GENERAL FORMULA OF FIBONACCI SEQUENCE AND LUCAS SEQUENCE'S REVERSE SUM

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Abstract. The sum $\sum_{k=1}^{n} a_k b_{n+1-k} = a_1 b_n + a_2 b_{n-1} + ... + a_n b_1$, where n are any positive integers, denoted by $R(a_n, b_n)$, are called *Reverse Sum* of a_n and b_n. Reverse Sum usually appears in Rearrangement Inequality, but not in normal Algebra. *Fibonacci Sequence* $\{F_n\}$ and *Lucas Sequence* $\{L_n\}$ are very similar sequences because they also have recurrence formula, but have $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 0$. Because of that similarity of sequences, we suggest that those sequences can be related as a function of Reverse Sum. In this paper it is shown that $R(F_n, L_n)$ can be written into general form within $\{F_n\}$ and some various constants.

Contents

1. Introduction

Consider real finite sequences $\{a_i\}_{i=1}^{n} : a_1, a_2, a_3, ..., a_n$ and $\{b_i\}_{i=1}^{n} : b_1, b_2, b_3, ..., b_n$. We will get their *Reverse Sum* value as,

$$R(a_n, b_n) = a_1 b_n + a_2 b_{n-1} + ... + a_{n-1} b_2 + a_n b_1$$

Clearly illustrating by, for example, $R(a_3, b_3) = a_1 b_3 + a_2 b_2 + a_3 b_1$ and $R(a_6, b_6) = a_1 b_6 + a_2 b_5 + a_3 b_4 + a_4 b_3 + a_5 b_2 + a_6 b_1$. We can see that, for natural number $1 \leq u \neq v \leq n$, if there were have $a_u b_v$, there will also have $a_v b_u$ too, except for case of odd numbers n yield only one term of $a_k b_k$. Next section, we will take a look for Fibonacci Sequence and Lucas Sequence, also theorems and lemmas for proving our main theorem.

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2. Preliminaries

These are some important theorems to proof the main theorem in section 3.

Definition 2.1 (Fibonacci Sequence)**.** The sequence $\{F_n\}$ is called *Fibonacci Sequence* if and only if,

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1$$

where n denote integer such that $n \geq 2$.

Definition 2.2 (Lucas Sequence)**.** Sequence $\{L_n\}$ is called *Lucas Sequence* if and only if,

$$L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1$$

where n denote integer such that $n \geq 2$.

Theorem 2.1 (Recursive Sequence Type IV)**.** Consider type IV of recurrence relation $\{a_n\}$,

$$a_{n+1} = pa_n + qa_{n-1}, n \geq 2, (q \neq 0).$$

Determine α and β. (3) gives $a_{n+1} = (\alpha + \beta)a_n - \alpha\beta a_{n-1}$, so let $\alpha + \beta = p$ and $\alpha\beta = -q$. Thus, α, β are the two roots of the quadratic equation $t^2 - pt - q = 0$, which is called the *Characteristic Equation* of the given recurrence formula.

(1) $a_n = A\alpha^n + B\beta^n$, if $\alpha \neq \beta$
(2) $a_n = (An + B)\alpha^n$, if $\alpha = \beta$

where A, B are constants determined by the initial values a_1 and a_2.

Definition 2.3 (Reverse Sum of Fibonacci Sequence and Lucas Sequence)**.** The following equation below shown their Reverse Sum of 2 sequences that we are considering,

$$R(F_n, L_n) = F_1L_n + F_2L_{n-1} + ... + F_nL_1$$

We can observe symmetry of the expression such as F_1L_n and F_nL_1. Which needed to be established the following lemma.

3. Road to the Main Theorem

Lemma 3.1. Suppose Fibonacci Sequence $\{F_n\}$. Then,

$$F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$

Proof. By Definition 1 gives $\alpha+\beta = 1$ and $\alpha\beta = -1$ respectively. Considering Characteristic Equation by substituting p, q: $t^2 - t + 1 = 0$, gives the solution

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}.$$

That mean $\alpha \neq \beta$, so

$$F_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n.$$

Because $F_0 = 0$ and $F_1 = 1$, it is easy to get $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. Therefore, the lemma have been proved. □

Lemma 3.2. Suppose Lucas Sequence $\{L_n\}$. Then,

$$L_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$$

Proof. Claiming Definition 2 gives $\alpha + \beta = 1$ and $\alpha\beta = -1$ respectively. By similarity of Lemma 2 gives,

$$L_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n.$$

From $L_0 = 2$ and $L_1 = 1$ yield $A = B = 1$. Thus, proving have been occurred. □

Lemma 3.3. Suppose that $\{F_n\}$ and $\{L_n\}$ be Fibonacci Sequence and Lucas Sequence. Let m and n are positive integer including 0. Then,

$$F_mL_n + F_nL_m = 2F_{m+n}$$

Proof. By assuming $\Psi_1 = \frac{1+\sqrt{5}}{2}$, $\Psi_2 = \frac{1-\sqrt{5}}{2}$ and $\eta = \frac{1}{\sqrt{5}}$. Then:

$$\begin{aligned} F_mL_n + F_nL_m &= (\eta\Psi_1^m - \eta\Psi_2^m)(\Psi_1^n + \Psi_2^n) + (\eta\Psi_1^n - \eta\Psi_2^n)(\Psi_1^m + \Psi_2^m) \\ &= 2\eta\Psi_1^m\Psi_2^n - 2\eta\Psi_1^m\Psi_2^n \\ &= 2(\eta\Psi_1^{m+n} - \eta\Psi_2^{m+n}) = 2F_{m+n} \end{aligned}$$

□

Definition 3.1 (Floor Function)**.** Let x be a real number. $\lfloor x \rfloor$ denote *Floor Function* or integer part of x. For example, $\lfloor 5.08 \rfloor = 5$ and $\lfloor 7 \rfloor = 7$.

Theorem 3.1. Let x be a real number. Then

$$\{x\} = x - \lfloor x \rfloor$$

where $\{x\}$ denote *Decimal Part* of x.

Definition 3.2 (Congruence of Integers)**.** Let a, b be integers. $a \equiv_m b$ meaning $a - b$ is divisible by m.

4. Main Theorem

By applying all definitions, theorems, and lemma;

Theorem 4.1. Let n be positive integer including 0. Then:

$$R(F_n, L_n) = (n + 2\{\frac{n}{2}\})F_{n+1}$$

Proof. We separate the value of n into 2 classes below,

(1) $n \equiv_2 0$ implies

$$R(F_n, L_n) = F_1L_n + ... + F_{\frac{n}{2}-1}L_{\frac{n}{2}+1} + F_{\frac{n}{2}+1}L_{\frac{n}{2}-1} + ... + F_nL_1$$

from there, using Lemma 4 yields $R(F_n, L_n) = (F_1L_n + F_nL_1) + (F_2L_{n-1} + F_{n-1}L_2) + ... + (F_{\frac{n}{2}-1}L_{\frac{n}{2}+1} + F_{\frac{n}{2}+1}L_{\frac{n}{2}-1}) = \frac{n}{2}(2F_{n+1}) = nF_{n+1}$.

(2) $n \equiv_2 1$

so $n + 1 \equiv_2 1 + 1 \equiv_2 2 \equiv_2 0$ gives

$$R(F_n, L_n) = F_1L_n + ... + F_{\frac{n+1}{2}}L_{\frac{n+1}{2}} + ... + F_nL_1$$

likewise (1), $R(F_n, L_n) = (F_1L_n+F_nL_1)+(F_2L_{n-1}+F_{n-1}L_2)+...+F_{\frac{n+1}{2}}L_{\frac{n+1}{2}}$. Because

$$F_{\frac{n+1}{2}}L_{\frac{n+1}{2}} = \frac{1}{2}F_{\frac{n+1}{2}}L_{\frac{n+1}{2}} + \frac{1}{2}F_{\frac{n+1}{2}}L_{\frac{n+1}{2}} = \frac{1}{2}(2F_{n+1}) = F_{n+1}$$

then, $R(F_n, L_n) = \frac{n}{2}(2F_{n+1}) + F_{n+1} = (n + 1)F_{n+1}$.

But you may wonder that where Decimal Part came from.
Because (1) we got,

$$R(F_n, L_n) = nF_{n+1} = (n + 0)F_{n+1} = (n + 2\{\frac{n}{2}\})F_{n+1}$$

and (2) also gives

$$R(F_n, L_n) = (n + 1)F_{n+1} = (n + 2\{\frac{n}{2}\})F_{n+1}$$

Therefore, the theorem have been proven. □