.
In the following, an equation $x^a + y^b - z^c = 0$ with $(x, y, z) \in E^3$ and $(a, b, c) \in F^3$ is considered at **close equivalence**, we say $x^a + y^b - z^c = 0$ is a **Fermat class**.
Example: The equation $x^{15} + y^{15} - z^{15} = 0$ with $(x, y, z) \in \mathbb{Q}^3$ is equivalent to the equation $X^3 + Y^3 - Z^3 = 0$ with $(X, Y, Z) \in \mathbb{Q}_5^3$ and where $\mathbb{Q}_5 = \{q^5, q \in \mathbb{Q}\}$

2 The proof of Fermat's last theorem

Theorem 1. *There are no non-zero a, b, and c three elements of E with $E \subset \mathbb{Q}$ such that: $a^n + b^n = c^n$, with n an integer strictly greater than 2*

Lemma 1. *If $n \in \mathbb{N}$, a, b and c are a non-zero three elements of $\mathbb{R}$ with $a^n + b^n = c^n$ then:*

$$\int_0^b x^{n-1} - \left(\frac{c-a}{b}x + a\right)^{n-1} \frac{c-a}{b} dx = 0$$

Proof.

$$a^n + b^n = c^n \Longleftrightarrow \int_0^a nx^{n-1}dx + \int_0^b nx^{n-1}dx = \int_0^c nx^{n-1}dx$$

But as :

$$\int_0^c nx^{n-1}dx = \int_0^a nx^{n-1}dx + \int_a^c nx^{n-1}dx$$

So :

$$\int_0^b nx^{n-1}dx = \int_a^c nx^{n-1}dx$$

And as by changing variables we have :

$$\int_a^c nx^{n-1}dx = \int_0^b n\left(\frac{c-a}{b}y + a\right)^{n-1} \frac{c-a}{b} dy$$

Then :

$$\int_0^b x^{n-1}dx = \int_0^b \left(\frac{c-a}{b}y + a\right)^{n-1} \frac{c-a}{b} dy$$

It results:

$$\int_0^b x^{n-1} - \left(\frac{c-a}{b}x + a\right)^{n-1} \frac{c-a}{b} dx = 0$$

□

Corollary 1. *If $N, n \in \mathbb{N}^*$, a, b and c are a non-zero three elements of $\mathbb{R}$ and $a^n + b^n = c^n$ then :*

$$\int_0^{\frac{b}{N}} x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} dx = 0$$

Proof. It results from **lemma 1** by replacing a, b and c respectively by $\frac{a}{N}$, $\frac{b}{N}$ and $\frac{c}{N}$

□

Lemma 2. *If $a^n + b^n = c^n$ is a* ***Fermat class****, where $n \in \mathbb{N}$, a, b and c are a non-zero three elements of $E \subset \mathbb{R}^+$ with $n > 2$ and $0 < a \leq b \leq c$. Then we can choose a not zero integer N, a, b , c and n in the class, such that :*

$$f(x) = x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} \leq 0 \ \forall x \in \left[0, \frac{b}{N}\right]$$

Proof.

$$\frac{df}{dx} = (n-1)x^{n-2} - (n-1)\left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-2} \left(\frac{c-a}{b}\right)^2$$

The function f decreases to the right of 0 in $[0, \epsilon[$.
But $f(x) = 0 \Longleftrightarrow x = \frac{\frac{a}{N}(\frac{c-a}{b})^{\frac{1}{n-1}}}{1-(\frac{c-a}{b})^{1+\frac{1}{n-1}}}$
So $f(x) \leq 0 \ \forall x$ such that $0 \leq x \leq \frac{\frac{a}{N}(\frac{c-a}{b})^{\frac{1}{n-1}}}{1-(\frac{c-a}{b})^{1+\frac{1}{n-1}}}$
And $f(x) \geq 0 \ \forall x$ such that $x \geq \frac{\frac{a}{N}(\frac{c-a}{b})^{\frac{1}{n-1}}}{1-(\frac{c-a}{b})^{1+\frac{1}{n-1}}}$
Otherwise if $\mu \in]0, 1]$ we have $\frac{b(1-\mu)}{N} \lneq \frac{\frac{a}{N}(\frac{c-a}{b})^{\frac{1}{n-1}}}{1-(\frac{c-a}{b})^{1+\frac{1}{n-1}}} \Longleftrightarrow 1-\mu(1-(\frac{c-a}{b})^{1+\frac{1}{n-1}}) \lneq (\frac{c-a}{b})^{1+\frac{1}{n-1}} + \frac{a}{b}(\frac{c-a}{b})^{\frac{1}{n-1}}$
By replacing a, b and c respectively with $a' = a^{\frac{1}{k}}$, $b' = b^{\frac{1}{k}}$, and $c^{\frac{1}{k}}$, we get another **Fermat class** : $a'^{kn} + b'^{kn} = c'^{kn}$

we will show for this class and for k large enough that $1-\mu(1-(\frac{c'-a'}{b'})^{1+\frac{1}{kn-1}}) \leq (\frac{c'-a'}{b'})^{1+\frac{1}{kn-1}} + \frac{a'}{b'}(\frac{c'-a'}{b'})^{\frac{1}{kn-1}}$:

First we have : $1-\mu(1-(\frac{c'-a'}{b'})^{1+\frac{1}{kn-1}}) \leq 1-\mu(1-(\frac{c'-a'}{b'})^{2}) \lneqq 1$

And as $(\frac{c'-a'}{b'})^{1+\frac{1}{kn-1}}+\frac{a'}{b'}(\frac{c'-a'}{b'})^{\frac{1}{kn-1}} = \frac{c'}{b'}(\frac{c'-a'}{b'})^{\frac{1}{kn-1}} \geq (\frac{c^{\frac{1}{k}}-a^{\frac{1}{k}}}{b^{\frac{1}{k}}})^{\frac{1}{kn-1}} \geq (1-(\frac{a}{b})^{\frac{1}{k}})^{\frac{1}{kn-1}}$

By using the **logarithm**, we have $\lim_{k\to+\infty}(1-(\frac{a}{b})^{\frac{1}{k}})^{\frac{1}{kn-1}} = \lim_{k\to+\infty}(1-(\frac{a}{b})^{\frac{1}{k}})^{\frac{1}{kn}} = 1$ because :

$(1-(\frac{a}{b})^{\frac{1}{k}})^{\frac{1}{k}} = e^{\frac{1}{k}ln(1-(\frac{a}{b})^{\frac{1}{k}})}$, by posing : $1-(\frac{a}{b})^{\frac{1}{k}} = e^{-\mathcal{N}}$, we will have : $\frac{1}{k} = \frac{ln(1-e^{-\mathcal{N}})}{ln(\frac{a}{b})}$ and $\lim_{k\to+\infty}(1-(\frac{a}{b})^{\frac{1}{k}})^{\frac{1}{k}} = \lim_{\mathcal{N}\to+\infty} e^{-\mathcal{N}\frac{ln(1-e^{-\mathcal{N}})}{ln(\frac{a}{b})}} = 1$ which shows the result.

So, for k large enough, we deduce that there exists a class $a'^{kn}+b'^{kn}=c'^{kn}$ such that :

$$f(x) = x^{kn-1} - \left(\frac{c'-a'}{b'}x+\frac{a'}{N}\right)^{kn-1}\frac{c'-a'}{b'} \lneqq 0 \ \forall x \in \left[0, \frac{b'(1-\mu)}{N}\right]$$

independently of N.

Let's fix an N and put $S = sup\{f(x), x \in \left[0, \frac{b'(1-\mu)}{N}\right]\}$

By replacing a', b' and c' respectively with $a'(1-\mu)=a''$, $b'(1-\mu)=b''$, and $c'(1-\mu)=c''$, we get another **Fermat class** : $a''^{kn}+b''^{kn}=c''^{kn}$

And we will have for M **large enough** :

$$f(x) = x^{kn-1} - \left(\frac{c''-a''}{b''}x+\frac{a''}{M}\right)^{kn-1}\frac{c''-a''}{b''} \lneqq 0 \ \forall x \in \left[0, \frac{b''}{M}\right]$$

Because $f(x) \leq S + Sup\{\frac{P(x,\mu)}{M}, x \in \left[0, \frac{b'(1-\mu)}{N}\right]\}$ where P is a polynomial, and as for M **large enough** $|Sup\{\frac{P(x,\mu)}{M}, x \in \left[0, \frac{b'(1-\mu)}{N}\right]\}| \lneqq |S|$ and $S \lneqq 0$, the result is deduced.

□

Proof of Theorem:

Proof. If $a^n + b^n = c^n$ is a **Fermat class**, where $n \in \mathbb{N}$, a, b and c are a non-zero three elements of $E \subset \mathbb{R}^+$ with $n > 2$ and $0 < a \leq b \leq c$. Then, by the **lemma 2**, for well chosen N, and a, b, c, and n in the class , we will have :

$$f(x) = x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} \leq 0 \ \forall x \in \left[0, \frac{b}{N}\right]$$

And by using the **corollary 1**, we have :

$$\int_0^{\frac{b}{N}} x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} dx = 0$$

So

$$x^{n-1} - \left(\frac{c-a}{b}x + \frac{a}{N}\right)^{n-1} \frac{c-a}{b} = 0 \ \forall x \in \left[0, \frac{b}{N}\right]$$

And therefore $\frac{c-a}{b} = 1$ because f(x) is a null polynomial as it have more than n zeros. So $c = a + b$ and $a^n + b^n \neq c^n$ which is absurde .

□

3 The proof of Beal conjecture

Corollary 2 (Beal conjecture)**.** *If $a^x + b^y = c^z$ where a, b, c, x, y and z are positive integers with x, y, z > 2, then a, b, and c have a common prime factor.*

Equivalently, there are no solutions to the above equation in positive integers a, b, c, x, y, z with a, b and c being pairwise coprime and all of x, y, z being greater than 2.

Proof. Let $a^x + b^y = c^z$

If a, b and c are not pairwise coprime, then by posing $a = ka'$, $b = kb'$, and

$c = kc'$.

Let $a' = u'^{yz}, b' = v'^{xz}, c' = w'^{xy}$ and $k = u^{yz}, k = v^{xz}, k = w^{xy}$

As $a^x + b^y = c^z$, we deduce that $(uu')^{xyz} + (vv')^{xyz} = (ww')^{xyz}$.

So :

$$k^x u'^{xyz} + k^y v'^{xyz} = k^z w'^{xyz}$$

This equation does not look like the one studied in the first theorem.

But if a, b and c are pairwise coprime, we have $k = 1$ and $u = v = w = 1$ and we will have to solve the equation :

$$u'^{xyz} + v'^{xyz} = w'^{xyz}$$

The equation $u'^{xyz} + v'^{xyz} = w'^{xyz}$ have a solution if and only if at least one of the equations : $(u'^{xy})^z + (v'^{xy})^z = (w'^{xy})^z$, $(u'^{xz})^y + (v'^{xz})^y = (w'^{xz})^y$, $(u'^{yz})^x + (v'^{yz})^x = (w'^{yz})^x$ have a solution .

So by the proof given in the proof of the first Theorem we must have : $z \leq 2$ or $y \leq 2$, or $x \leq 2$.

We therefore conclude that if $a^x + b^y = c^z$ where a, b, c, x, y, and z are positive integers with x, y, z > 2, then a, b, and c have a common prime factor.

□

4 Important notes

1- If a, b, and c are not pairwise coprime, someone, by applying the proof given in the corollary like this : $a = u^{yz}, b = v^{xz}, c = w^{xy}$ we will have $u^{xyz} + v^{xyz} = w^{xyz}$, and could say that all the x, y and z are always smaller than 2. What is false: $7^3 + 7^4 = 14^3$

The reason is sipmle: it is the common factor k which could increase the power, for example if $k = c'^r$ in the proof, then $c^z = (kc')^z = c'^{(r+1)z}$. You can take the example : $2^r + 2^r = 2^{r+1}$ where $k = 2^r$.

2- These techniques do not say that the equation $a^n + b^n = c^n$ where $a, b, c \in]0, +\infty[$ has no solution since in the proof the **Fermat class** $X^2 + Y^2 = Z^2$ can have a sloution(We take $a = X^{\frac{2}{n}}$ $b = Y^{\frac{2}{n}}$ and $C = Z^{\frac{2}{n}}$).

3- In [1] I proved the abc conjecture which implies only that the equation $a^x + b^y = c^z$ has only a finite number of solutions with a, b, c, x, y, z a positive integers and a, b and c being pairwise coprime and all of x, y, z being greater than 2.

5 Conclusion

The **Fermat class** used in this article have allowed to prove both the Fermat' last theorem and the Beal' conjecture and have shown that the Beal conjecture is only a corollary of the Fermat' last theorem.

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