Trigonometry with nested radicals

We will see what we can do with these functions

$$\sqrt{2_1-\sqrt{2_2\pm\sqrt{2_3\pm\ldots\pm\sqrt{2_n}}}}=2\sin\left(\frac{90^\circ(2a+1)}{2^n}\right)$$

$$\sqrt{2_1+\sqrt{2_2\pm\sqrt{2_3\pm\ldots\pm\sqrt{2_n}}}}=2\cos\left(\frac{90^\circ(2a+1)}{2^n}\right)$$

where

$$n=1,2\ \Rightarrow\ a=0$$
$$n\geq 2\ \Rightarrow\ 0\leq a\leq 2^{n-2}-1$$

If n,a are known then the signs $S_k=\pm 1$ are given by relation

$$S_k=(-1)^{\operatorname{round}(a/2^{n-k})},\ \ k=2,3,\ldots,n-1$$

If every sign S_k is replaced with the d_k digit according to the

$$d_k=(1-S_k)/2\ \ (+=0,-=1)$$

relation, then the binary representation of a number

$$b=(d_2d_3\ldots d_{n-1})_{(2)}$$

which is closely associated with the a number will be formed:

- The binary representations of the a,b numbers always have the same number of digits.
- The numbers a,b are linked to one another one-by-one, regardless of the value of n.

The following table shows the characteristic matching pattern in the area of the four-digit binary numbers (8-15). There are two ways of transferring groups of numbers, both cross-sectional and parallel. For example, if $a=11$ then $b=14$ (and vice versa).

a			b
08	12	12	12
09	13	13	13
10	14	14	15
11	15	15	14
12	08	10	10
13	09	11	11
14	10	08	09
15	11	09	08

Example

In practice, calculation of the S_k sign is very easy and can be done without the help of a computer.

For example, if $n = 6$ and $a = 12$ then we will have the following equation of signs:

$$\sqrt{2_1 - \sqrt{2_2 \pm \sqrt{2_3 \pm \sqrt{2_4 \pm \sqrt{2_5 \pm \sqrt{2_6}}}}}} = 2\sin\left(\frac{90^\circ\,(2 \cdot 12 + 1)}{2^6}\right)$$

In order to determine the unknown signs, we first divide the a with the numbers $2^{n-2}, 2^{n-3}, \ldots, 2$ in this order, and we mark under each fraction the quotient rounded to the nearest integer. If this is an even number then you write under the fraction $+$, otherwise you put $-$. Thus, the following table is formed.

12			
16	8	4	2
1	2	3	6
$-$	$+$	$-$	$+$

So the solution is

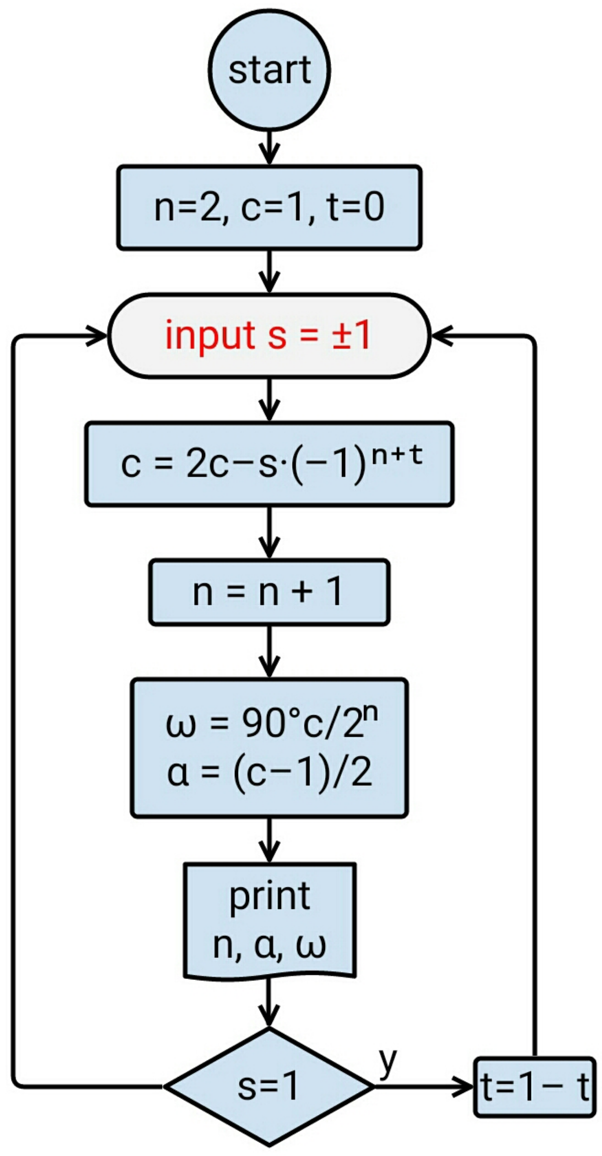
$$\sqrt{2_1 - \sqrt{2_2 - \sqrt{2_3 + \sqrt{2_4 - \sqrt{2_5 + \sqrt{2_6}}}}}} = 2\sin\left(\frac{90^\circ \cdot 25}{64}\right)$$

As shown in the previous matching table, it will be $b = 1010_{(2)} = 10_{(10)}$.

Algorithm for constructing a radical function

With the following algorithm you can construct a radical step-by-step function by inserting the signs s that follow the 2_2 term. After each insertion you can see how the values of n, a and angle $\omega = 90^\circ c/2^n$ are modeled, where

$$c = 2a + 1 = 2^0 \pm 2^1 \pm 2^2 \pm \ldots \pm 2^{n-2}$$



The auxiliary parameter t is dependent on s and takes values 0 and 1.

Generalization

We will now extend the radical function so that we can include more general trigonometric terms in it.

The following results are not fully proven, so caution is needed!

$$\sqrt{2_1 - \sqrt{2_2 \pm \sqrt{2_3 \pm \ldots \pm \sqrt{2_n \pm 2f(r)}}}} = 2\sin\omega$$

$$\sqrt{2_1 + \sqrt{2_2 \pm \sqrt{2_3 \pm \ldots \pm \sqrt{2_n \pm 2f(r)}}}} = 2\cos\omega$$

where

$$f(r) = \sin r, -90^\circ \leq r \leq 90^\circ \;\; or$$
$$f(r) = \cos r, \quad 0^\circ \leq r \leq 180^\circ$$
$$r \in \mathbb{R}$$

and

$$\omega = \frac{45^\circ(2a+1) + (-1)^a(45^\circ - r)}{2^n} \;\; \text{if} \;\; f(r) = \sin r$$
$$\omega = \frac{45^\circ(2a+1) - (-1)^a(45^\circ - r)}{2^n} \;\; \text{if} \;\; f(r) = \cos r$$

where

$$n \geq 2, \;\; 0 \leq a \leq 2^{n-1} - 1$$

for which the signs S_k are computed by the relation

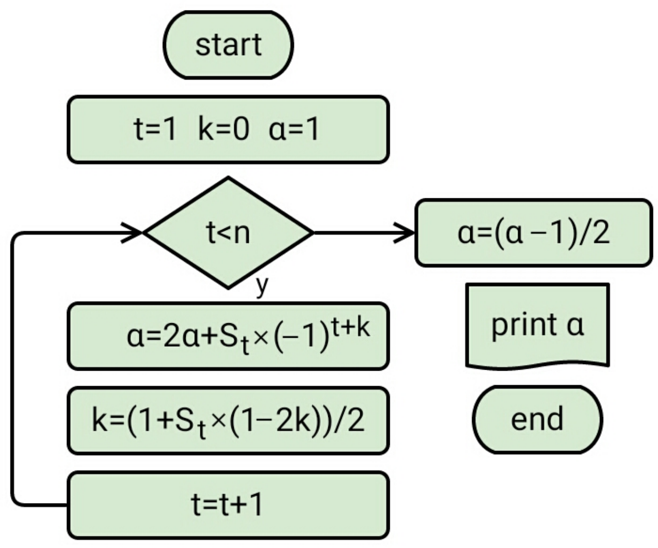
$$S_k = (-1)^{\text{round}(a/2^{n-k})}, \;\; k = 1, 2, \ldots, n-1$$

Note that these signs are not dependent on $f(r)$.

Equations of special form

$$x = \sqrt{2_1 \pm \sqrt{2_2 \pm \sqrt{2_3 \pm \ldots \pm \sqrt{2_n \pm x}}}}$$

In this equation, x is unknown and all the signs S_t for $t = 0,1,2,\ldots,n-1$ are known, with $n \geq 1$. We first find the value of an integer a through the following algorithm



Then,

if $S_0 = +1$ the solution will be in the form $x = 2\cos r$

if $S_0 = -1$ the solution will be in the form $x = 2\sin r$

where

$$r = \frac{45^\circ\,(2a+1-(-1)^a S_0)}{2^n - (-1)^a S_0}$$

If the equation is

$$2\sin r = \sqrt{2_1 - \sqrt{2_2 \pm \sqrt{2_3 \pm \ldots \pm \sqrt{2_n \pm 2\sin r}}}}$$

where r, n are known and the signs unknown, then we first determine the integer a that verifies equality

$$r = \frac{45^\circ\,(2a+1+(-1)^a)}{2^n + (-1)^a}, \quad 0 \leq a \leq 2^{n-1} - 1$$

so the signs S_t are taken through the relationship

$$S_t = (-1)^{\mathrm{round}(a/2^{n-t})}, \quad t = 1, 2, \ldots, n-1$$

The same procedure is followed to solve the above equation if in it we replace $\sin r$ with $\cos r$, except that a is determined by the relation

$$r = \frac{45^\circ\,(2a+1-(-1)^a)}{2^n - (-1)^a}, \quad 0 \leq a \leq 2^{n-1} - 1$$

If we have a solution, then we can have infinite of them. For example, let's look at equality

$$2\sin\left(\frac{630^\circ}{2^4+1}\right)=\sqrt{2_1-\sqrt{2_2-\sqrt{2_3+\sqrt{2_4-2\sin\left(\frac{630^\circ}{2^4+1}\right)}}}}$$

The $-+-=101_{(2)}=5$ motif following 2_2 corresponds to $a=6$. If we insert any number of positive signs between the terms 2_2 and $2_{n\text{-}2}$, the value of b will not change (because $00\ ...\ 00101_{(2)}=101_{(2)}=5=$ fixed), so the same will apply to the a value. That is, it will be

$$2\sin\left(\frac{630^\circ}{2^n+1}\right)=\sqrt{2_1-\sqrt{2_2+\ldots+\sqrt{2_{n-2}-\sqrt{2_{n-1}+\sqrt{2_n-2\sin\left(\frac{630^\circ}{2^n+1}\right)}}}}}$$