GENERAL FORMULA OF FIBONACCI SEQUENCE AND LUCAS SEQUENCE'S REVERSE SUM

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1. Introduction

The sum $\sum_{k=1}^{n} a_k b_{n+1-k} = a_1 b_n + a_2 b_{n-1} + ... + a_n b_1$, where n are any positive integers, denoted by $R(a_n, b_n)$, are called *Reverse Sum* of a_n and b_n. Reverse Sum usually appears in Rearrangement Inequality, but not in normal Algebra. *Fibonacci Sequence* $\{F_n\}$ and *Lucas Sequence* $\{L_n\}$ are very similar sequences because they also have recurrence formula, but have $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 0$. Because of that similarity of sequences, we suggest that those sequences can be related as a function of Reverse Sum. In this paper it is shown that $R(F_n, L_n)$ can be written into general form within $\{F_n\}$ and some various constants.

2. Preliminaries

These are some important theorems to proof the following main theorem.

Definition 1: The sequence $\{F_n\}$ is called *Fibonacci Sequence* if and only if,

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1$$

where n denote integer such that $n \geq 2$.

Definition 2: Sequence $\{L_n\}$ is called *Lucas Sequence* if and only if,

$$L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1$$

where n denote integer such that $n \geq 2$.

Lemma 1: Consider type IV of recurrence relation $\{a_n\}$ like this,

$$a_{n+1} = pa_n + qa_{n-1}, n \geq 2, (q \neq 0).$$

Determine α and β. (3) gives $a_{n+1} = (\alpha + \beta)a_n - \alpha\beta a_{n-1}$, so let $\alpha + \beta = p$ and $\alpha\beta = -q$. Thus, α, β are the two roots of the quadratic equation $t^2 - pt - q = 0$, which is called the *Characteristic Equation* of the given recurrence formula.

(1) $a_n = A\alpha^n + B\beta^n$, if $\alpha \neq \beta$
(2) $a_n = (An + B)\alpha^n$, if $\alpha = \beta$

where A, B are constants determined by the initial values a_1 and a_2.

Now, We start with considering the following Reverse Sum definition

Definition 3:

$$R(F_n, L_n) = F_1L_n + F_2L_{n-1} + ... + F_nL_1$$

We can observe symmetry of the expression such as F_1L_n and F_nL_1. Which needed to be established the following lemma;

Lemma 2: Suppose Fibonacci Sequence $\{F_n\}$. Then,

$$F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$

Proof: By Definition 1 gives $\alpha + \beta = 1$ and $\alpha\beta = -1$ respectively. Considering Characteristic Equation by substituting p, q: $t^2 - t + 1 = 0$, gives the solution

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}.$$

That mean $\alpha \neq \beta$, so

$$F_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n.$$

Because $F_0 = 0$ and $F_1 = 1$, it is easy to get $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. Therefore, the lemma have been proved. $\square$

Lemma 3: Suppose Lucas Sequence $\{L_n\}$. Then,

$$L_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$$

Proof: Claiming Definition 2 gives $\alpha+\beta = 1$ and $\alpha\beta = -1$ respectively. By similarity of Lemma 2 gives

$$L_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n.$$

From $L_0 = 2$ and $L_1 = 1$ yield $A = B = 1$. Thus, proving have been occured. $\square$

Lemma 4: Suppose that $\{F_n\}$ and $\{L_n\}$ be Fibonacci Sequence and Lucas Sequence. Let m and n are positive integer including 0. Then,

$$F_mL_n + F_nL_m = 2F_{m+n}$$

Proof: Make it easy by assuming $\Psi_1 = \frac{1+\sqrt{5}}{2}$, $\Psi_2 = \frac{1-\sqrt{5}}{2}$ and $\eta = \frac{1}{\sqrt{5}}$. Then:

$$\begin{aligned} F_mL_n + F_nL_m &= (\eta\Psi_1^m - \eta\Psi_2^m)(\Psi_1^n + \Psi_2^n) + (\eta\Psi_1^n - \eta\Psi_2^n)(\Psi_1^m + \Psi_2^m) \\ &= 2\eta\Psi_1^m\Psi_2^n - 2\eta\Psi_1^m\Psi_2^n \\ &= 2(\eta\Psi_1^{m+n} - \eta\Psi_2^{m+n}) = 2F_{m+n} \quad \square \end{aligned}$$

Definition 4: Let x be a real number. $\lfloor x \rfloor$ denote *Floor Function* or integer part of x. For example, $\lfloor 5.08 \rfloor = 5$ and $\lfloor 7 \rfloor = 7$.

Definition 5: Let x be a real number. Then

$$\{x\} = x - \lfloor x \rfloor$$

where $\{x\}$ denote *Decimal Part* of x.

Definition 6: Let a, b be integers. $a \equiv_m b$ meaning $a - b$ is divisible by m.

3. Main Theorem

By applying all Definitions and Lemma,

Theorem 1: Let n be positive integer including 0. Then:

$$R(F_n, L_n) = (n + 2\{\frac{n}{2}\})F_{n+1}$$

Proof: We separate the value of n into 2 classes below,

(1) $n \equiv_2 0$ implies

$$R(F_n, L_n) = F_1 L_n + ... + F_{\frac{n}{2}-1} L_{\frac{n}{2}+1} + F_{\frac{n}{2}+1} L_{\frac{n}{2}-1} + ... + F_n L_1$$

from there, using Lemma 4 yields $R(F_n, L_n) = (F_1 L_n + F_n L_1) + (F_2 L_{n-1} + F_{n-1} L_2) + ... + (F_{\frac{n}{2}-1} L_{\frac{n}{2}+1} + F_{\frac{n}{2}+1} L_{\frac{n}{2}-1}) = \frac{n}{2}(2F_{n+1}) = nF_{n+1}$.

(2) $n \equiv_2 1$

so $n + 1 \equiv_2 1 + 1 \equiv_2 2 \equiv_2 0$ gives

$$R(F_n, L_n) = F_1 L_n + ... + F_{\frac{n+1}{2}} L_{\frac{n+1}{2}} + ... + F_n L_1$$

likewise (1), $R(F_n, L_n) = (F_1 L_n + F_n L_1) + (F_2 L_{n-1} + F_{n-1} L_2) + ... + F_{\frac{n+1}{2}} L_{\frac{n+1}{2}}$. Because $F_{\frac{n+1}{2}} L_{\frac{n+1}{2}} = \frac{1}{2} F_{\frac{n+1}{2}} L_{\frac{n+1}{2}} + \frac{1}{2} F_{\frac{n+1}{2}} L_{\frac{n+1}{2}} = \frac{1}{2}(2F_{n+1}) = F_{n+1}$, so $R(F_n, L_n) = \frac{n}{2}(2F_{n+1}) + F_{n+1} = (n+1)F_{n+1}$.

But you may wonder that where Decimal Part came from.
Because (1) we got $R(F_n, L_n) = nF_{n+1} = (n+0)F_{n+1} = (n + 2\{\frac{n}{2}\})F_{n+1}$
and (2) makes $R(F_n, L_n) = (n+1)F_{n+1} = (n + 2\{\frac{n}{2}\})F_{n+1}$
Then the theorem have been proven. □