New mathematical connections between Riemann-Ricci-Einstein models and String Theory, Cosmological Constant, Dark Matter and Dark Energy.

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Riassunto.

Scopo della presente tesi è quello di evidenziare le interessanti correlazioni ottenute tra i modelli matematici di Riemann, Ricci ed Einstein. Sull’unificazione delle forze gravitazionali ed elettromagnetiche, per quanto riguarda i modelli di Riemann ed Einstein, sulla teoria matematica dell’elasticità applicata ai fenomeni luminosi, per quanto riguarda il modello di Ricci. In tutti e tre i modelli si giunge all’unificazione delle due interazioni postulando l’esistenza di un “mezzo”, che nella moderna visione fisica può identificarsi con la materia/energia oscura. Per quanto concerne il modello di Einstein, è il termine cosmologico, contenuto nelle equazioni di campo della relatività generale, che viene correlato all’energia del vuoto quantistico, quindi all’energia oscura. Viene poi evidenziato come in tutti e tre i modelli sia possibile ottenere delle interessanti correlazioni con la teoria di stringa, precisamente con il modello di Palumbo applicato alla teoria di stringa, che mette in relazione l’azione di stringa bosonica con quella di superstringa.

Verrà anche trattato il tema degli assioni, correlati alla materia oscura ed al modello di Palumbo, e la loro connessione in teoria di stringa, prendendo spunto dal lavoro di Witten e Svrcek “Axions in String Theory”.

Evidenzieremo inoltre, nel corso della trattazione, le correlazioni ottenute tra alcune equazioni inerenti la teoria di stringa ed alcune formule che riguardano la Teoria dei Numeri, precisamente, la funzione zeta di Riemann, il Numero di Legendre, la serie di Fibonacci, il fattore medio di crescita delle partizioni, le funzioni modulari ed alcune identità di Rogers-Ramanujan.

Modello di Riemann

Teorema 1.

In ogni punto dello spazio esiste in ogni istante una causa, determinata in grandezza e direzione (forza d’accelerazione), che ad ogni punto ponderabile lì presente comunica un determinato moto, uguale per tutti, che si somma geometricamente al moto che già possiede.

La causa, determinata in base a grandezza e direzione (forza di gravità accelerante), che in base al Teorema 1 ha luogo in ogni punto dello spazio, viene cercata nella forma dinamica di una “materia diffusa uniformemente in tutto lo spazio infinito”, quindi si ipotizza che la direzione del moto sia uguale alla direzione della forza da spiegare in base ad essa e che la sua velocità sia proporzionale alla grandezza della forza. Questa materia può dunque essere rappresentata come uno spazio fisico, i cui punti si muovono nello spazio geometrico. I due fenomeni, gravitazione e movimento della luce nello spazio vuoto, sono gli unici che possono essere spiegati semplicemente in base ai moti di questa materia. Adesso ipotizziamo che il movimento reale della materia nello spazio vuoto sia composto dal moto che deve essere assunto per spiegare la gravitazione e da quello per spiegare i fenomeni elettromagnetici.

Il moto che deve essere assunto nello spazio vuoto per spiegare i fenomeni luminosi (elettromagnetici) può essere considerato come composto da onde piane, cioè da moti tali che lungo ogni piano di una famiglia di piani paralleli (piani ondulati) la forma del moto sia costante.
Ognuno di questi sistemi ondulatori consiste dunque di moti paralleli al piano ondulatorio, che si propagano lungo la normale al piano ondulatorio con una identica velocità costante c per tutte le forme di moto (tipi di luce). Siano ξ_1, ξ_2, ξ_3 coordinate ortogonali di un punto dello spazio per un tale sistema ondulatorio, la prima normale, le altre parallele al piano ondulatorio; $\omega_1, \omega_2, \omega_3$ le componenti di velocità ad esse parallele in questo punto, al tempo t; allora si ha:

$$\frac{\partial \omega}{\partial \xi_2} = 0, \qquad \frac{\partial \omega}{\partial \xi_3} = 0 \quad \text{(a).}$$

Si ha innanzitutto $\omega_1 = 0$. Inoltre, il moto è composto da un moto che procede verso la parte positiva del piano ondulatorio e da uno che procede verso quella negativa con velocità c. Se ω' sono le componenti di velocità del primo e ω'' quelle del secondo, le ω' rimangono invariate quando t aumenta di dt e ξ_1 di cdt, le ω'' quando t aumenta di dt e ξ_1 di –cdt, e si ha $\omega = \omega' + \omega''$.
Quindi, le interazioni gravitazionali ed elettromagnetiche possono essere spiegate in chiave unitaria, facendo riferimento esclusivamente ai moti di una "materia diffusa uniformemente in tutto lo spazio infinito", che nella nostra moderna visione fisica può benissimo identificarsi con i moti di un "mezzo" (da identificarsi con il vuoto quantistico, con il vuoto perturbativo di stringa) che genera membrane vibranti. Quindi, in termini di teoria di stringa, avremo che l'azione di stringa bosonica, quindi l'energia ordinaria e l'energia oscura (quest'ultima collegata alla costante cosmologica) è matematicamente correlata all'azione di superstringa, quindi alla materia ordinaria e alla materia oscura. Quindi anche in questa tesi, come vedremo, alcune equazioni fondamentali di questa tesi possono essere ben correlate al modello di Palumbo applicato alla teoria di stringa.

Modello di Ricci

Nella dinamica dei sistemi elastici si considerano oltre alle "forze di massa", le quali agiscono sugli elementi di volume, anche delle "forze di superficie", che si considerano applicate ai diversi elementi delle superfici, che limitano il mezzo. A queste viene dato il nome di "pressioni" o di "tensioni" secondo che le loro direzioni vanno dall'esterno verso l'interno del mezzo o viceversa.
Riferiamoci ad un sistema di coordinate generali, per le quali il quadrato dell'elemento lineare dello spazio assume l'espressione

$$ds^2 = \sum_{rs} a_{rs} dx_r dx_s = \varphi, \quad \text{(b)}$$

ed indichiamo con X_r e P_r gli elementi dei due sistemi covarianti generatori rispettivamente del vettore, che rappresenta le forze di massa riferite all'unità di volume e di quello, che rappresenta le forze di superficie riferite alla unità di area. Si supponga che il mezzo elastico sia in equilibrio sotto l'azione di quelle forze dopo aver subito una deformazione (u_r), cioè dopo che ogni suo punto P da una posizione primitiva corrispondente ai valori x_r delle coordinate è passato alla vicinissima $\left(x_r + u^{(r)}\right)$. Al punto P si immagini dato un nuovo spostamento per il quale esso passi da questa ad un'altra posizione $x_r + u^{(r)} + \delta u^{(r)}$. Se con dS si indica l'elemento di volume, che si trova intorno a P, il lavoro fatto per questo spostamento dalle forze di massa applicate a dS avrà l'espressione $dS \sum_r X_r \delta u^{(r)}$. Per i punti P delle superfici σ, che limitano il mezzo, indicando con $d\sigma$ l'elemento di area $d\sigma$ intorno a P, sarà da aggiungere a questo il lavoro delle forze di superficie che avrà per espressione $d\sigma \sum_r P_r \delta u^{(r)}$. Bisogna infine tener conto del lavoro fatto per lo

spostamento considerato dalle forze elastiche interne, il quale per ogni elemento di volume dS sarà rappresentato dal prodotto $\delta\Pi dS$, designando con $\delta\Pi$ la variazione, che il potenziale elastico Π del mezzo subisce per gli incrementi $\delta u^{(r)}$ dati agli spostamenti $u^{(r)}$.
Concludendo, Ricci afferma che nella teoria meccanica della luce i fenomeni elettromagnetici sono attribuiti alle vibrazioni di "un mezzo indefinito, elastico ed isotropo".
Anche le affermazioni di Ricci, come quelle già trattate del Riemann, possono essere reinterpretate nel linguaggio della fisica moderna. Un'onda elettromagnetica si propaga attraverso il vuoto. Ma il vuoto è "pieno" di energia: nel vuoto quantistico, o vuoto perturbativo di stringa, si creano e annichilano "continuamente" coppie di particelle-antiparticelle. Questa energia del vuoto, inoltre, può essere correlata all'energia oscura.
Anche alcune fondamentali equazioni della tesi del Ricci possono, come vedremo, essere ottimamente correlate con la teoria di stringa, precisamente con il modello di Palumbo.

Modelli di Einstein

Dopo aver descritto l'azione di Hilbert-Einstein, vengono studiate le equazioni di campo di Einstein comprensive della costante cosmologica. Quindi vengono descritte alcune applicazioni della teoria di Kaluza-Klein che vedremo essere ottimamente correlate con il modello di Palumbo applicato alla teoria di stringa.

Assioni

Infine, vengono evidenziate alcune recenti applicazioni dell'assione in teoria di stringa.
Ricordiamo che l'assione è una ipotetica particella avente una massa pari a circa 10^{-5} eV, ossia circa un decimiliardesimo della massa dell'elettrone. Gli assioni sono strettamente connessi alle particelle scalari introdotte nella teoria di Peccei-Quinn, le quali si condensano nel vuoto e rompono la simmetria detta di "Peccei-Quinn". Ogni volta che particelle scalari si condensano nel vuoto e rompono una simmetria continua, dovrebbe essere associato ad esse un altro grado di libertà, che si manifesta come una particella priva di massa: nella teoria di Peccei-Quinn questa particella è l'assione. Una volta che l'universo si sia raffreddato abbastanza perché cominci a diventare importante l'interazione forte, il termine nelle equazioni della cromodinamica quantistica che altrimenti condurrebbe alla violazione della parità della coniugazione di carica, produce interazioni che rompono esplicitamente la simmetria di Peccei-Quinn. Ne segue che vengono indotte interazioni fra gli assioni le quali impongono alla produzione di un assione reale un costo in termini di energia: queste interazioni finiscono col dare all'assione una massa, mentre in precedenza ne era stato privo.
Poiché queste particelle sono create inizialmente dalla meccanica quantistica in una configurazione di energia minima, e non attraverso un processo termico, ciò significa che in tempi molto antichi esse dovettero comportarsi in modo non relativistico, persino quando la temperatura del bagno di radiazione superava di gran lunga la loro massa. Questa è la principale richiesta che si deve soddisfare perché la materia oscura sia "fredda", così che una fluttuazione iniziale di densità non abbia alcun problema a collassare per effetto della gravità al tempo giusto. Questo fatto riveste notevole importanza: significa che non si richiede che la materia oscura fredda sia costituita da particelle pesanti. Purchè siano consentiti meccanismi di produzione non termica, come in questo caso, la gamma di possibilità è molto più ricca: gli assioni sono un esempio primario di questa nuova libertà.
Vedremo come anche alcune equazioni di quest'ultima tesi, siano ottimamente correlabili con il modello di Palumbo applicato alla teoria di stringa.

Riemann's Model [1]

1. Gravitation and Electromagnetism.

1.1 Motion from whom rise only gravitational phenomena.

The gravitational strength is defined in each point from the potential function V, whose partial differentials $\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3}$ are the components of the gravity strength.
When all the attraction bodies are inside of a finite space and r denote the infinite distance from a point of this space, then $r\frac{\partial V}{\partial x_1}, r\frac{\partial V}{\partial x_2}, r\frac{\partial V}{\partial x_3}$ are infinitesimal. If $\frac{\partial V}{\partial x} = u$, we have that $dV = u_1 dx_1 + u_2 dx_2 + u_3 dx_3$. The, we have the following conditions:

$$\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} = 0, \quad \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} = 0, \quad \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = 0 \quad (1.1)$$

$$\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) dx_1 dx_2 dx_3 = -4\pi dm, \quad (1.2)$$

$$ru_1 = 0, \quad ru_2 = 0, \quad ru_3 = 0, \quad \text{per} \quad r = \infty. \quad (1.3)$$

Inversely, also the quantities u are equal to the components of the gravity strength when satisfy these conditions, because the conditions (1.1) comprises the possibility of a function U, whose differential is equal to $dU = u_1 dx_1 + u_2 dx_2 + u_3 dx_3$, hence the partial differentials $\frac{\partial U}{\partial x} = u$ and the remaining give $U = V + \cos t$.

1.2 Motion from whom rise only electromagnetic phenomena.

From (a) we have that:

$$\left(\frac{\partial \omega'}{\partial t} + c\frac{\partial \omega'}{\partial \xi_1}\right) dt = 0, \quad \left(\frac{\partial \omega''}{\partial t} - c\frac{\partial \omega''}{\partial \xi_1}\right) dt = 0, \quad (1.4)$$

$$\frac{\partial^2 \omega'}{\partial t^2} = -c\frac{\partial^2 \omega'}{\partial \xi_1 \partial t} = c^2 \frac{\partial^2 \omega'}{\partial \xi_1^2}, \quad \frac{\partial^2 \omega''}{\partial t^2} = c\frac{\partial^2 \omega''}{\partial \xi_1 \partial t} = c^2 \frac{\partial^2 \omega''}{\partial \xi_1^2} \text{ and hence } \frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial \xi_1^2}. \quad (1.5)$$

From these equations, we have the following symmetric:

$$\frac{\partial \omega_1}{\partial \xi_1} + \frac{\partial \omega_2}{\partial \xi_2} + \frac{\partial \omega_3}{\partial \xi_3} = 0, \quad \frac{\partial^2 \omega}{\partial t^2} = c^2 \left(\frac{\partial^2 \omega}{\partial \xi_1^2} + \frac{\partial^2 \omega}{\partial \xi_2^2} + \frac{\partial^2 \omega}{\partial \xi_3^2}\right), \quad (1.6)$$

that, with the original coordinates system become :

$$\frac{\partial\omega_1}{\partial x_1}+\frac{\partial\omega_2}{\partial x_2}+\frac{\partial\omega_3}{\partial x_3}=0, \quad \frac{\partial^2\omega}{\partial t^2}=c^2\left(\frac{\partial^2\omega}{\partial x_1^2}+\frac{\partial^2\omega}{\partial x_2^2}+\frac{\partial^2\omega}{\partial x_3^2}\right). \quad (1.7)$$

These equations are good for each wave passing for the point (x_1, x_2, x_3) to the time t, and consequently also for the motion composed from the sum of these.

1.3 Motion from whom rise gravitational and electromagnetic phenomena.

From the conditions found for u and w, we have the following conditions for v, therefore the motion's laws of the matter in the vacuum space:

$$\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}=0, \quad (1.8)$$

$$\left(\partial_t^2-c^2\left(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2\right)\right)\left(\frac{\partial v_2}{\partial x_3}-\frac{\partial v_3}{\partial x_2}\right)=0, \quad \left(\partial_t^2-c^2\left(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2\right)\right)\left(\frac{\partial v_3}{\partial x_1}-\frac{\partial v_1}{\partial x_3}\right)=0$$

$$\left(\partial_t^2-c^2\left(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2\right)\right)\left(\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}\right)=0, \quad (1.9)$$

hence:

$$\partial_t^2-c^2\left(\partial x_1^2+\partial x_2^2+\partial x_3^2\right)\left[\left(\frac{\partial v_2}{\partial x_3}-\frac{\partial v_3}{\partial x_2}\right)+\left(\frac{\partial v_3}{\partial x_1}-\frac{\partial v_1}{\partial x_3}\right)+\left(\frac{\partial v_1}{\partial x_2}-\frac{\partial v_2}{\partial x_1}\right)\right]=0. \quad (1.10)$$

These equations denote that the motion of a material point derive always only from the motions of the neighbouring parts of space and time. The equation (1.8) denote the assertion that in the motion of matter the density is unchanged, because

$\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)dx_1dx_2dx_3dt$, (1.11) that consequently to this equation is equal to 0, give the quantity of matter that "run" in the spatial element $dx_1dx_2dx_3$, in the temporal element dt, thence the quantity of matter inside in it is constant. The conditions (1.9) are similar to the condition that the total differential dW is equal to

$$dW=\left(\partial_t^2-c^2\left(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2\right)\right)\left(v_1dx_1+v_2dx_2+v_3dx_3\right). \quad (1.12)$$

Now $\left(\partial_t^2-c^2\left(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2\right)\right)\left(w_1dx_1+w_2dx_2+w_3dx_3\right)=0$, (1.13) and consequently

$$dW=\left(\partial_t^2-c^2\left(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2\right)\right)\left(u_1dx_1+u_2dx_2+u_3dx_3\right)=\left(\partial_t^2-c^2\left(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2\right)\right)dV, \quad (1.14)$$

or, because $\left(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2\right)dV=0$, $\quad dW=d\frac{\partial^2 V}{\partial t^2}$. (1.15)

1.4 Expression for the laws of the motion of matter and of the gravity strength effect on the motion of ponderable bodies.

The laws of these phenomena can be express with the condition that the change of the integral

$$\frac{1}{2}\int\sum\left(\frac{\partial\eta_\iota}{\partial t}\right)^2 - c^2\left[\left(\frac{\partial\eta_2}{\partial x_3}-\frac{\partial\eta_3}{\partial x_2}\right)^2+\left(\frac{\partial\eta_3}{\partial x_1}-\frac{\partial\eta_1}{\partial x_3}\right)^2+\left(\frac{\partial\eta_1}{\partial x_2}-\frac{\partial\eta_2}{\partial x_1}\right)^2\right]dx_1dx_2dx_3dt +$$
$$+\int V\left(\sum\frac{\partial^2\eta_\iota}{\partial x_\iota\partial t}dx_1dx_2dx_3+4\pi dm\right)dt+2\pi\int dm\sum\left(\frac{\partial x_\iota}{\partial t}\right)^2 dt \quad (1.16)$$

for fixed conditions bound is 0. In this expression both first integrals are extended to the whole geometrical space, the other integrals are extended to the ponderable material elements. Furthermore, this expression arise from the relations (1.1), (1.2), (1.9) and (1.10).

Therefore, the quantities $\frac{\partial\eta}{\partial t}(=v)$ are equal to the components of the velocity of motion of matter, and V is equal to the potential for t time in the point (x_1, x_2, x_3).

Now, for the expression (1.16) equal to zero and inverting the sides, we obtain:

$$-\int V\left(\sum\frac{\partial^2\eta_\iota}{\partial x_\iota\partial t}dx_1dx_2dx_3+4\pi dm\right)dt+2\pi\int dm\sum\left(\frac{\partial x_\iota}{\partial t}\right)^2 dt =$$
$$=\frac{1}{2}\int\sum\left(\frac{\partial\eta_\iota}{\partial t}\right)^2 - c^2\left[\left(\frac{\partial\eta_2}{\partial x_3}-\frac{\partial\eta_3}{\partial x_2}\right)^2+\left(\frac{\partial\eta_3}{\partial x_1}-\frac{\partial\eta_1}{\partial x_3}\right)^2+\left(\frac{\partial\eta_1}{\partial x_2}-\frac{\partial\eta_2}{\partial x_1}\right)^2\right]dx_1dx_2dx_3dt \Rightarrow$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right], \quad (1.17)$$

Therefore, this expression, for the Palumbo's model applied to the string theory, can be related, with regard the left-hand side, to the bosonic string action, with regard the right-hand side, to the superstring action. The sign minus indicates the expansion force: i.e. the Einstein cosmological constant.

Ricci's model [2]

2. <u>Theory of elesticity applied to the electromagnetics phenomena deriving from the vibrations of an indefinite elastic and isotropic "diffuse substance"</u>.

For the application of the principle of virtual velocities, we have the equations of the elastic stability, in the following form:

$$\int_S\left(\delta\Pi+\sum_p X^{(p)}\delta u_p\right)dS+\int_\sigma d\sigma\sum_p P^{(p)}\delta u_p=0, \quad (2.2)$$

where S is the space regarding the elastic substance, and σ are the surfaces that limit it.

Now, we compute the variation $\delta\Pi$. For the general expression of elastic potential $-2\Pi=\sum_{rstu}c_{rs,tu}\zeta^{(rs)}\zeta^{(tu)}$, we have

$$2\Pi = -\sum_{rstu} c^{(rs,tu)} \zeta_{rs} \zeta_{tu} \ . \ (2.3)$$

Putting

$$\Pi^{(pq)} = \frac{\partial \Pi}{\partial \zeta_{pq}}, \ (2.4)$$

and considering ζ_{pq} distinct from ζ_{qp} if the index p and q are distinct, we have

$$\Pi^{(pq)} = -\sum_{rs} c^{(rs,pq)} \zeta_{rs}, \ (2.5)$$

from (2.5) we have that the $\Pi^{(pq)}$ are a "double system symmetric controvariant". We have also

$$\delta\Pi = \sum_{pq} \Pi^{(pq)} \delta\zeta_{pq} \quad (2.6) \quad \text{hence, for} \quad 2\zeta_{pq} = u_{pq} + u_{qp},$$

$$\delta\Pi = \sum_{pq} \Pi^{(pq)} \delta u_{pq} \ . \ (2.7)$$

Remembering the expression for derivative covariant

$$u_{pq} = \frac{\partial u_p}{\partial x_q} - \sum_t a_{pq,t} u^{(t)}, \ (2.8) \quad \text{that for} \quad u_p = \sum_t a_{pt} u^{(t)} \quad \text{become}$$

$$u_{pq} = \sum_t \left(a_{pt} \frac{\partial u^{(t)}}{\partial x_q} + a_{qt,p} u^{(t)} \right), \ (2.9) \quad \text{for these, we obtain}$$

$$\delta u_{pq} = \sum_t \left(a_{pt} \frac{\partial \delta u^{(t)}}{\partial x_q} + a_{qt,p} \delta u^{(t)} \right). \ (2.10) \quad \text{Therefore, we have}$$

$$\delta\Pi = \sum_{pqt} \left(a_{pt} \frac{\partial \delta u^{(t)}}{\partial x_q} + a_{qt,p} \delta u^{(t)} \right) \Pi^{(pq)}, \ (2.11) \quad \text{and thence}$$

$$\int_S \delta\Pi dS = \sum_{pqt} \left(\int_S a_{pt} \Pi^{(pq)} \frac{\partial \delta u^{(t)}}{\partial x_q} dS + \int_S a_{qt,p} \Pi^{(pq)} \delta u^{(t)} dS \right). \ (2.12)$$

For the expression $\int_S \frac{1}{\sqrt{a}} \frac{\partial\left(\sqrt{a}u\right)}{\partial x_q} dS = -\int_\sigma u \sqrt{a_{qq}} \cos \hat{n}\hat{x}_q d\sigma$, (2.13) changing u with $u \cdot v$, we obtain

$$\int_S u \frac{\partial v}{\partial x_q} dS = -\int_\sigma uv \sqrt{a_{qq}} \cos \hat{n}\hat{x}_q d\sigma - \int_S \frac{v}{\sqrt{a}} \frac{\partial\left(\sqrt{a}u\right)}{\partial x_q} dS \ . \ (2.14)$$

Putting $\quad u = a_{pt} \Pi^{(pq)}, \quad v = \delta u^{(t)}, \quad$ we obtain:

$$\int_S a_{pt} \Pi^{(pq)} \frac{\partial \delta u^{(t)}}{\partial x_q} dS = -\int_\sigma \Pi^{(pq)} a_{pt} \delta u^{(t)} \sqrt{a_{qq}} \cos \hat{n}\hat{x}_q d\sigma - \int_S \frac{\delta u^{(t)}}{\sqrt{a}} \frac{\partial}{\partial x_q} \left(\sqrt{a} a_{pt} \Pi^{(pq)} \right) dS, \ (2.15)$$

and thence:

$$\int_S \delta\Pi dS = \sum_{pqt} \int_S \delta u^{(t)} \left(a_{qt,p}\Pi^{(pq)} - \frac{1}{\sqrt{a}} \frac{\partial\left(\sqrt{a}a_{pt}\Pi^{(pq)}\right)}{\partial x_q} \right) dS - \sum_{pq} \int_\sigma \delta u_p \Pi^{(pq)} \sqrt{a_{qq}} \cos\hat{n}\hat{x}_q d\sigma, \quad (2.16)$$

for the relationships: $\frac{\partial a_{pt}}{\partial x_q} = a_{pq,t} + a_{tq,p}$, (2.17a)

$$\int_S \delta\Pi dS = -\sum_{pqt} \int_S \delta u^{(t)} \left(\Pi^{(pq)} a_{pq,t} + \frac{a_{pt}}{\sqrt{a}} \frac{\partial\left(\sqrt{a}\Pi^{(pq)}\right)}{\partial x_q} \right) dS - \sum_{pq} \int_\sigma \delta u_p \Pi^{(pq)} \sqrt{a_{qq}} \cos\hat{n}\hat{x}_q d\sigma. \quad (2.17b)$$

Now, for the expression for derivative controvariant, we have the following relations:

$\Pi^{(rsu)} = \sum_h a^{(hu)} \left\{ \frac{\partial \Pi^{(rs)}}{\partial x_h} + \sum_{vw} a_{hv,w} \left(a^{(rw)}\Pi^{(vs)} + a^{(sw)}\Pi^{(vr)} \right) \right\}$, (2.18) from these and from

$\sum_{sw} a^{(sw)} a_{sv,w} = \frac{\partial \log \sqrt{a}}{\partial x_v}$, (2.19) we obtain

$$\sum_{su} a_{su} \Pi^{(rsu)} = \sum_{qst} a^{(rq)} \Pi^{(ts)} a_{st,q} + \frac{1}{\sqrt{a}} \sum_q \frac{\partial\left(\sqrt{a}\Pi^{(rq)}\right)}{\partial x_q}, \quad (2.20)$$

$$\sum_{rsu} a_{rt} a_{su} \Pi^{(rsu)} = \sum_{pq} \Pi^{(pq)} a_{pq,t} + \frac{1}{\sqrt{a}} \sum_{pq} a_{pt} \frac{\partial\left(\sqrt{a}\Pi^{(pq)}\right)}{\partial x_q}. \quad (2.21)$$

In conclusion, we obtain:

$$\int_S \delta\Pi dS = -\int_S dS \sum_p \delta u_p \sum_{su} a_{su} \Pi^{(psu)} - \int_\sigma d\sigma \sum_p \delta u_p \sum_q \Pi^{(pq)} \sqrt{a_{qq}} \cos\hat{n}\hat{x}_q d\sigma, \quad (2.22)$$

the eq. (2.2) become:

$$\int_S dS \sum_p \delta u_p \left(X^{(p)} - \sum_{su} a_{su} \Pi^{(psu)} \right) + \int_\sigma d\sigma \sum_p \delta u_p \left(P^{(p)} - \sum_q \Pi^{(pq)} \sqrt{a_{qq}} \cos\hat{n}\hat{x}_q \right) = 0. \quad (2.23)$$

This equation must be satisfied for arbitrary values of δu_p and is equal to

$$X^{(p)} = \sum_{su} a_{su} \Pi^{(psu)}, \quad (2.24)$$

that must be satisfied in the S space and are defined "indefinite equations" and to

$$P^{(p)} = \sum_q \Pi^{(pq)} \sqrt{a_{qq}} \cos\hat{n}\hat{x}_q, \quad (2.25)$$

that must be satisfied on the surfaces σ, that bound S, and are defined "boundary equations". If, $f(x_1, x_2, x_3) = 0$ is the equation of the surfaces σ, and
$\sqrt{a_{qq}} \cos\hat{n}\hat{x}_q = \frac{1}{\Delta_1 f} \frac{\partial f}{\partial x_q}$, the eqs. (2.25) can be written also

$$P^{(p)} = \frac{1}{\underset{1}{\Delta f}} \sum\nolimits_q \Pi^{(pq)} \frac{\partial f}{\partial x_q}. \quad (2.26)$$

The eqs. (2.24) and (2.26) for the "duality principle" can be written also

$$X_r = \sum\nolimits_{su} a^{(su)} \Pi_{rsu}, \quad (2.27) \qquad\qquad P_r = \frac{1}{\underset{1}{\Delta f}} \sum\nolimits_s \Pi_{rs} f^{(s)}, \quad (2.28)$$

because $f^{(s)} = \sum\nolimits_p a^{(ps)} \frac{\partial f}{\partial x_p}$.

For homogeneous and isotropic "diffuse substances", the elastic potential is given by

$2\Pi = (2B - A)\Theta^2 - 2BI$, (2.29) (with A and B constants) being, in general coordinates,

$\Theta = \sum\nolimits_{rs} a^{(rs)} \zeta_{rs}$, $\quad I = \sum\nolimits_{pqrs} a^{(pr)} a^{(qs)} \zeta_{pq} \zeta_{rs}$. From these, we have

$$\frac{\partial \Theta}{\partial \zeta_{rs}} = a^{(rs)}, \quad \frac{\partial I}{\partial \zeta_{rs}} = 2\zeta^{(rs)}, \quad \Pi^{(rs)} = \frac{\partial \Pi}{\partial \zeta_{rs}} = (2B - A)\Theta \frac{\partial \Theta}{\partial \zeta_{rs}} - B \frac{\partial I}{\partial \zeta_{rs}}, \quad (2.30)$$

hence $\quad \Pi^{(rs)} = (2B - A)\Theta a^{(rs)} - 2B\zeta^{(rs)}$, (2.31) $\quad$ from whose derivatives one obtain

$$\Pi^{(rsp)} = (2B - A) a^{(rs)} \Theta^{(p)} - 2B\zeta^{(rsp)}. \quad (2.32)$$

The eqs. (2.27) e (2.28) for elastics homogeneous and isotropic "diffuse substances", become

$$X_r = (2B - A)\Theta_r - 2B \sum\nolimits_{su} a^{(su)} \zeta_{rsu}, \quad (2.33) \qquad P_r = \frac{1}{\underset{1}{\Delta f}} \left\{ (2B - A)\Theta \frac{\partial f}{\partial x_r} - 2B \sum\nolimits_s \zeta_{rs} f^{(s)} \right\}, \quad (2.34)$$

If we have in the space a system of orthogonal coordinates, whose expression ds^2 of the space is $ds^2 = H_1^2 dx_1^2 + H_2^2 dx_2^2 + H_3^2 dx_3^2$, then, the general equations of the elastic stability for the homogeneous and isotropic "diffuse substances" are

$$X_r = (2B - A)\Theta_r - 2B \sum\nolimits_s \frac{1}{H_s^2} \zeta_{rss}, \ (2.33a) \quad P_r = \frac{1}{\underset{1}{\Delta f}} \left\{ (2B - A)\Theta \frac{\partial f}{\partial x_r} - 2B \sum\nolimits_s \frac{1}{H_s^2} \zeta_{rs} \frac{\partial f}{\partial x_s} \right\}, \ (2.34b)$$

If the orthogonal coordinates x_1, x_2, x_3 coincide with a system of cartesian orthogonal coordinates y_1, y_2, y_3 putting the coefficients $H_r = 1$ we obtain the equations of the elastic stability for an arbitrary substance:

$$X_r = \sum\nolimits_s \frac{\partial \Pi_{rs}}{\partial y_s}, \quad (2.35) \qquad\qquad P_r = \sum\nolimits_s \Pi_{rs} \frac{\partial y_s}{\partial n}, \quad (2.36)$$

where n denote the normal to the surfaces σ that bound the substance. While, the eqs. (2.33) and (2.34), for the homogeneous and isotropic "diffuse substances" are:

$$X_t = -A\Theta_t + 2B\left(\Theta_t - \sum_s \zeta_{tss}\right), \text{ (2.37)} \qquad P_t = (2B - A)\Theta \frac{\partial y_t}{\partial n} - 2B\sum_s \zeta_{ts} \frac{\partial y_s}{\partial n}. \text{ (2.38)}$$

Because

$$\Theta = \sum_s \frac{\partial u_s}{\partial y_s}, \qquad 2\zeta_{tss} = \frac{\partial^2 u_t}{\partial y_s^2} + \frac{\partial^2 u_s}{\partial y_s \partial y_t}, \qquad 2\mu_t = \frac{\partial u_{t+1}}{\partial y_{t+2}} - \frac{\partial u_{t+2}}{\partial y_{t+1}}, \text{ (2.39)}$$

we have also

$$X_t = -A\frac{\partial \Theta}{\partial y_t} + 2B\left(\frac{\partial u_{t+1}}{\partial y_{t+2}} - \frac{\partial \mu_{t+2}}{\partial y_{t+1}}\right), \text{ (2.40)}$$
$$P_t = (2B - A)\Theta\frac{\partial y_t}{\partial n} - 2B\left(\frac{\partial u_t}{\partial n} + \mu_{t+1}\frac{\partial y_{t+2}}{\partial n} - \mu_{t+2}\frac{\partial y_{t+1}}{\partial n}\right). \text{ (2.41)}$$

From the indefinite equations of the elastic stability, we obtain the following equations of motion:

$$X^{(r)} - \rho\frac{\partial^2 u^{(r)}}{\partial t^2} = \sum_{pq} a_{pq}\Pi^{(rpq)} . \text{ (2.42)}$$

In cartesian orthogonal coordinates, we have the following indefinite equations of the elastic motion:

$$X_r - \rho\frac{\partial^2 u_r}{\partial t^2} = \sum_q \frac{\partial \Pi_{rq}}{\partial x_q}. \text{ (2.43)}$$

With regard the propagation of rectilinear displacements for plane waves, the indefinite equations of the motion of elastic substance, if the elastic potential Π have the most general expression, putting $\Pi = -P$ and, supposing null the external strengths, and the substance homogeneous and of density $\rho = 1$, for the eqs. (2.43), become:

$$\frac{\partial^2 u_r}{\partial t^2} = \sum_1^3 {}_q \frac{\partial}{\partial y_q}\frac{\partial P}{\partial \zeta_{rq}}. \text{ (2.44)}$$

Thence, we obtain from the (2.23), (2.40) and (2.41) the following expression:

$$\int_S dS \sum_p \delta u_p \left[-A\frac{\partial \Theta}{\partial y_t} + 2B\left(\frac{\partial \mu_{t+1}}{\partial y_{t+2}} - \frac{\partial \mu_{t+2}}{\partial y_{t+1}}\right)\right] =$$
$$= -\int_\sigma d\sigma \sum_p \delta u_p \left[(2B - A)\Theta\frac{\partial y_t}{\partial n} - 2B\left(\frac{\partial u_t}{\partial n} + \mu_{t+1}\frac{\partial y_{t+2}}{\partial n} - \mu_{t+2}\frac{\partial y_{t+1}}{\partial n}\right)\right]. \text{ (2.45)}$$

The left-hand side of this expression give the strength of mass (matter, electromagnetic field), therefore is related to the superstring action, while the right-hand side give the strengths of surfaces (gravity strength), therefore is related to the bosonic string action. Then, for the Palumbo's model applied to the string theory, we have that the eq. (2.45) is related to the expression:

$$\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]=$$
$$=-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right].\ (2.46)$$

Einstein's models [3]

3. <u>The Hilbert-Einstein action</u>.

The equations of the gravitational field, the Einstein's equations, can be written in the following form:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{8\pi G}{c^4}T_{\mu\nu}.\ (3.1)$$

Here $g_{\mu\nu}$ represent the metric tensor of the gravitational field, $R_{\mu\nu}$ the Ricci's tensor and $T_{\mu\nu}$ the energy-momentum tensor.
From the eqs. (3.1) having the source $T_{\mu\nu}$, it is possible to obtain the metric of the space-time. Furthermore, the eqs. (3.1) are non-linear equations in $g_{\mu\nu}$.
The Einstein's equations (3.1) can be obtained from a variational principle. The action of the gravitational field, the Hilbert-Einstein action, is:

$$S_g=-\frac{c^3}{16\pi G}\int d^4x\sqrt{-g(x)}R(x),\ (3.2)$$

where g is the determinant of $g_{\mu\nu}$ and R is the scalar curvature,

$$R=g^{\mu\nu}R_{\mu\nu}=g^{\mu\nu}\left(\frac{\partial\Gamma^\rho_{\mu\nu}}{\partial x^\rho}-\frac{\partial\Gamma^\rho_{\mu\rho}}{\partial x^\nu}+\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho}-\Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\mu\rho}\right).\ (3.3)$$

The variation to fixed bulk of the integral (3.2) is

$$\delta\int d^4x\sqrt{-g}R=\int d^4x\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}+\int d^4xR\delta\sqrt{-g}+\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}.\ (3.4)$$

Now, we have that $\delta\sqrt{-g}$:

$$\delta\sqrt{-g}=-\frac{1}{2\sqrt{-g}}\delta g=-\frac{1}{2\sqrt{-g}}\frac{\partial g}{\partial g^{\mu\nu}}\delta g^{\mu\nu}.\ (3.5)$$

The derivative $\partial g / \partial g^{\mu\nu}$ is the algebraic complement of $g^{\mu\nu}$. Because the elements of the inverse matrix of $g^{\mu\nu}$, that is $g_{\mu\nu}$, are the algebraic complements of $g^{\mu\nu}$ divided for g, hence

$g_{\mu\nu} = \frac{1}{g}\frac{\partial g}{\partial g^{\mu\nu}}$, (3.6) we have $\frac{\partial g}{\partial g^{\mu\nu}} = g g_{\mu\nu}$, (3.7) and the (3.5) become

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \quad (3.8)$$

Replacing this expression in the eq. (3.4), we obtain:

$$\delta\int d^4x\sqrt{-g}R = \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \int d^4x\sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu}. \quad (3.9)$$

If the Ricci's tensor $R_{\mu\nu}$ is a function of $g_{\mu\nu}$ through the Christoffel's symbols, we obtain that the second term in eq. (3.9) is a surface term that can be neglected. Then, we have

$$\delta\int d^4x\sqrt{-g}R = \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}. \quad (3.10)$$

From the principle of stationary action

$$\delta S_g = -\frac{c^3}{16\pi G}\delta\int d^4x\sqrt{-g}R = 0 \quad (3.11)$$

for the eq. (3.10), we have

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \quad (3.12)$$

This is the Einstein's equation of gravitational field without sources .
Now we consider a source of energy represented from a generic field φ. The action of φ is

$$S_\varphi = \frac{1}{c}\int d^4x\sqrt{-g}L, \quad (3.13)$$

where L is the Lagrangian density. Putting equal to zero the variation of S_φ with respect to φ we obtain the equations of motion for the field φ. The variation of S_φ with respect to $g^{\mu\nu}$, is

$$\delta S_\varphi = \frac{1}{c}\int d^4x\left[\frac{\partial\left(\sqrt{-g}L\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial\left(\sqrt{-g}L\right)}{\partial\partial_\rho g^{\mu\nu}}\delta\partial_\rho g^{\mu\nu}\right] = \frac{1}{c}\int d^4x\left[\frac{\partial\left(\sqrt{-g}L\right)}{\partial g^{\mu\nu}} - \partial_\rho\frac{\partial\left(\sqrt{-g}L\right)}{\partial\partial_\rho g^{\mu\nu}}\right]\delta g^{\mu\nu}.$$
(3.14)

Introducing the tensor $T_{\mu\nu}$ defined by

$$\frac{1}{2}\sqrt{-g}T_{\mu\nu} = \frac{\partial\left(\sqrt{-g}L\right)}{\partial g^{\mu\nu}} - \partial_\rho \frac{\partial\left(\sqrt{-g}L\right)}{\partial\partial_\rho g^{\mu\nu}}, \quad (3.15)$$

the (3.14) can be written

$$\delta S_\varphi = \frac{1}{2c}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}. \quad (3.16)$$

The total action of the system constituted from the field φ and from the gravitational field $g_{\mu\nu}$ is

$$S = S_g + S_\varphi = -\frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R + \frac{1}{c}\int d^4x\sqrt{-g}L. \quad (3.17)$$

The interaction between the two fields is incorporated in the second term. For the (3.10) and (3.16), we obtain that the stationary condition for S is

$$\delta S = \int d^4x\sqrt{-g}\left[-\frac{c^3}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{2c}T_{\mu\nu}\right]\delta g^{\mu\nu} = 0, \quad (3.18)$$

and from this we obtain the Einstein's equations in presence of sources,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (3.19)$$

The tensor $T_{\mu\nu}$ is therefore the energy-momentum tensor of the field described from the Lagrangian L. This tensor is symmetric. Now, we see the interaction between the electromagnetic field and the gravitational field. Here, the action is:

$$S = S_g + S_{em} = -\frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{16\pi G}\int d^4x\sqrt{-g}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}, \quad (3.20)$$

Now, because

$$\frac{\partial\left(\sqrt{-g}L_{em}\right)}{\partial g^{\alpha\beta}} = \frac{\partial\sqrt{-g}}{\partial g^{\alpha\beta}}L_{em} + \sqrt{-g}\frac{\partial L_{em}}{\partial g^{\alpha\beta}} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}L_{em} - \frac{\sqrt{-g}}{8\pi}F_{\mu\alpha}F_{\rho\beta}g^{\mu\rho}, \quad (3.21)$$

for $T^{em}_{\alpha\beta}$ we have, using the (3.15),

$$T^{em}_{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}L_{em}\right)}{\partial g^{\alpha\beta}} = \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}g_{\alpha\beta} - \frac{1}{4\pi}F_{\alpha\rho}F^{\rho}_{\beta}, \quad (3.22)$$

and this is effectively the symmetrized energy-momentum tensor of the electromagnetic field. Indeed, the expression for the energy-momentum tensor of the electromagnetic field is:

$$T^{\mu\nu}_{em} = -\frac{1}{4\pi} F^{\mu\rho} F^{\nu}_{\rho} + \frac{1}{16\pi} F^{\rho\sigma} F_{\rho\sigma} g^{\mu\nu}. \quad (3.23)$$

4. Einstein's equations with cosmological constant and theoretical models with time dependent dark energy.

Einstein's equations with cosmological constant [4]

Einstein's equations, which determine the dynamics of the space-time, can be derived from the action:

$$A = \frac{1}{16\pi G} \int R \sqrt{-g} d^4 x + \int L_m (\phi, \partial\phi) \sqrt{-g} d^4 x, \quad (4.1)$$

where L_m is the Lagrangian for matter depending on some dynamical variables generically denoted as ϕ. (We are using units with c = 1.) The variation of this action with respect to ϕ will lead to the equation of motion for matter $(\delta L_m / \delta\phi) = 0$, in a given background geometry, while the variation of the action with respect to the metric tensor g_{ik} leads to the Einstein's equation

$$R_{ik} - \frac{1}{2} g_{ik} R = 16\pi G \frac{\delta L_m}{\delta g^{ik}} \equiv 8\pi G T_{ik}, \quad (4.2)$$

where the last equation defines the energy momentum tensor of matter to be $T_{ik} \equiv 2(\delta L_m / \delta g^{ik})$.
Let us now consider a new matter action $L'_m = L_m - (\Lambda / 8\pi G)$ where Λ is a real constant. Equation of motion for the matter $(\delta L_m / \delta\phi) = 0$, does not change under this transformation since Λ is a constant; but the action now picks up an extra term proportional to Λ

$$A = \frac{1}{16\pi G} \int R \sqrt{-g} d^4 x + \int \left(L_m - \frac{\Lambda}{8\pi G} \right) \sqrt{-g} d^4 x = \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-g} d^4 x + \int L_m \sqrt{-g} d^4 x, \quad (4.3)$$

and equation (4.2) gets modified. This innocuous looking addition of a constant to the matter Lagrangian leads to one of the most fundamental and fascinating problems of theoretical physics. The nature of this problem and its theoretical backdrop acquires different shades of meaning depending which of the two forms of equations in (4.3) is used.
The first interpretation, based on the left-hand side of equation (4.3), treats Λ as the shift in the matter Lagrangian which, in turn, will lead to a shift in the matter Hamiltonian. This could be thought of as a shift in the zero point energy of the matter system. Such a constant shift in the energy does not affect the dynamics of matter while gravity – which couples to the total energy of the system – picks up an extra contribution in the form of a new term Q_{ik} in the energy-momentum tensor, leading to:

$$R^i_k - \frac{1}{2} \delta^i_k R = 8\pi G (T^i_k + Q^i_k); \qquad Q^i_k \equiv \frac{\Lambda}{8\pi G} \delta^i_k \equiv \rho_\Lambda \delta^i_k. \quad (4.4)$$

The right-hand side in equation (4.3) can be interpreted as gravitational field, described by the Lagrangian of the form $L_g \propto (1/G)(R-2\Lambda)$, interacting with matter described by the Lagrangian L_m. In this interpretation, gravity is described by two constants, the Newton's constant G and the *cosmological constant* Λ. It is then natural to modify the left-hand side of Einstein's equation and write (4.4) as:

$$R_k^i - \frac{1}{2}\delta_k^i R - \delta_k^i \Lambda = 8\pi G T_k^i . \quad (4.5)$$

In this interpretation, the spacetime is treated as curved even in the absence of matter $(T_{ik} = 0)$ since the equation $R_{ik} - (1/2)g_{ik}R - \Lambda g_{ik} = 0$ does not admit flat spacetime as a solution.
The action principle for gravity in the presence of a cosmological constant

$$A = \frac{1}{16\pi G}\int (R-2\Lambda)\sqrt{-g}\,d^4x = \frac{1}{16\pi G}\int R\sqrt{-g}\,d^4x - \frac{\Lambda}{8\pi G}\int \sqrt{-g}\,d^4x, \quad (4.6)$$

can be thought of as a variational principle extremizing the integral over R, subject to the condition that the 4-volume of the universe remains constant. To implement the constraint that the 4-volume is a constant, one will add a Lagrange multiplier term which is identical in structure to the second term in the above equation. Hence, mathematically, one can think of the cosmological constant as a Lagrange multiplier ensuring the constancy of the 4-volume of the universe when the metric is varied.
Several people have suggested modifying the basic structure of general relativity so that the cosmological constant will appear as a constant of integration. One simple way of achieving this is to assume that the determinant g of g_{ab} is not dynamical and admit only those variations which obeys the condition $g^{ab}\delta g_{ab} = 0$ in the action principle. This is equivalent to eliminating the trace part of Einstein's equations. Instead of the standard result, we will now be led to the equation

$$R_k^i - \frac{1}{4}\delta_k^i R = 8\pi G\left(T_k^i - \frac{1}{4}\delta_k^i T\right), \quad (4.7)$$

which is just the traceless part of Einstein's equation. The general covariance of the action, however, implies that $T^{ab}_{;b} = 0$ and the Bianchi identities $\left(R_k^i - \frac{1}{2}\delta_k^i R\right)_{;i} = 0$ continue to hold. These two conditions imply that $\partial_i R = -8\pi G \partial_i T$ requiring $R + 8\pi G T$ to be a constant. Calling this constant (-4Λ) and combining with equation (4.7), we get

$$R_k^i - \frac{1}{2}\delta_k^i R - \delta_k^i \Lambda = 8\pi G T_k^i , \quad (4.8)$$

which is precisely Einstein's equation in the presence of cosmological constant. In this approach, the cosmological constant has nothing to do with any term in the action or vacuum fluctuations and is merely an integration constant. Like any other integration constant its value can be fixed by invoking suitable boundary conditions for the solutions.

Now, we consider a system consisting of the gravitational fields g_{ab}, radiation fields, and a scalar field ϕ which couples to the trace of the energy-momentum tensor of all fields, including its own. The "zeroth order" action for this system is given by

$$A^{(0)} = A_{grav} + A_{\phi}^{(0)} + A_{\text{int}}^{(0)} + A_{radn}\,, \quad (4.9)$$

where

$$A_{grav} = \frac{1}{(16\pi g)} \int R\sqrt{-g}\,d^4x - \int \Lambda\sqrt{-g}\,d^4x\,, \qquad A_{\phi}^{(0)} = \frac{1}{2}\int \phi^i\phi_i\sqrt{-g}\,d^4x\,,$$

$$A_{\text{int}}^{(0)} = \eta\int Tf(\phi/\phi_0)\sqrt{-g}\,d^4x\,, \quad (4.10)$$

therefore:

$$A^{(0)} = \frac{1}{(16\pi G)}\int R\sqrt{-g}\,d^4x - \int \Lambda\sqrt{-g}\,d^4x + \frac{1}{2}\int \phi^i\phi_i\sqrt{-g}\,d^4x + \eta\int Tf(\phi/\phi_0)\sqrt{-g}\,d^4x + A_{radn}\,. \quad (4.11)$$

Here, we have explicitly included the cosmological constant term and η is a dimensionless number which "switches on" the interaction. In the zeroth order action, T represents the trace of all fields other than ϕ. Since the radiation field is traceless, the only zeroth-order contribution to T comes from the Λ term, so that we have $T = 4\Lambda$. The coupling to the trace is through a function f of the scalar field, and one can consider various possibilities for this function. The constant ϕ_0 converts ϕ to a dimensionless variable, and is introduced for dimensional convenience.
To take into account the back-reaction of the scalar field on itself, we must add to T the contribution $T_\phi = -\phi^l\phi_l$ of the scalar field. If we now add T_ϕ to T in the interaction term $A_{\text{int}}^{(0)}$ further modifies T_ϕ^{ik}. This again changes T_ϕ. Thus to arrive at the correct action an infinite iteration will have to be performed and the complete action can be obtained by summing up all the terms. The full action ca be found more simply by a consistency argument.
Since the effect of the iteration is to modify the expression for A_ϕ and A_Λ, we consider the following ansatz for the full action:

$$A = \frac{1}{16\pi G}\int R\sqrt{-g}\,d^4x - \int \alpha(\phi)\Lambda\sqrt{-g}\,d^4x + \frac{1}{2}\int \beta(\phi)\phi^i\phi_i\sqrt{-g}\,d^4x + A_{rad}\,. \quad (4.12)$$

Here $\alpha(\phi)$ and $\beta(\phi)$ are functions of ϕ to be determined by the consistency requirement that they represent the effect of the iteration of the interaction term. The energy-momentum tensor for ϕ and Λ is now given by

$$T^{ik} = \alpha(\phi)\Lambda g^{ik} + \beta(\phi)\left[\phi^i\phi_k - \frac{1}{2}g^{ik}\phi^\alpha\phi_\alpha\right], \quad (4.13)$$

so that the total trace is $T_{tot} = 4\alpha(\phi)\Lambda - \beta(\phi)\phi^i\phi_i$. The functions $\alpha(\phi)$ and $\beta(\phi)$ can now be determined by the consistency requirement

$$-\int \alpha(\phi)\Lambda\sqrt{-g}\,d^4x + \frac{1}{2}\int \beta(\phi)\phi^i\phi_i\sqrt{-g}\,d^4x =$$

$$= -\int \Lambda \sqrt{-g} d^4x + \frac{1}{2}\int \phi^i \phi_i \sqrt{-g} d^4x + \eta \int T_{tot} f(\phi/\phi_0)\sqrt{-g} d^4x . \quad (4.14)$$

Using T_{tot} and comparing terms in the above equation we find that

$$\alpha(\phi) = [1 + 4\eta f]^{-1}, \qquad \beta(\phi) = [1 + 2\eta f]^{-1} . \quad (4.15)$$

Thus the complete action can be written as

$$A = \frac{1}{16\pi G}\int R\sqrt{-g} d^4x - \int \frac{\Lambda}{1+4nf}\sqrt{-g} d^4x + \frac{1}{2}\int \frac{\phi^i \phi_i}{1+2nf}\sqrt{-g} d^4x + A_{rad} . \quad (4.16)$$

The action in (4.16) leads to the following fields equations,

$$R_{ik} - \frac{1}{2} g_{ik} R = -8\pi G\left[\beta(\phi)\left(\phi^i \phi^k - \frac{1}{2} g^{ik}\phi^\alpha \phi_\alpha\right) + \frac{\Lambda}{8\pi G}\alpha(\phi) g_{ik} + T_{ik}^{traceless}\right] \quad (4.17)$$

$$\Box\phi + \frac{1}{2}\frac{\beta'(\phi)}{\beta(\phi)}\phi^i \phi_i + \frac{\Lambda}{8\pi G}\frac{\alpha'(\phi)}{\beta(\phi)} = 0 \quad (4.18).$$

Here, $\Box$ stands for a covariant d'Lambertian, $T_{ik}^{traceless}$ is the stress tensor of all fields with traceless stress tensor and a prime denotes differentiation with respect to ϕ. In the cosmological context, this reduces to

$$\ddot{\phi} + \frac{3\dot{a}}{a}\dot{\phi} = \eta\dot{\phi}^2 \frac{f'}{1+2\eta f} + \eta\frac{\Lambda}{2\pi G}\frac{f'(1+2\eta f)}{(1+4\eta f)^2} \quad (4.19)$$

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\left[\frac{1}{2}\frac{\dot{\phi}^2}{1+2\eta f} + \frac{\Lambda}{8\pi G}\frac{1}{(1+4\eta f)} + \frac{\rho_0}{a^4}\right] . \quad (4.20)$$

It is obvious that the effective cosmological constant can decrease if f increases in an expanding universe. The result can be easily generalized for a scalar field with a potential by replacing Λ by $V(\phi)$. This model in conceptually attractive since it correctly accounts for the coupling of the scalar field with the trace of the stress tensor.

Theoretical models with time dependent dark energy

The distribution of matter in the universe is homogeneous and isotropic at sufficiently large scales. The assumption of isotropy and homogeneity implies that the large scale geometry can be described by a metric of the form

$$ds^2 = dt^2 - a^2(t) d\vec{x}^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \quad (4.21)$$

in a suitable set of coordinates called comoving coordinates. Here $a(t)$ is an arbitrary function of time (called *expansion factor*) and $k = 0, \pm 1$. Defining a new coordinate χ through $\chi = (r, \sin^{-1} r, \sinh^{-1} r)$ for $k = (0, +1, -1)$ this line element becomes

$$ds^2 \equiv dt^2 - a^2 d\vec{x}^2 \equiv dt^2 - a^2(t)\left[d\chi^2 + S_k^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \quad (4.22)$$

where $S_k(\chi) = (\chi, \sin\chi, \sinh\chi)$ for $k = (0, +1, -1)$. The line element in terms of $[a, \chi, \theta, \phi]$ or $[z, \chi, \theta, \phi]$ is:

$$ds^2 = H^{-2}(a)\left(\frac{da}{a}\right)^2 - a^2 d\vec{x}^2 = \frac{1}{(1+z)^2}\left[H^{-2}(z)dz^2 - dx^2\right], \quad (4.23)$$

where $H(a) = (\dot{a}/a)$, called the *Hubble parameter*, measures the rate of expansion of the universe. The following equation

$$\rho_i = \rho_i(a_0)\left(\frac{a_0}{a}\right)^3 \exp\left[-3\int_{a_0}^{a} \frac{d\bar{a}}{\bar{a}} w_i(\bar{a})\right], \quad (4.24)$$

determines the evolution of the energy density of each of the species in terms of the functions $w_i(a)$. This description determines $\rho(a)$ for different sources but not $a(t)$. To determine the latter we can use one of the Einstein's equations:

$$H^2(a) = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\sum_i \rho_i(a) - \frac{k}{a^2}. \quad (4.25)$$

This equation shows that, once the evolution of the individual components of energy density $\rho_i(a)$ is known, the function $H(a)$ and thus the line element in equation (4.23) is known.
The simplest model for the universe is based on the assumption that each of the sources which populate the universe has a constant w_i; then equation (4.25) becomes

$$\frac{\dot{a}^2}{a^2} = H_0^2 \sum_i \Omega_i \left(\frac{a_0}{a}\right)^{3(1+w_i)} - \frac{k}{a^2}, \quad (4.26)$$

where each of these species is identified by density parameter Ω_i and the equation of state characterized by w_i. The most familiar form of energy densities are those due to pressure-less matter with $w_i = 0$ and radiation with $w_i = (1/3)$.
The term (k/a^2) in equation (4.26) can be thought of as contributed by a hypothetical species of matter with $w = -(1/3)$. Hence equation (4.26) can be written in the form

$$\frac{\dot{a}^2}{a^2} = H_0^2 \sum_i \Omega_i \left(\frac{a_0}{a}\right)^{3(1+w_i)}, \quad (4.27)$$

with a term having $w_i = -(1/3)$ added to the sum. Let $\alpha \equiv 3(1+w)$ and $\Omega(\alpha)$ denote the fraction of the critical density contributed by matter with $w = (\alpha/3) - 1$. In the continuum limit, equation (4.27) can be rewritten as

$$H^2 = H_0^2 \int_{-\infty}^{\infty} d\alpha \Omega(\alpha) e^{-\alpha q}, \quad (4.28)$$

where $(\alpha/\alpha_0) = \exp(q)$. Let us divide the source energy density into two components: $\rho_k(a)$, which is known from independent observations and a component $\rho_X(a)$ which is not known and where "X" is the unknown dark energy component. From (4.25), it follows that

$$\frac{8\pi G}{3}\rho_X(a) = H^2(a)(1 - Q(a)); \quad Q(a) \equiv \frac{8\pi G \rho_k(a)}{3H^2(a)}. \quad (4.29)$$

Taking a derivative of $\ln \rho_X(a)$ and using (4.24), it is easy to obtain the relation

$$w_X(a) = -\frac{1}{3}\frac{d}{d\ln a}\ln\left[(1 - Q(a))H^2(a)a^3\right]. \quad (4.30)$$

Note that the value $w \neq 0$ is a clear indication of a dark energy component which is evolving.
A simple form of the source with variable w are scalar fields with Lagrangians of different forms, of which we will discuss two possibilities:

$$L_{quin} = \frac{1}{2}\partial_a\phi\partial^a\phi - V(\phi); \quad L_{tach} = -V(\phi)\left[1 - \partial_a\phi\partial^a\phi\right]^{1/2}. \quad (4.31)$$

Both these Lagrangians involve one arbitrary function $V(\phi)$. The first one, L_{quin}, which is a natural generalisation of the Lagrangian for a non-relativistic particle, $L = (1/2)\dot{q}^2 - V(q)$, is usually called "quintessence". When it acts as a source in Friedman universe, it is characterized by a time dependent $w(t)$ with

$$\rho_q(t) = \frac{1}{2}\dot{\phi}^2 + V; \quad P_q(t) = \frac{1}{2}\dot{\phi}^2 - V; \quad w_q = \frac{1 - \left(2V/\dot{\phi}^2\right)}{1 + \left(2V/\dot{\phi}^2\right)}. \quad (4.32)$$

The stress tensor for the tachyonic scalar field can be written in a perfect fluid form

$$T_k^i = (\rho + p)u^i u_k - p\delta_k^i, \quad (4.33)$$

with

$$u_k = \frac{\partial_k\phi}{\sqrt{\partial^i\phi\partial_i\phi}}; \quad \rho = \frac{V(\phi)}{\sqrt{1 - \partial^i\phi\partial_i\phi}}; \quad p = -V(\phi)\sqrt{1 - \partial^i\phi\partial_i\phi}, \quad (4.34) \quad \text{thence}$$

$$T_k^i = \left[\frac{V(\phi)}{\sqrt{1 - \partial^i\phi\partial_i\phi}} - V(\phi)\sqrt{1 - \partial^i\phi\partial_i\phi}\right]u^i \frac{\partial_k\phi}{\sqrt{\partial^i\phi\partial_i\phi}} + V(\phi)\sqrt{1 - \partial^i\phi\partial_i\phi}\,\delta_k^i. \quad (4.35)$$

The remarkable feature of this stress tensor is that it could be considered as *the sum of a pressure less dust component and a cosmological constant*.
In the cosmological context, the tachyonic field is described by:

$$\rho_t(t) = V\left[1-\dot{\phi}^2\right]^{-1/2}; \quad P_t = -V\left[1-\dot{\phi}^2\right]^{1/2}; \quad w_t = \dot{\phi}^2 - 1. \quad (4.36)$$

Now, we assume that the universe has two forms of energy density with $\rho(a) = \rho_{known}(a) + \rho_\phi(a)$ where $\rho_{known}(a)$ arises from any known forms of source (matter, radiation,…) and $\rho_\phi(a)$ is due to a scalar field. When $w(a)$ is given, one can determine the $V(\phi)$ using either (4.32) or (4.36). For quintessence, (4.32) along with (4.29) gives

$$\dot{\phi}^2(a) = \rho(1+w) = \frac{3H^2(a)}{8\pi G}(1-Q)(1+w); \quad 2V(a) = \rho(1-w) = \frac{3H^2(a)}{8\pi G}(1-Q)(1-w). \quad (4.37)$$

For tachyonic scalar field, (4.36) along with (4.29) gives

$$\dot{\phi}^2(a) = (1+w); \quad V(a) = \rho(-w)^{1/2} = \frac{3H^2(a)}{8\pi G}(1-Q)(-w)^{1/2}. \quad (4.38)$$

Given $Q(a), w(a)$ these equations implicitly determine $V(\phi)$. Combining (4.30) with either (4.37) or (4.38), one can completely solve the problem.
Now we consider quintessence. Here, using (4.30) to express w in terms of H and Q, the potential is given implicitly by the form

$$V(a) = \frac{1}{16\pi G}H(1-Q)\left[6H + 2aH' - \frac{aHQ'}{1-Q}\right], \quad (4.39)$$

$$\phi(a) = \left[\frac{1}{8\pi G}\right]^{1/2}\int\frac{da}{a}\left[aQ' - (1-Q)\frac{d\ln H^2}{d\ln a}\right]^{1/2}, \quad (4.40)$$

where $Q(a) \equiv \left[8\pi G\rho_{known}(a)/3H^2(a)\right]$.
Similar results exists for the tachyonic scalar field. For example, given any $H(t)$, one can construct a tachyonic potential $V(\phi)$ so that the scalar field is the source for the cosmology. The equations determining $V(\phi)$ are now given by:

$$\phi(a) = \int\frac{da}{aH}\left(\frac{aQ'}{3(1-Q)} - \frac{2}{3}\frac{aH'}{H}\right)^{1/2}; \quad (4.41) \qquad V = \frac{3H^2}{8\pi G}(1-Q)\left(1 + \frac{2}{3}\frac{aH'}{H} - \frac{aQ'}{3(1-Q)}\right)^{1/2}. \quad (4.42)$$

Equations (4.41) and (4.42) completely solve the problem. Given any $H(t)$, these equations determine $V(t)$ and $\phi(t)$, and thus the potential $V(\phi)$.

5. Kaluza-Klein theories and their applications. [5]

Kaluza unified electromagnetism with gravity by applying Einstein's general theory of relativity to a *five*-, rather than four-dimensional spacetime manifold.

The Einstein equations in five dimensions with *no five-dimensional energy-momentum tensor* are:

$$\hat{G}_{AB} = 0, \quad (5.1)$$

or, equivalently:

$$\hat{R}_{AB} = 0, \quad (5.2)$$

where $\hat{G}_{AB} \equiv \hat{R}_{AB} - \hat{R}\hat{g}_{AB}/2$ is the Einstein tensor, $\hat{R}_{AB}$ and $\hat{R} = \hat{g}_{AB}\hat{R}^{AB}$ are the five-dimensional Ricci tensor and scalar respectively, and $\hat{g}_{AB}$ is the five-dimensional metric tensor. These equations can be derived by varying a five-dimensional version of the usual Einstein action:

$$S = -\frac{1}{16\pi\hat{G}}\int \hat{R}\sqrt{-\hat{g}}\,d^4x dy, \quad (5.3)$$

with respect to the five-dimensional metric, where $y = x^4$ represents the new (fifth) coordinate and $\hat{G}$ is a "five-dimensional gravitational constant". The absence of matter sources in these equations reflects the Kaluza's first key assumption (i), inspired by Einstein: *the universe in higher dimensions is empty*.
The five-dimensional Ricci tensor and Christoffel symbols are defined in terms of the metric exactly as in four dimensions:

$$\hat{R}_{AB} = \partial_C \hat{\Gamma}^C_{AB} - \partial_B \hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC}, \quad \hat{\Gamma}^C_{AB} = \frac{1}{2}\hat{g}^{CD}\left(\partial_A \hat{g}_{DB} + \partial_B \hat{g}_{DA} - \partial_D \hat{g}_{AB}\right). \quad (5.4)$$

Note that, aside from the fact that tensor indices run over 0 – 4 instead of 0 – 3, all is exactly as it was in Einstein's theory: this is the second key feature (ii) of Kaluza's approach to unification. In general, one identifies the $\alpha\beta$-part of $\hat{g}_{AB}$ with $g_{\alpha\beta}$ (the four-dimensional metric tensor), the $\alpha 4$-part with A_α (the electromagnetic potential), and the 44-part with ϕ (a scalar field). A convenient way to parametrize things is as follows:

$$\left(\hat{g}_{AB}\right) = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\phi^2 A_\alpha A_\beta & \kappa\phi^2 A_\alpha \\ \kappa\phi^2 A_\beta & \phi^2 \end{pmatrix}. \quad (5.5)$$

If one then applies the third key feature (iii) of Kaluza's theory (the cylinder condition), which means dropping all derivatives with respect to the fifth coordinate, then one finds, using the metric (5.5) and the definitions (5.4), that the $\alpha\beta$-, $\alpha 4$-, and 44-components of the five-dimensional field equation (5.2) reduce respectively to the following field equations in four dimensions:

$$G_{\alpha\beta} = \frac{\kappa^2\phi^2}{2}T^{EM}_{\alpha\beta} - \frac{1}{\phi}\left[\nabla_\alpha\left(\partial_\beta\phi\right) - g_{\alpha\beta}\Box\phi\,\right], \quad \nabla^\alpha F_{\alpha\beta} = -3\frac{\partial^\alpha\phi}{\phi}F_{\alpha\beta}, \quad \Box\phi = \frac{\kappa^2\phi^3}{4}F_{\alpha\beta}F^{\alpha\beta}, \quad (5.6)$$

where $G_{\alpha\beta} \equiv R_{\alpha\beta} - Rg_{\alpha\beta}/2$ is the Einstein tensor, $T^{EM}_{\alpha\beta} \equiv g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}/4 - F^\gamma_\alpha F_{\beta\gamma}$ is the electromagnetic energy-momentum tensor, and $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$. There are a total of 10+4+1 = 15 equations, as expected since there are fifteen independent elements in the five-dimensional metric (5.5).

If the scalar field ϕ is constant throughout spacetime, then the first two of eqs. (5.6) are just the Einstein and Maxwell equations:

$$G_{\alpha\beta} = 8\pi G\phi^2 T_{\alpha\beta}^{EM}, \quad \nabla^\alpha F_{\alpha\beta} = 0, \; (5.7)$$

where we have identified the scaling parameter κ in terms of the gravitational constant G (in four dimension) by:

$$\kappa \equiv 4\sqrt{\pi G}. \; (5.8)$$

This is the result originally obtained by Kaluza and Klein, who set $\phi = 1$. The condition $\phi =$ constant is, however, only consistent with the *third* of the field equations (5.6) when $F_{\alpha\beta}F^{\alpha\beta} = 0$.

Using the metric (5.5) and the definitions (5.4), and invoking the cylinder condition not only to drop derivatives with respect to y, but also to pull $\int dy$ out of the action integral, one finds that eq. (5.3) contains three components:

$$S = -\int d^4x\sqrt{-g}\phi\left(\frac{R}{16\pi G} + \frac{1}{4}\phi^2 F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{3\kappa^2}\frac{\partial^\alpha\phi\partial_\alpha\phi}{\phi^2}\right), \; (5.9)$$

where G is defined in terms of its five-dimensional counterpart $\hat{G}$ by:

$$G \equiv \hat{G} / \int dy, \; (5.10)$$

and where we have used equation (5.8) to bring the factor of $16\pi G$ inside the integral. If one takes $\phi =$ constant, then the first two components of this action are just the Einstein-Maxwell action for gravity and electromagnetic radiation (scaled by factors of ϕ). The third component is the action for a massless Klein-Gordon scalar field. The fact that the action (5.3) leads to (5.9), or – equivalently – that the source-less field equations (5.2) lead to (5.6) *with* source matter, constitutes the central miracle of Kaluza-Klein theory. Four-dimensional matter (electromagnetic radiation, al least) has been shown to arise purely from the geometry of empty five-dimensional spacetime. The goal of all subsequent Kaluza-Klein theories has been to extend this success to *other* kinds of matter.

If one does not set ϕ = constant, then Kaluza's five-dimensional theory contains besides electromagnetic effects a Brans-Dicke-type scalar field theory, as becomes clear when one considers the case in which the electromagnetic potentials vanish, $A_\alpha = 0$. This is acceptable in some contexts, for example, in early-universe models which are dynamically dominated by scalar fields. Neglecting the A_α-fields, then, eq. (5.5) becomes:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \phi^2 \end{pmatrix}. \; (5.11)$$

With this metric, the field equations (5.2), and Kaluza's assumptions (i) – (iii) as before, the action (5.3) reduces to:

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R\phi . \quad (5.12)$$

This is the special case $\omega = 0$ of the Brans-Dicke action:

$$S_{BD} = -\int d^4x\sqrt{-g}\left(\frac{R\phi}{16\pi G} + \omega\frac{\partial^\alpha\phi\partial_\alpha\phi}{\phi}\right) + S_m , \quad (5.13)$$

where ω is the dimensionless Brans-Dicke constant and the term S_m refers to the action associated with any matter fields which may be coupled to the metric or scalar field.
Now we take the five-dimensional metric:

$$\hat{g}_{AB} \rightarrow \hat{g}'_{AB} = \Omega^2\hat{g}_{AB} , \quad (5.14)$$

where $\Omega^2 > 0$ is the conformal (or Weyl) factor, a function of the first four coordinates only (assuming Kaluza's cylinder condition). The four-dimensional metric tensor is rescaled by the same factor as the five-dimensional one ($g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$), and this has the following effect on the four-dimensional Ricci scalar:

$$R \rightarrow R' = \Omega^{-2}\left(R + 6\frac{\Box..\Omega}{\Omega}\right). \quad (5.15)$$

A convenient parametrization is obtained by making the trivial redefinition $\phi^2 \rightarrow \phi$ and then introducing the conformal factor $\Omega^2 = \phi^{-1/3}$, so that the five-dimensional metric reads:

$$\left(\hat{g}'_{AB}\right) = \phi^{-1/3}\begin{pmatrix} g_{\alpha\beta} + \kappa^2\phi A_\alpha A_\beta & \kappa\phi A_\alpha \\ \kappa\phi A_\beta & \phi \end{pmatrix}, \quad (5.16)$$

Then, we have the following *conformally rescaled* action instead of eq. (5.9) above:

$$S' = -\int d^4x\sqrt{-g'}\left(\frac{R'}{16\pi G} + \frac{1}{4}\phi F'_{\alpha\beta}F'^{\alpha\beta} + \frac{1}{6\kappa^2}\frac{\partial'^\alpha\phi\partial'_\alpha\phi}{\phi^2}\right), \quad (5.17)$$

where primed quantities refer to the rescaled metric (ie., $\partial'^\alpha\phi = g'^{\alpha\beta}\partial_\beta\phi$), where G and κ are defined as before. The gravitational part of the action then has the conventional form, as desired.
The Brans-Dicke case, obtained by putting $A_\alpha = 0$ in the metric, is also modified by the presence of the conformal factor. One finds (again making the redefinition $\phi^2 \rightarrow \phi$ and using $\Omega^2 = \phi^{-1/3}$) that the action (5.12) becomes:

$$S' = -\int d^4x\sqrt{-g'}\left(\frac{R'}{16\pi G} + \frac{1}{6\kappa^2}\frac{\partial'^\alpha\phi\partial'_\alpha\phi}{\phi^2}\right). \quad (5.18)$$

In terms of the "dilaton" field $\sigma \equiv \frac{\ln\phi}{\left(\sqrt{3}\kappa\right)}$, or $\sigma \equiv \frac{1}{\sqrt{3}\kappa}\ln\phi$, this action can be written:

$$S' = -\int d^4x\sqrt{-g'}\left(\frac{R'}{16\pi G} + \frac{1}{2}\partial'^{\alpha}\sigma\partial'_{\alpha}\sigma\right), \quad (5.19)$$

which is the canonical action for a minimally coupled scalar field with no potential.

Klein assumed that the fifth coordinate was to be a lengthlike one (like the first three), and assigned it two properties: (1) a *circular topology* (S^1); and (2) a *small scale*. Under property (1), any quantity f(x, y) (where $x = (x^0, x^1, x^2, x^3)$ and $y = x^4$) becomes *periodic*; $f(x, y) = f(x, y + 2\pi r)$ where r is the scale parameter or "radius" of the fifth dimension. Therefore all the fields can be Fourier-expanded:

$$g_{\alpha\beta}(x, y) = \sum_{n=-\infty}^{n=\infty} g_{\alpha\beta}^{(n)}(x) e^{iny/r}, \quad A_{\alpha}(x, y) = \sum_{n=-\infty}^{n=\infty} A_{\alpha}^{(n)}(x) e^{iny/r}, \quad \phi(x, y) = \sum_{n=-\infty}^{n=\infty} \phi^{(n)} e^{iny/r}, \quad (5.20)$$

where the superscript $^{(n)}$ refers to the *n*th Fourier mode. Thanks to quantum theory, these modes carry a momentum in the y-direction of the order $|n|/r$. This is where property (2) comes in: if r is small enough, then the y-momenta of even the n = 1 modes will be so large as to put them beyond the reach of experiment. Hence only the n = 0 modes, which are independent of y, will be observable, as required in Kaluza's theory.
The expansion of fields into Fourier modes suggests a possible mechanism to explain *charge quantization*. The simplest kind of matter is a massless five-dimensional scalar field $\hat{\psi}(x, y)$. Its action would have a kinetic part only:

$$S_{\hat{\psi}} = -\int d^4x dy\sqrt{-\hat{g}}\,\partial^A\hat{\psi}\partial_A\hat{\psi}. \quad (5.21)$$

The field can be expanded like those in eq. (5.20):

$$\hat{\psi}(x, y) = \sum_{n=-\infty}^{n=\infty} \hat{\psi}^{(n)} e^{iny/r}. \quad (5.22)$$

When this expansion is put into the action (5.21), one finds (using eq. (5.16)) the following result, analogous to eq. (5.9):

$$S_{\hat{\psi}} = -\left(\int dy\right)\sum_n \int d^4x\sqrt{-g}\left[\left(\partial^{\alpha} + \frac{in\kappa A^{\alpha}}{r}\right)\hat{\psi}^{(n)}\left(\partial_{\alpha} + \frac{in\kappa A_{\alpha}}{r}\right)\hat{\psi}^{(n)} - \frac{n^2}{\phi r^2}\hat{\psi}^{(n)2}\right]. \quad (5.23)$$

From this action one can read off both the charge and mass of the scalar modes $\hat{\psi}^{(n)}$. Comparison with the minimal coupling rule $\partial_{\alpha} \to \partial_{\alpha} + ieA_{\alpha}$ of quantum electrodynamics (where "e" is the electron charge) shows that in this theory the nth Fourier mode of the scalar field $\hat{\psi}$ also carries a *quantized charge*:

$$q_n = \frac{n\kappa}{r}\left(\phi\int dy\right)^{-1/2} = \frac{n\sqrt{16\pi G}}{r\sqrt{\phi}}, \quad (5.24)$$

where we have normalized the definition of A_α in the action (5.17) by dividing out the factor $\left(\phi\int dy\right)^{1/2}$, and made use of the definitions (5.8) and (5.10) for κ and G respectively. As a corollary to this result one can also come close to *predicting* the value of the fine structure constant, simply by identifying the charge q_1 of the first Fourier mode with the electron charge "e". Taking $r\sqrt{\phi}$ to be on the order of the Planck length $l_{pl}=\sqrt{G}$, one has:

$$\alpha \equiv \frac{q_1^2}{4\pi} \approx \frac{\left(\sqrt{16\pi G}/\sqrt{G}\right)^2}{4\pi} = 4. \quad (5.25)$$

The possibility of thus explaining an otherwise "fundamental constant" would have made compactified five-dimensional Kaluza-Klein theory very attractive.

We note that the result of the expression (5.25), i.e. 4, expression related to the eqs. (5.8), (5.9) and (5.10), is connected with some expressions concerning the mean increase factor of the partition function p(n), the Legendre constant "c" = 1,08366 , related to the prime numbers and, indirectly, to the Riemann zeta function, and the "aurea" section and "aurea" ratio, related to the Fibonacci's numbers. In fact, we have the following expressions:

$$\frac{1}{2}\left[(1{,}375)^4+(1{,}375)^5\right]=\frac{1}{2}\cdot 8{,}48934 \cong 4.2 \quad (5.26)$$

$$\left[(c)^{17}+\frac{1}{4}(c)\right]=\left[(1{,}08366)^{17}+\frac{1}{14}(1{,}08366)\right]=3{,}919058+0{,}077404=3{,}996462\cong 4 \quad (5.27)$$

$$\left[\left(\frac{\sqrt{5}+1}{2}\right)^3-\frac{1}{3}\left(\frac{\sqrt{5}-1}{2}\right)\right]=\left[(1{,}618033)^3-\frac{1}{3}(0{,}618033)\right]=4{,}236060-0{,}206011=4{,}030049\cong 4.$$

(5.28).

Is very important remember that $1{,}375=c^4\cdot\sqrt[8]{c}=1{,}08^4\cdot 1{,}01009\cong 1{,}375$ and that $c^6=1{,}0836^6\cong 1{,}618$. Then "c" is related to the mean increase factor of p(n) and to the Fibonacci's numbers.

Furthermore, the eq. (5.28) is connected with two fundamental equations offered from the mathematician S. Ramanujan [9], also related to the "aurea" section, "aurea" ratio and the partition function, and precisely the following expressions:

$$\frac{1}{\sqrt{R(q)}}-\left(\frac{1+\sqrt{5}}{2}\right)\sqrt{R(q)}=\frac{1}{q^{1/10}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1+\left(\frac{1+\sqrt{5}}{2}\right)q^{n/5}+q^{2n/5}}, \quad (5.29)$$

$$R(q)=\frac{\sqrt{5}-1}{2}-\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}. \quad (5.30)$$

Then, we have the interesting connections:

$$-\int d^4x\sqrt{-g}\phi\left(\frac{R}{16\pi G}+\frac{1}{4}\phi^2 F_{\alpha\beta}F^{\alpha\beta}+\frac{2}{3\kappa^2}\frac{\partial^\alpha\phi\partial_\alpha\phi}{\phi^2}\right)\Rightarrow\alpha\equiv\frac{q_1^2}{4\pi}\approx\frac{\left(\sqrt{16\pi G}/\sqrt{G}\right)^2}{4\pi}=4=$$

$$=\left[\left(\frac{1}{\sqrt{R(q)}}-\frac{1}{q^{1/10}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1+\left(\frac{1+\sqrt{5}}{2}\right)q^{n/5}+q^{2n/5}}\right)\cdot\frac{1}{\sqrt{R(q)}}\right]^3+$$

$$-\frac{1}{3}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right].\quad(5.31)$$

Then, also for the eqs. (5.9) and (5.25) there are very important connections with "c", p(n), "aurea" section, "aurea" ratio, related to the Fibonacci's numbers, and the Riemann zeta function.

Now, also the eq. (5.19) and the expression $\sigma=\frac{1}{\sqrt{3}\kappa}\ln\phi$ are related with Palumbo's model and with some expression concerning the Riemann zeta function. Indeed, we have the following connections:

$$S'=-\int d^4x\sqrt{-g'}\left(\frac{R'}{16\pi g}+\frac{1}{2}\partial'^\alpha\sigma\partial'_\alpha\sigma\right)\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right],\quad(5.32)$$

with regard the connection concerning the Palumbo's model.

Now, we take the Lemma 3 of Goldston-Montgomery theorem [10]. Let $f(t)\geq 0$ a continuous function defined on $[0,+\infty)$ so that $f(t)<<\log^2(t+2)$. If $I(k)=\int_0^\infty\left(\frac{\sin ku}{u}\right)^2 f(u)du=\left(\frac{\pi}{2}+\varepsilon'(k)\right)k\log\frac{1}{k}$, then

$$J(T)=\int_0^T f(t)dt=(1+\varepsilon')T\log T,\quad(5.33)$$

with $|\varepsilon'|$ small if $|\varepsilon(k)|\leq\varepsilon$ uniformly for $\frac{1}{T\log T}\leq k\leq\frac{1}{T}\log^2 T$. If now we take the expression $\sigma=\frac{1}{\sqrt{3}\kappa}\ln\phi$, and the eq. (5.33), we have the following connection:

$$J(T)=\int_0^T f(t)dt=(1+\varepsilon')T\log T\Rightarrow\frac{1}{\sqrt{3}\kappa}\ln\phi,\quad(5.34)$$

therefore the connection with the equation concerning the Goldston-Montgomery theorem, related with the Riemann zeta function.

6. The dimensional reduction process inherent to Kaluza-Klein theories. [6]

Let $\mathrm{M}(V_n, G, \pi, \Phi)$ be an m-dimensional C^{ν}-differentiable principal fibre bundle defined over the base space V_n, an n-dimensional C^{ν}-manifold, and that possesses as structure group the Lie group G. We will choose an affine connection for the Riemannian manifold M that is metric compatible. The Christoffel symbols are then given in the chosen local coordinates by

$$\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} = \frac{1}{2}\gamma^{\hat{\alpha}\hat{\delta}}\left(\partial_{\hat{\beta}}\gamma_{\hat{\gamma}\hat{\delta}} + \partial_{\hat{\gamma}}\gamma_{\hat{\delta}\hat{\beta}} - \partial_{\hat{\delta}}\gamma_{\hat{\beta}\hat{\gamma}}\right).$$

The Riemann's tensor components follow, in the usual way, from

$$R^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}\hat{\delta}} = \partial_{\hat{\gamma}}\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\delta}} - \partial_{\hat{\delta}}\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} + \Gamma^{\hat{\varepsilon}}_{\hat{\beta}\hat{\delta}}\Gamma^{\hat{\alpha}}_{\hat{\varepsilon}\hat{\gamma}} - \Gamma^{\hat{\varepsilon}}_{\hat{\beta}\hat{\gamma}}\Gamma^{\hat{\alpha}}_{\hat{\varepsilon}\hat{\delta}}.$$

We then construct the Ricci tensor, $R_{\hat{\alpha}\hat{\beta}} = R^{\hat{\lambda}}_{\hat{\alpha}\hat{\lambda}\hat{\beta}}$,

$$R_{ab} = R_{ab}(G) + \frac{1}{4}F_{\mu\nu a}F^{\mu\nu}_{b} + \frac{1}{2}\xi^{cd}D_{\mu}\xi_{ac}D^{\mu}\xi_{bd} - \frac{1}{4}\xi^{cd}D_{\mu}\xi_{ab}D^{\mu}\xi_{cd} - \frac{1}{2}D_{\mu}\left(D^{\mu}\xi_{ab}\right),$$

$$R_{\mu\nu} = R_{\mu\nu}(V_n) - \frac{1}{2}F_{\mu\sigma a}F^{\mu a}_{\nu} - \frac{1}{4}\xi^{ab}\xi^{cd}D_{\mu}\xi_{ac}D_{\nu}\xi_{bd} - \frac{1}{2}D_{\mu}\left(\xi^{ab}D_{\nu}\xi_{ab}\right),$$

$$R_{\mu a} = \frac{1}{2}D^{\sigma}F_{\mu\sigma a} + \frac{1}{4}F_{\mu\sigma a}\xi^{cd}D^{\sigma}\xi_{cd} - \frac{1}{2}C^{c}_{ab}\xi^{bd}D_{\mu}\xi_{cd},$$

where $R(G)$ and $R(V_n)$ refers to the Ricci tensor of G and of V_n, respectively. From the Ricci tensor we determine the scalar of curvature, $R = \gamma^{\hat{\alpha}\hat{\beta}}R_{\hat{\alpha}\hat{\beta}}$:

$$R = R(G) + R(V_n) - \frac{1}{4}F_{\mu\nu a}F^{\mu\nu a} - \frac{1}{4}D_{\mu}\xi_{ab}D^{\mu}\xi^{ab} - \frac{1}{4}\xi^{ab}\xi^{cd}D_{\mu}\xi_{ab}D^{\mu}\xi_{cd} - D^{\mu}\left(\xi^{ab}D_{\mu}\xi_{ab}\right). \quad (6.1)$$

The Hilbert-Einstein-Yang-Mills lagrangian density in m dimensions with a cosmological term is defined as

$$L^{(m)}_{HEYM} = \sqrt{-\gamma}\left(R - 2\Lambda_m\right) = \sqrt{-g}\,\xi^{1/2}\Big[R(G) + R(V_n) - \frac{1}{4}F_{\mu\nu\alpha}F^{\mu\nu\alpha} - \frac{1}{4}D_{\mu}\xi_{ab}D^{\mu}\xi^{ab} + \\ -\frac{1}{4}\xi^{ab}\xi^{cd}D_{\mu}\xi_{ab}D^{\mu}\xi_{cd} - D^{\mu}\left(\xi^{ab}D_{\mu}\xi_{ab}\right) - 2\Lambda_m\Big], \quad (6.2)$$

where Λ_m is the cosmological constant in M and γ, g and ξ are the determinants of the metrics of M, V_n and G, respectively. Since γ depends only on $x \in V_n$, R, g, ξ will only depend on $x \in V_n$. The same happens for the Yang-Mills strength tensor field F. We can then reduce the previous lagrangian density to a n-dimensional one:

$$\int_{\mathrm{M}} L^{(m)}_{HEYM} d^m x = vol(G)\int_{V_n} L^{(n)}_{HEYM} d^n x \,.$$

The Einstein-Yang-Mills equations, in the absence of matter fields and considering the cosmological term, are then the Euler-Lagrange equations of the following action

$$S[\gamma] = \int_{\mathrm{M}} d^m x\sqrt{-\gamma}\left(R - 2\Lambda_m\right) = \int_{\mathrm{M}} d^m \sqrt{-g}\,\xi^{1/2}\left(R - 2\Lambda_m\right).$$

Taking $\delta S = 0$, one finds

$$\left(R_{\hat{\alpha}\hat{\beta}} - \frac{1}{2}\gamma_{\hat{\alpha}\hat{\beta}}\left(R - 2\Lambda\right)\right)\delta\gamma^{\hat{\alpha}\hat{\beta}} = 0\,.$$

The variations $\delta\gamma^{\hat{\alpha}\hat{\beta}}$ are not completely arbitrary: they should be in such way that the special form that was given to $\gamma^{\hat{\alpha}\hat{\beta}}$ as a composition of the metrics of the submanifolds of the bundle would be preserved in those variations. If we varies δS not in order to γ but separatively with respect to g, A and ξ we get:

$$R_{\mu\nu} - \frac{1}{2}\left(R - 2\Lambda_m\right)g_{\mu\nu} = 0\,, \ (6.3) \qquad R_{\mu a} = 0\,, \ (6.4) \qquad R_{ab} - \frac{1}{2}\left(R - 2\Lambda_m\right)\xi_{ab}\,. \ (6.5)$$

7. Realistic model within the Kaluza-Klein frame: the Randall-Sundrum model.

Within the Randall-Sundrum model matter fields are located in a particular brane, being only gravity the one allowed to propagate in the bulk. We will then consider a matter field configuration as a section of a fibre bundle $F_I\left(V_{n_I}, f, \Upsilon\right)$ defined over a brane $V_{n_I} \subset \mathrm{M}$.
The classical action of an Randall-Sundrum scheme will be given by an Hilbert-Einstein term with a cosmological constant Λ_m in M,b Gibbons-Hawking terms with the respective brane cosmological constants $\left\{\Lambda^I_{n_I}\right\}_{I=1,\dots,b}$, and a matter field action of fields located on the branes (we consider $n_I = n = m-1$) and on the bulk,

$$S[\gamma, g_1,\dots, g_b, \phi] = S_{HE}[\gamma] + S_{GH}[g_1,\dots, g_b] + S_{BM}[g_1,\dots, g_b, \phi] + S_M[\gamma, \phi] \quad (7.1)$$

with

$$S_{HE}[\gamma] = \frac{1}{2\kappa^2}\int_{\mathrm{M}} d^m z\sqrt{-\gamma}\left(R - 2\Lambda_m\right), \ (7.2)$$

$$S_{GH}[g_1,\dots, g_b] = \frac{1}{\kappa^2}\sum_{I=1}^{b}\int_{V^I_{n_I}} d^{n_I} x_I \sqrt{-g_I}\left(K^I - \Lambda^I_{n_I}\right), \ (7.3)$$

$$S_{BM}[g_1,\dots, g_b, \phi] = \sum_{I=1}^{b}\int_{V^I_{n_I}} d^{n_I} x_I L^I_{BM}\left(\phi, \partial\phi, g_I\right), \ (7.4)$$

$$S_M[\gamma, \phi] = \int_{\mathrm{M}} d^m z L_M\left(\phi, \partial\phi, \gamma\right), \ (7.5)$$

therefore:

$$S[\gamma, g_1,\dots, g_b, \phi] = \frac{1}{2\kappa^2}\int_{\mathrm{M}} d^m z\sqrt{-\gamma}\left(R - 2\Lambda_m\right) + \frac{1}{\kappa^2}\sum_{I=1}^{b}\int_{V^I_{n_I}} d^{n_I} x_I \sqrt{-g_I}\left(K^I - \Lambda^I_{n_I}\right) +$$

$$+\sum_{I=1}^{b}\int_{V_{n_I}^I} d^{n_I}x_I L_{BM}^I(\phi,\partial\phi,g_I)+\int_{\mathrm{M}} d^m z L_M(\phi,\partial\phi,\gamma),\ (7.6)$$

and where K^I is the trace of the extrinsic curvature of the I brane, that can be written in terms of a normal vector n^I to $V_{n_I}^I$

$$K_{\hat{\alpha}\hat{\beta}}^I=\tilde{\nabla}_{\hat{\alpha}}n_{\hat{\beta}}^I.\ (7.7)$$

If we take Gaussian normal coordinates to the brane V_{n_I}, we can simply take

$$K_{\hat{\alpha}\hat{\beta}}^I=\frac{1}{2}\partial_\eta\gamma_{\hat{\alpha}\hat{\beta}},\ (7.8)$$

with δ_η the derivative along the normal coordinate to the brane.

By taking the variation of this action (7.1) with respect to $\{\gamma,g_I\}_{I=1,\dots,b}$,

$$\delta S[\gamma,g_I]=\int_{\mathrm{M}} d^m z\sqrt{-\gamma}\left[-\frac{1}{2\kappa^2}\left(R_{\hat{\alpha}\hat{\beta}}-\frac{1}{2}\gamma_{\hat{\alpha}\hat{\beta}}R+\Lambda_m\gamma_{\hat{\alpha}\hat{\beta}}\right)+\frac{1}{2}T_{\hat{\alpha}\hat{\beta}}\right]\delta\gamma^{\hat{\alpha}\hat{\beta}}+$$
$$+\sum_{I=1}^{b}\int_{V_{n_I}^I} d^{n_I}x_I\sqrt{-g_I}\left[-\frac{1}{2\kappa^2}\left(K_{\alpha\beta}^I-g_{\alpha\beta}^I K^I+\Lambda_{n_I}^I g_{\alpha\beta}^I\right)+\frac{1}{2}T_{\alpha\beta}^I\right]\delta g^{\alpha\beta},\ (7.9)$$

where $$T_{\hat{\alpha}\hat{\beta}}=\frac{2}{\sqrt{-\gamma}}\frac{\delta S_M[\gamma,\phi]}{\delta\gamma^{\hat{\alpha}\hat{\beta}}},\ (7.10)\qquad T_{\alpha\beta}^I=\frac{2}{\sqrt{-g^I}}\frac{\delta S_{BM}[g_1,\dots,g_b,\phi]}{\delta g_I^{\alpha\beta}},\ (7.11)$$

are the m-dimensional stress-energy tensor of the bulk matter fields and the n_I dimensional stress-energy tensor of the matter fields on $V_{n_I}^I$, we obtain the Einstein equations,

$$\frac{\delta S[\gamma,g_I,\phi]}{\delta\gamma^{\hat{\alpha}\hat{\beta}}}=-\frac{1}{2\kappa^2}\sqrt{-\gamma}\left(R_{\hat{\alpha}\hat{\beta}}-\frac{1}{2}\gamma_{\hat{\alpha}\hat{\beta}}R+\Lambda_m\gamma_{\hat{\alpha}\hat{\beta}}\right)+\frac{1}{2}\sqrt{-\gamma}T_{\hat{\alpha}\hat{\beta}}=0,\ (7.12)$$

for I = 1,…,b.

In order to find a solution we have to solve the Einstein equation

$$R_{\hat{\alpha}\hat{\beta}}-\frac{1}{2}\gamma_{\hat{\alpha}\hat{\beta}}R+\Lambda_m\gamma_{\hat{\alpha}\hat{\beta}}=\kappa^2T_{\hat{\alpha}\hat{\beta}},\ (7.13)$$

in the space between the branes - the bulk - , and then assure the jump over all the branes,

$$\Delta\left[K_{\alpha\beta}^I-g_{\alpha\beta}^I K^I\right]+\Lambda_{n_I}^I g_{\alpha\beta}^I=\kappa^2T_{\alpha\beta}^I.\ (7.14)$$

Another way of writing the Israel junction conditions is

$$\Delta K_{\alpha\beta}^I+\frac{1}{3}\Lambda_{n_I}^I g_{\alpha\beta}^I=\kappa^2\left(T_{\alpha\beta}^I-\frac{1}{3}T_\alpha^{\alpha I}g_{\alpha\beta}^I\right),\ (7.15)$$

where $\Delta K^{I}_{\alpha\beta} = K^{I+}_{\alpha\beta} - K^{I-}_{\alpha\beta}$, (7.16) with $K^{I\pm}_{\alpha\beta} = \lim_{z\to(x^I,0^\pm)} K^{I}_{\alpha\beta}$,

being the limit taken along the normal n^I to the brane.

Let us consider a 5-dimensional universe of the form $\mathrm{M} = \overline{\mathrm{M}} \times S^1 / \mathrm{Z}_2$, where the internal space is a orbifold with a Z_2 symmetry. We will consider that there are two four-dimensional branes in this universe, and by choosing a coordinate $y \in [0, d]$ for the internal space we will suppose that they lie at the fixed points y = 0 and y = d. In order to find the ground state for this model, we consider the following action,

$$S[\gamma, g^{\pm}] = \frac{1}{2\kappa^2}\left(\int d^4x \int_{-d}^{d} dy \sqrt{-\gamma}(R - 2\Lambda_5) - 2\Lambda_4^{+} \int d^4x \sqrt{-g^{+}} - 2\Lambda_4^{-} \int d^4x \sqrt{-g^{-}} \right), \quad (7.17)$$

where we have represented the brane at y = 0 by the plus sign and that at y = d by the minus sign. We will then get the following system of equations,

$$R_{\hat{\alpha}\hat{\beta}} - \frac{1}{2}\gamma_{\hat{\alpha}\hat{\beta}} R + \gamma_{\hat{\alpha}\hat{\beta}} \Lambda_5 = 0, \quad (7.18) \quad \text{and} \quad \Delta K^{\pm}_{\alpha\beta} + \frac{1}{3}\Lambda_4^{\pm} g^{\pm}_{\alpha\beta} = 0, \quad (7.19) \quad \text{with}$$

$$\Delta K^{\pm}_{\alpha\beta} = \frac{1}{2}\left(\partial_y g^{\pm}_{\alpha\beta}\left(y^{\pm}\big|_{\mp}\right) - \partial_y g^{\pm}_{\alpha\beta}\left(y^{\pm}\big|_{\mp}\right) \right), \quad (7.20)$$

being $y^{\pm}\big|_{\mp}$ the limit taken from both sides of the position of the brane. Since we have chosen Gaussian normal coordinates for both branes that overlap, we can simply take

$$g^{\pm} = \gamma\left(y^{\pm}\right)\big|_{\pm}. \quad (7.21)$$

The equations can then be written in terms of γ only,

$$R_{\hat{\alpha}\hat{\beta}} - \frac{1}{2}\gamma_{\hat{\alpha}\hat{\beta}} R + \gamma_{\hat{\alpha}\hat{\beta}} \Lambda_5 = 0, \quad (7.22) \qquad \partial_y \gamma_{\alpha\beta}\left(y^{\pm}\big|_{\mp}\right)\Big|_{\pm} - \partial_y \gamma_{\alpha\beta}\left(y^{\pm}\big|_{\mp}\right)\Big|_{\pm} + \frac{2}{3}\Lambda_4^{\pm}\gamma\left(y^{\pm}\right)\big|_{\pm} = 0. \quad (7.23)$$

The Randall-Sundrum ansatz is

$$\gamma = \begin{pmatrix} e^{-2\phi(y)}\eta_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix}. \quad (7.24)$$

By inserting it in the previous equations, we get for $\phi(y)$,

$$\phi(y) = |y|/l, \quad (7.25)$$

with $l = \sqrt{-\frac{6}{\Lambda_5}}$ (7.26) and for $\Lambda_4^{\pm}$, $\Lambda_4^{\pm} = \pm\frac{\sqrt{-6\Lambda_5}}{\kappa^2}$. (7.27)

We then conclude that the bulk of this model must be a slice of an AdS_5 geometry, being the observed universe the brane located at $y = y^- = d$.
Now we note that the eq. (7.9) can be related to the Palumbo's model and to the Ramanujan's modular function [8]-[9]. Indeed, the Veneziano Model of the Eulero beta function describes the strong nuclear force. When a string moves in space-time by splitting and recombining, a large number of mathematical identities must be satisfied. These are the identities of Ramanujan's modular function. The KSV (Kikkawa-Sakita-Virasoro) loop diagrams of interacting strings can be described using modular functions. The "Ramanujan function", [8] an elliptic modular function that satisfies the "conformal symmetry", has 24 "modes" (24 + 2 = 26) that correspond to the physical vibrations of a bosonic string. Furthermore, when the Ramanujan function is generalized, 24 is replaced by 8 (8 + 2 = 10), hence, has 8 "modes" that correspond to the physical vibrations of a superstring.
Ramanujan's function τ is defined by the expansion

$$x\prod_1^\infty \left(1-x^n\right)^{24} = \sum_1^\infty \tau(n)x^n , \quad (7.28)$$

which is valid for each complex number x such that $|x|<1$.
The Ramanujan's modular function is also related to the Rogers-Ramanujan identity (5.30). Hence, we have the following interesting connections:

$$\delta S[\gamma, g_I] = \int_M d^m z\sqrt{-\gamma}\left[-\frac{1}{2\kappa^2}\left(R_{\hat\alpha\hat\beta} - \frac{1}{2}\gamma_{\hat\alpha\hat\beta}R + \Lambda_m\gamma_{\hat\alpha\hat\beta}\right) + \frac{1}{2}T_{\hat\alpha\hat\beta}\right]\delta\gamma^{\hat\alpha\hat\beta} +$$
$$+\sum_{I=1}^b \int_{V_{n_I}^I} d^{n_I}x_I\sqrt{-g_I}\left[-\frac{1}{2\kappa^2}\left(K_{\alpha\beta}^I - g_{\alpha\beta}^I K^I + \Lambda_{n_I}^I g_{\alpha\beta}^I\right) + \frac{1}{2}T_{\alpha\beta}^I\right]\delta g^{\alpha\beta} \Rightarrow$$
$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde H_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] =$$
$$= \int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] \Rightarrow$$
$$\Rightarrow x\prod_1^\infty \left(1-x^n\right)^{24} = \sum_1^\infty \tau(n)x^n \Rightarrow \frac{\sqrt{5}-1}{2} - \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)} . \quad (7.29)$$

Furthermore, we note that the pure number 24 is connected at some expressions concerning the aurea ratio $\Phi = 1{,}618033$ and aurea section $\phi = 0{,}618033$, related to the Fibonacci's numbers, the Legendre constant "c" = 1,08366 related to the prime numbers and the number F_m = 1,375 that is the mean increase factor of the partition function p(n). Indeed, we have

$$\left[\left(\frac{\sqrt{5}+1}{2}\right)^6 + \left(\frac{\sqrt{5}+1}{2}\right)^7\right]\cdot\frac{1}{2} + \left(\frac{\sqrt{5}-1}{2}\right) \cong 24; \quad (c)^{40} - \frac{1}{2}(c)^7 \cong 24; \; \left(F_m\right)^{10} - \frac{1}{3^2}\left(F_m\right) \cong 24 . \quad (7.30)$$

Recent cosmological observations suggest the existence of a positive cosmological constant Λ with the magnitude $\Lambda\left(G\hbar/c^3\right) \approx 10^{-123}$. Now, we take the pure numbers 10 and 123, i.e. the base and the exponent of cosmological constant magnitude. They are related to "c", F_m, Φ and ϕ, from the following expressions:

$$F_m - \left[\left(\frac{1}{3^2}\cdot F_m + \frac{1}{2\cdot 5}\cdot F_m\right)\cdot\frac{1}{2}\right]\cdot\left(2^2\cdot 5^2\right) \cong 123; \quad \left[\left(c^3\right) - \frac{1}{2\cdot 13}\cdot c\right]\cdot\left(2^2\cdot 5^2\right) \cong 123;$$

$$\left[\left(\frac{\sqrt{5}+1}{2}\right) - \frac{1}{2^3/5}\left(\frac{\sqrt{5}-1}{2}\right)\right]\cdot\left(2^2\cdot 5^2\right) \cong 123. \quad (7.31)$$

Also in these cases is very interesting remember that $1{,}375 = c^4 \cdot \sqrt[8]{c} = 1{,}08^4 \cdot 1{,}009666552 \cong 1{,}375$ and that $c^6 = 1{,}0836^6 \cong 1{,}618$. Hence, the Legendre constant "c" is related to the mean increase factor of p(n) and to the aurea ratio and aurea section, Φ and ϕ.

8. Connections between axions in string theory, dark matter, Palumbo's model, Riemann zeta function and Number Theory (Legendre constant, Mean increase factor of p(n), Aurea ratio Φ and Aurea section ϕ). [7]

The relevant part of the ten-dimensional low energy Lagrangian of the heterotic string, from the equation

$$S_{het} = \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2} e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_V\left(\left|F_2\right|^2\right)\right], \quad (8.1)$$

is

$$L = \frac{1}{2\kappa_{10}^2}\sqrt{-g}R - \frac{1}{4\kappa_{10}^2}H\wedge *H - \frac{\alpha'}{8\kappa_{10}^2}trF\wedge *F = \frac{2\pi}{g_s^2 l_s^8}\sqrt{-g}R - \frac{2\pi}{g_s^2 l_s^4}\frac{1}{2}H\wedge *H - \frac{1}{4(2\pi)g_s^2 l_s^6}trF\wedge *F$$

(8.2).

Here R is the Ricci scalar, H the field strength of the two-form field B, and F the $E_8\times E_8$ or SO(32) curvature. To reduce to four dimensions, we compactify on a six-manifold Z with volume V_Z. The four-dimensional spacetime (which might be Minkowski spacetime) we call M. The relevant terms in the four-dimensional effective action include

$$S = \frac{M_P^2}{2}\int d^4x(-g)^{1/2}R - \frac{1}{4g_{YM}^2}\int d^4x\sqrt{-g}trF_{\mu\nu}F^{\mu\nu} - \frac{2\pi V_Z}{g_s^2 l_s^4}\int\left(\frac{1}{2}H\wedge *H\right), \quad (8.3)$$

where the four-dimensional reduced Planck mass M_P and Yang-Mills coupling g_{YM} are

$$M_P^2 = 4\pi\frac{V_Z}{g_s^2 l_s^8} \quad (8.4) \quad \text{and} \quad g_{YM}^2 = 4\pi\frac{g_s^2 l_s^6}{V_Z}. \quad (8.5)$$

So $\alpha_{YM} = g_{YM}^2/4\pi$ is given by $\alpha_{YM} = \frac{g_s^2 l_s^6}{V_Z}$. (8.6)

Now we consider the model-independent axion. The Bianchi identity for the gauge-invariant field strength of H is

$$dH = \frac{1}{16\pi^2}\left(trR \wedge R - trF \wedge F\right). \quad (8.7)$$

(The normalization can be extracted from equation $\tilde{H}_3 = dB_2 - \frac{\kappa_{10}^2}{g_{10}^2}\omega_3$, $\delta B_2 = \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\lambda dA_1\right)$ related to eq. (8.1), bearing in mind that our H is l_s^2 times the H-field used there). The four-dimensional component of the B-field can be dualized by introducing a field a that is a Lagrange multiplier for the Bianchi identity, the coupling being $\int a\left(dH + \frac{1}{16\pi^2}\left(trF \wedge F - trR \wedge R\right)\right)$. Including also the B-field kinetic energy from (8.3), the action is

$$-\frac{2\pi V_Z}{g_s^2 l_s^4}\int d^4x \frac{1}{2} H \wedge *H + \int a\left(dH + \frac{1}{16\pi^2}\left(trF \wedge F - trR \wedge R\right)\right). \quad (8.8)$$

Here H is an independent field variable (which can be expressed in terms of B if one integrates first over a to impose the Bianchi identity).
As H has integer periods, a should have period 2π. Instead, we integrate out H to get an effective action for a:

$$S(a) = \frac{g_s^2 l_s^4}{2\pi V_Z}\int d^4x\left(-\frac{1}{2}\partial_\mu a \partial^\mu a\right) + \int a \frac{1}{16\pi^2}\left(trF \wedge F - trR \wedge R\right). \quad (8.9)$$

Since $trF \wedge F = 2ktrF \wedge F$, we see that for this particular axion, the integer r characterizing the axionic coupling is equal to the current algebra level k. So we can read off the axion decay constant:

$F_a = \frac{g_s^2 l_s^2}{\sqrt{2\pi V_Z}} = \frac{k\alpha_G}{2\pi}\frac{M_P}{\sqrt{2}}$. (8.10) The axion couplings are proportional to F_a / k, which is $\frac{F_a}{k} = \frac{\alpha_C M_P}{2\pi\sqrt{2}}$. (8.11) If we take $\alpha_C = 1/25$, this gives $F_a / k \approx 1.1 \times 10^{16} GeV$.

Gauge instantons at the string scale have action $\frac{2\pi}{\alpha_{YM}} = \frac{2\pi}{k\alpha_C} \approx \frac{157}{k}$, (8.12) by analogy with the action of an instanton, that is $I = \frac{8\pi^2}{g^2} = \frac{2\pi}{\alpha_s}$, where $\alpha_s = g^2 / 4\pi$.

Model-dependent heterotic string axions arise from zero modes of the B-field on the compact manifold Z. Let there be $n = \dim H^2(Z, R)$ such zero modes $\beta_1, ..., \beta_n$. We normalize them so that

$\int_{C_j}\beta_i = \delta_{ij}$, (8.13) where the C_j are two-cycles representing a basis of $H_2(Z, Z)$ modulo torsion. Then we make an ansatz $B = \frac{1}{2\pi}\sum_i \beta_i b_i$, (8.14) where b_i are four-dimensional fields. The factor of $1/2\pi$ is included so that the fields b_i have periods 2π, as is conventional for axions. Set

$\gamma_{ij} = \int_Z \beta_i \wedge * \beta_j$. (8.15) By dimensional reduction from the B-field kinetic energy in (8.2), the kinetic energy of the b_i fields in four dimensions comes out to be

$$S_{kin} = -\frac{1}{2\pi g_s^2 l_s^4} \int d^4x \frac{\gamma_{ij}}{2} \partial_\mu b_i \partial^\mu b_j . \quad (8.16)$$

These modes acquire axionic couplings from the one-loop couplings that enter the Green-Schwarz anomaly cancellation mechanism. The relevant couplings are

$$\frac{-1}{4(2\pi)^3 4!} \int B \left\{ -\frac{TrF \wedge F trR \wedge R}{30} + \frac{TrF^4}{3} - \frac{(TrF \wedge F)^2}{900} \right\} . \quad (8.17)$$

To proceed further, we consider the $E_8 \times E_8$ heterotic string, embedding the Standard Model in the first E_8, and write tr_1 and tr_2 for traces in the first or second E_8. The couplings in four dimensions of the axion modes to $tr_1 F \wedge F$ come out to be

$$-\frac{1}{2\pi^2 4!} \sum_i \int_Z \beta_i \left\{ -\frac{trR \wedge R}{2} + 2tr_1 F \wedge F - tr_2 F \wedge F \right\} \int_M b_i \frac{tr_1 F \wedge F}{16\pi^2} . \quad (8.18)$$

Using the Bianchi identity (8.7), one can alternatively write these couplings as

$$-\sum_i \int_Z \beta_i \wedge \frac{1}{16\pi^2} \left(tr_1 F \wedge F - \frac{1}{2} trR \wedge R \right) \int_M b_i \frac{tr_1 F \wedge F}{16\pi^2} . \quad (8.19)$$

Now, we consider intersecting D-brane models in type IIA string theory. We assume that gauge symmetry lives on D(3+q)-branes which are extended along the four noncompact dimensions and wrap a q-cycle Q in the compactification manifold. In Type IIA, one takes a stack of five D6-branes wrapped around $M \times Q$ where M is the Minkowski space and Q is a compact special Lagrangian three-cycle in the compact manifold X. The low energy effective supergravity action contains the gravitational term

$$S_{GR} = \frac{2\pi}{g_s^2 l_s^8} \int \sqrt{-g} R . \quad (8.20)$$

Dimensionally reducing the Einstein action to four dimensions determines the Planck mass

$$M_P^2 = 4\pi \frac{V_X}{g_s^2 l_s^8} . \quad (8.21)$$

The low energy action of the RR q-form field C_q, from the following equation

$$-\frac{1}{4\kappa_{10}^2} \int d^{10}x (-G)^{1/2} \left| F_{p+2} \right|^2 + \mu_p \int C_{p+1} \text{, is}$$

$$-\frac{2\pi}{l_s^{8-2q}} \int \frac{1}{2} F_{q+1} \wedge * F_{q+1} + 2\pi \int C_q , \quad (8.22)$$

where the second term is the coupling of the q form to D(q – 1) branes. The effective action of the gauge theory living on the D-branes, from the following equation

$$g_{Dp}^2 = \frac{1}{(2\pi\alpha')^2 \tau_p} = (2\pi)^{p-2} g\alpha'^{(p-3)/2} \text{, is}$$

$$S_{YM} = -\frac{1}{4(2\pi) g_s l_s^{4+q}} \int d^q x \sqrt{-g}\, trF_{\mu\nu}F^{\mu\nu}, \quad (8.23)$$

where the trace is in the fundamental N representation of SU(N). Reducing the gauge action on Q to four dimensions leads to the action

$$-\frac{V_Q}{8(2\pi) g_s l_s^q} \int d^4 x \sqrt{-g}\, F_{\mu\nu}^a F^{\mu\nu a}, \quad (8.24)$$

where we used the normalization of SU(N) generators $\mathbf{tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$.

The axions come from the ansatz $C_q = \frac{1}{2\pi}\sum_\alpha a_\alpha \omega_\alpha, \quad \alpha = 1,\ldots,b_q(X)$. (8.25)

We included a factor of $1/2\pi$ so that a_α have period 2π. Substituting this into the RR-field effective action (8.22) we get the kinetic energy of the axions

$$S = -\frac{1}{2}\sum_{\alpha,\beta} \gamma_{\alpha\beta} \partial_\mu a_\alpha \partial^\mu a_\beta, \quad (8.26)$$

where $\gamma_{\alpha\beta} = \frac{1}{2\pi l_s^{8-2q}} \int_X \omega_\alpha \wedge * \omega_\beta$, (8.27) hence

$$S = -\frac{1}{2}\sum_{\alpha,\beta} \frac{1}{2\pi l_s^{8-2q}} \int_X \omega_\alpha \wedge * \omega_\beta \partial_\mu a_\alpha \partial^\mu a_\beta .$$

The axions acquire axionic couplings from the D-brane Chern-Simons term,

$i\mu_p \int_{p+1} Tr\left[\exp(2\pi\alpha' F_2 + B_2) \wedge \sum_q C_q\right]$, hence, we have

$$2\pi \int_{M\times Q} C_q \wedge \frac{1}{8\pi^2} trF \wedge F . \quad (8.28)$$

Dimensionally reducing this to four dimensions using the ansatz (8.25) leads to the couplings

$\sum_\alpha r_\alpha \int a_\alpha \frac{trF \wedge F}{8\pi^2}$, (8.29) where $r_\alpha = \int_Q \omega_\alpha$ are integers.

We consider branes wrapping a q-cycle Q with q > 0. We define R to be linear size of X, so that $V_X = R^6$. In terms of R, the Planck mass is $M_P^2 = \frac{4\pi R^6}{g_s^2 l_s^8}$. (8.30) For a generic axion, $\int_X \omega_Q \wedge * \omega_Q = x R^{6-2q}$, where x is of order one, so

$$F_a = \sqrt{\frac{xR^{6-2q}}{2\pi l_s^{8-2q}}} = M_P \left(\frac{l_s}{R}\right)^q \sqrt{\frac{x g_s^2}{8\pi^2}}, \quad (8.31)$$

where x is a dimensionless number of order one. To estimate the parameters of the compactification that lead to phenomenologically preferred axion decay constants, we express R and $M_s = l_s^{-1}$ in terms of F_α, M_P from (8.30) and (8.31):

$$R = l_s \left(\frac{M_P}{F_a}\right)^{1/q} \left(\frac{x g_s^2}{8\pi^2}\right)^{1/2q}, \quad M_s = F_a \left(\frac{F_a}{M_P}\right)^{3-q/q} \left(\frac{2\pi}{x}\right)^{1/2} \left(\frac{8\pi^2}{x g_s^2}\right)^{3-q/2q}. \quad (8.32)$$

The low energy gauge group on N D3-branes at a generic point in X is U(N). The gauge coupling is fixed by the string coupling (8.23): $\alpha_C = g_s$. (8.33)

The axions are four dimensional fields coming from reduction of the RR zero-form. A harmonic zero-form is just a constant, so we use the ansatz $C_0 = \frac{a}{2\pi}$, (8.34) where “a” is a four-dimensional pseudo-scalar field. It follows from the D-brane Chern-Simons coupling (8.28) that the axion has r = 1 coupling to the QCD instanton density $\int a \frac{trF \wedge F}{8\pi^2}$. (8.35)

The kinetic energy of the RR zero-form (8.22) is easily reduced to four dimensions, giving the axion kinetic energy

$$\frac{V_X}{2\pi l_s^8} \int d^4 x \left(-\frac{1}{2} \partial_\mu a \partial^\mu a\right), \quad (8.36)$$

whence the axion coupling constant is $F_a = \sqrt{\frac{V_X}{2\pi l_s^8}} = \frac{\alpha_C}{2\pi} \frac{M_P}{\sqrt{2}}$. (8.37)

If we take $\alpha_C \approx 1/25$, we get $F_a = 1.1 \times 10^{16}$ GeV, which is the same as the axion coupling parameter of the model-independent axion in weakly coupled heterotic string theory. The shift symmetry of the axion is explicitly broken by D(- 1)-brane instantons that are located on the D-3 brane worldvolume. These instantons are equivalent the SU(N) gauge theory instantons. Their action is

$$I = \frac{2\pi}{g_s} = \frac{2\pi}{\alpha_C} \cong 157, \quad (8.37b) \qquad \text{for } \alpha_C \approx 1/25.$$

On the conifold, there are harmonic two- and three-forms

$$\omega_2 = \frac{1}{2}\left(g^1 \wedge g^2 + g^3 \wedge g^4\right), \qquad \omega_3 = g^5 \wedge \omega_2. \quad (8.38)$$

Integrating the explicit expressions (8.38) for the harmonic two and three-forms over the cycle representatives gives

$\int_{S^2} \omega_2 = 4\pi$, $\int_{S^3} \omega_3 = 8\pi^2$, (8.39) hence $\int_{S^2} \frac{1}{2}\left(g^1 \wedge g^2 + g^3 \wedge g^4\right) = 4\pi$ and $\int_{S^3} g^5 \wedge \frac{1}{2}\left(g^1 \wedge g^2 + g^3 \wedge g^4\right) = 8\pi^2$.

In IIA string theory, we get gauge symmetry by wrapping D6-branes around the small S^3 of the deformed conifold. If the S^3 has radius r_0, the gauge coupling is

$$\alpha_{GUT} = \frac{g_s l_s^3}{2\pi^2 r_0^3}. \quad (8.40)$$

The axions is a four-dimensional scalar b coming from a zero mode of the RR three-form field C_3:

$C_3 = \frac{\omega_3}{8\pi^2}\frac{b}{2\pi}$. (8.41) ω_3 is a harmonic three-form on X with a nonzero flux through the vanishing S^3. We approximate it by a harmonic form on the cone, which is a pullback of the harmonic form ω_3 (8.38) on $T^{1,1}$. With the help of (8.39), we normalized the C-field so that the axion b has period 2π. We find F_b from the general formula for the decay constant of an RR-axion (8.27)

$$F_b^2 = \frac{1}{2\pi l_s^2}\left(\frac{1}{8\pi^2}\right)^2 \int_X \omega_3 \wedge * \omega_3 = \frac{3x}{4\pi^2 l_s^2}\ln\left(\frac{R}{r_0}\right), \quad (8.42)$$

where x is a dimensionless number of order one.
If we assume that the gauge coupling at the string scale is $\alpha_{GUT} \approx 1/25$, it follows from (8.40) that $r_0 \approx l_s$. Furthermore, for (8.37b), we have: $I = 2\pi / \alpha_{GUT} = 2\pi / 1/25 = 2\pi \cdot 25 \cong 157$. But we already know from our estimate (8.32) that $R >> l_s$, whence it follows that $R >> r_0$ and our approximations are self-consistent. To find the range of the string compactification parameters that lead to phenomenologically acceptable axion, we express M_s and R from (8.42) and (8.21), as

$$M_s = \frac{2\pi F_b}{\sqrt{3x\ln(R/r_0)}}, \quad R = l_s\left(\frac{M_P}{F_a}\right)^{1/3} g_s^{1/3}\left[\frac{3x\ln(R/r_0)}{2^4\pi^3}\right]^{1/6}. \quad (8.43)$$

For $10^9 GeV \leq F_b \leq 10^{12} GeV$ and x = 1, we have $1.4\times10^9 GeV \leq M_s \leq 1.8\times10^{12} GeV$,

$$800 l_s \geq \frac{R}{g_s^{1/3}} \geq 73 l_s. \quad (8.44)$$

With regard the Palumbo's model, if we take the eq. (8.1) related to (8.2) and to the following equations, we note that this equation is the right-hand side of the fundamental relation of the Palumbo's model. Then, we have:

$$\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=$$

$$=\int_{0}^{\infty}\frac{1}{2\kappa_{10}^{2}}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_{\mu}\Phi\partial^{\mu}\Phi-\frac{1}{2}\left|\tilde{H}_{3}\right|^{2}-\frac{\kappa_{10}^{2}}{g_{10}^{2}}Tr_{\nu}\left(\left|F_{2}\right|^{2}\right)\right],$$

hence the connection with the Palumbo's model.

Correlations with Number Theory. [9]-[11]

Now, we take the pure numbers 73, 157 and 800. These are prime numbers and are related at some expressions concerning the Legendre constant, the mean increase factor of the partition function p(n), the "aurea" ratio and the "aurea" section (concerning the Fibonacci's numbers). Indeed, we have:

$$\left[(\sigma)^{14}-(\sigma)^{8}-\frac{1}{3}(\sigma)\right]=\left[(1{,}375)^{14}-(1{,}375)^{8}-\frac{1}{3}(1{,}375)\right]\cong 73;\ (8.45)$$

$$\left[(c)^{53}+(c)^{11}\right]=\left[(1{,}08366)^{53}+(1{,}08366)^{11}\right]\cong 73;\ (8.46)$$

$$\left[(\Phi)^{9}-(\Phi)^{2}-\frac{1}{2}(\phi)\right]=\left[\left(\frac{\sqrt{5}+1}{2}\right)^{9}-\left(\frac{\sqrt{5}+1}{2}\right)^{2}-\frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)\right]\cong 73;\ (8.47)$$

$$\left[(\sigma)^{16}-(\sigma)^{6}+\frac{1}{2}(\sigma)\right]=\left[(1{,}375)^{16}-(1{,}375)^{6}+\frac{1}{2}(1{,}375)\right]\cong 157;\ (8.48)$$

$$\left[(c)^{63}-\frac{1}{2}(c)^{6}\right]=\left[(1{,}08366)^{63}-\frac{1}{2}(1{,}08366)\right]\cong 157;\ (8.49)$$

$$\left[(\Phi)^{10}+(\Phi)^{7}+(\Phi)^{3}+(\phi)\right]=\left[\left(\frac{\sqrt{5}+1}{2}\right)^{10}+\left(\frac{\sqrt{5}+1}{2}\right)^{7}+\left(\frac{\sqrt{5}+1}{2}\right)^{3}+\left(\frac{\sqrt{5}-1}{2}\right)\right]\cong 157.\ (8.50)$$

With regard the pure number 800, we have: $800=32\cdot 25=2^{5}\cdot 5^{2}$, thence the following expressions:

$$\left[(1{,}08366)^{43}+\frac{1}{3}(1{,}08366)\right]\cdot\left[(1{,}08366)^{40}+\frac{1}{3^{2}}(1{,}08366)\right]\cong 2^{5}\cdot 5^{2}\cong 800;\ (8.51)$$

$$\left[(1{,}375)^{10}+\frac{3}{5}(1{,}375)\right]\cdot\left[(1{,}375)^{11}-(1.375)^{1/2}\right]\cong 5^{2}\cdot 2^{5}\cong 800;\ (8.52)$$

$$\left[\left(\frac{\sqrt{5}+1}{2}\right)^6+\left(\frac{\sqrt{5}+1}{2}\right)^4+\frac{1}{3}\left(\frac{\sqrt{5}-1}{2}\right)\right]\cdot\left[\left(\frac{\sqrt{5}+1}{2}\right)^7+\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)\right]\cong 5^2\cdot 2^5\cong 800. \quad (8.53)$$

Also here, we have the connections with the Rogers-Ramanujan identity. Thence, from the expressions (8.3)-(8.6),(8.12) and (8.50), we obtain:

$$S=\frac{M_P^2}{2}\int d^4x(-g)^{1/2}R-\frac{1}{4g_{YM}^2}\int d^4x\sqrt{-g}trF_{\mu\nu}F^{\mu\nu}-\frac{2\pi V_Z}{g_s^2 l_s^4}\int\left(\frac{1}{2}H\wedge *H\right)\Rightarrow$$

$$\Rightarrow\left[\left(\frac{\sqrt{5}+1}{2}\right)^{10}+\left(\frac{\sqrt{5}+1}{2}\right)^7+\left(\frac{\sqrt{5}+1}{2}\right)^3+R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\cong 157; \quad (8.54)$$

With regard the pure numbers 800 and 73, we have the following expressions:

$$\left[\left(\frac{\sqrt{5}+1}{2}\right)^6+\left(\frac{\sqrt{5}+1}{2}\right)^4+\frac{1}{3}\left(R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)\right]\times$$

$$\times\left[\left(\frac{\sqrt{5}+1}{2}\right)^7+\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2}\left(R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)\right]\cong 5^2\cdot 2^5\cong 800; \quad (8.55)$$

$$\left[\left(\frac{\sqrt{5}+1}{2}\right)^9-\left(\frac{\sqrt{5}+1}{2}\right)^2-\frac{1}{2}\left(R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)\right]\cong 73. \quad (8.56)$$

Now, from the equation (8.42), we have the following interesting connections:

$$F_b^2 = \frac{1}{2\pi l_s^2}\left(\frac{1}{8\pi^2}\right)\int_X \omega_3 \wedge *\omega_3 = \frac{3x}{4\pi^2 l_s^2}\ln\left(\frac{R}{r_0}\right) \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T \Rightarrow$$

$$\Rightarrow \left[\left(\frac{\sqrt{5}+1}{2}\right)^6 + \left(\frac{\sqrt{5}+1}{2}\right)^4 + \frac{1}{3}\left(R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)\right] \times$$

$$\times \left[\left(\frac{\sqrt{5}+1}{2}\right)^7 + \left(\frac{\sqrt{5}+1}{2}\right)^2 + \frac{1}{2}\left(R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)\right] l_s \geq \frac{R}{g_s^{1/3}} \geq$$

$$\geq \left[\left(\frac{\sqrt{5}+1}{2}\right)^9 - \left(\frac{\sqrt{5}+1}{2}\right)^2 - \frac{1}{2}\left(R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)\right] l_s; \quad (8.57)$$

for $(1+\varepsilon')T = 3x/4\pi^2 l_s^2$ and $T = R/r_0$.

Thence, we have obtained new connections with the Lemma 3 of Goldston-Montgomery theorem, related to the Riemann zeta function, and with the equation concerning the Rogers-Ramanujan continued fraction.

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