Riemann Hypothesis

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1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},\ Re(s)>1$

The Zeta function is holomorphic in the complex plane except for a simple pole at $s = 1$. *The trivial zeros of* $\zeta(s)$ *are* $-2, -4, -6, ...$ *Its non trivial zeros lie in the critical strip* $0 < Re(s) < 1$.

The Riemann Hypothesis states that all the non trivial zeros lie on the critical line $Re(s) = 1/2$.

In this article we disprove the Riemann Hypothesis.

2 Proof

We give the proof by contradction.

Let us assume that the Riemann Hypothesis is true.

Riemann Hypothesis is equivalent to the integral

equation (see [3, p.136, Corollary 8.7])

$\int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt = 0$

$Let,\ I = \int_{-\infty}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt$

$$I = \int_{-\infty}^{0} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt + \int_{0}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt \qquad (1)$$

$Let, I_1 = \int_{-\infty}^{0} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt$

$Substitute,\ t = -u.$

$\Rightarrow dt = -du$

$I_1 = \int_{0}^{\infty} \frac{log|\zeta(1/2-iu)|}{1+4u^2}\ du$

$By\ Schwarz\ Reflection\ principle$ $[see\ 1,\ p.474]$, $\zeta(\bar{s}) = \overline{\zeta(s)}$

$\mid \zeta(1/2 - iu) \mid = \mid \zeta\ (\overline{1/2 + iu}) \mid = \mid \overline{\zeta(1/2 + iu)} \mid = \mid \zeta(1/2 + iu) \mid$

$I_1 = \int_{0}^{\infty} \frac{log|\zeta(1/2+iu)|}{1+4u^2}\ du$

$I_1 = \int_{0}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt$

$Putting\ this\ value\ of\ I_1\ in\ (1),$

$I = 2\int_{0}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt$

$Since\ I = 0,$

$$Thus,\ I = 2\int_{0}^{\infty} \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt = 0 \qquad (2)$$

$Riemann\ Hypothesis\ is\ true\ if\ and\ only\ if$ $(see\ [1,\ p.496\ , Theorem\ 7.26])$

$$\frac{1}{\pi}\int_0^\infty \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2} = \frac{\pi}{8}+\frac{\gamma}{4}+\frac{log\ 8\pi}{4}-2 \qquad (3)$$

$From\ equation(2),$

$$I = 2\int_0^\infty \frac{log|\zeta(1/2+it)|}{1+4t^2}\ dt$$

$$I = 2\int_0^\infty \frac{log|\zeta(1/2+it)|-log|\zeta(1/2)|+log|\zeta(1/2)|}{1+4t^2}\ dt$$

$$I = 2\int_0^\infty \frac{log\frac{|\zeta(1/2+it)|}{|\zeta(1/2)|}+log|\zeta(1/2)|}{1+4t^2}\ dt$$

$$I = 2\int_0^\infty \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{1+4t^2} + 2log\mid\zeta(1/2)\mid\int_0^\infty \frac{1}{1+4t^2}dt$$

$$I = 2\int_0^\infty \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{1+4t^2} + log\mid\zeta(1/2)\mid\frac{\pi}{2}$$

$$1+4t^2 > t^2$$

$$\frac{1}{1+4t^2} < \frac{1}{t^2}$$

$$I < 2\int_0^\infty \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2} + log\mid\zeta(1/2)\mid\frac{\pi}{2}$$

$$I < 2\pi(\frac{1}{\pi}\int_0^\infty \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2}) + log\mid\zeta(1/2)\mid\frac{\pi}{2}$$

$Putting\ the\ value\ of$ $\frac{1}{\pi}\int_0^\infty \frac{log|\frac{\zeta(1/2+it)}{\zeta(1/2)}|}{t^2}$ $from\ equation(3)$

$$I < 2\pi(\frac{\pi}{8}+\frac{\gamma}{4}+\frac{log\ 8\pi}{4}-2) + log\mid\zeta(1/2)\mid\frac{\pi}{2}$$

$Putting\ I = 0\ from\ equation(2),$

$$2\pi(\frac{\pi}{8}+\frac{\gamma}{4}+\frac{log\ 8\pi}{4}-2) + log\mid\zeta(1/2)\mid\frac{\pi}{2} > 0.$$

$$2(\frac{\pi}{8}+\frac{\gamma}{4}+\frac{log\ 8\pi}{4}-2)+log\mid\zeta(1/2)\mid\frac{1}{2} > 0. \qquad (4)$$

$Euler's\ constant\ (see\ [6])\ \gamma \approx 0.5772156649$

From, (see [7, Equation (91)]) ,

$\zeta(1/2) \approx$ -1.46035450880

Putting these values of γ *and* $\zeta(1/2)$ *in equation* (4),

$$2(0.3926990817{+}0.14430391622{+}0.8060425688{-}2){+}0.18933961025 > 0$$

$$2(-0.6569544332) + 0.18933961025 > 0$$

$$-1.3139088664 + 0.18933961025 > 0$$

$$-1.12456925615 > 0$$

which is a contradiction.

So, our assumption that Riemann Hypothesis is true is wrong.

Thus, we have disproved the Riemann Hypothesis.

3 References

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