Particle Transport by Turbulent Fluids

Rolf Warnemünde

E-Mail: rolf.warnemuende@t-online.de

Abstract

It is stated that moving fluids can be described as fluctuating continua although their material distribution is always discontinuous. A stochastic particle transport is then considered by an imaginary ensemble of any number of equivalent turbulent fluids existing in parallel. This leads to exspectation values of the densities of turbulently transported particles.
First a transport equation for a molecular self-diffusion is found. It is used as a reference for the difference between self-moving diffusing particles and transport through turbulent moving continua (e.g. aerosols). This is followed by a transport theory for longitudinal continuum fluctuations to provide an easier transition to the more complicated turbulent particle transport.

The following transport equations arise:

1. *-transport equation of molecular self-diffusion as partial differential equation as well as integral equation. The transition probabillity of velocities is calculated, explicitly.*
2. *-transport equation of a passive particle transport by longitudinal continuum-fluktuations as partial differential equation as well as integral equation. The transition probabillity of velocities is calculated, explicitly.*
3. *-transport equation of a passive particle transport by turbulent continuum-fluktuations as partial differential equation as well as integral equation. The transition probabillity of velocities is calculated, explicitly.*

Contents

1. Introduction

> Feynman[5]: "*Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences. It is the analysis of circulating or turbulent fluids.*"

The description of turbulent movements within the framework of continuum mechanics turned out to be difficult since more than 160 years. However, laminar fluid movements can be calculated by the known basic equations successfully confirmed in experiments: equation of continuity, Navier-Stokes-Equations and energy equation. The efforts, treating movements of turbulence in a similar way, must be considered as failures. There are substantial reasons for believing, that the above equations describing turbulent collective movements of **non-homogenously distributed molecular matter** are inadequate. This was the situation that inspired the idea, to explain the phenomenon of turbulence by stochastic methods. In that context, particularly approaches of Kolmogorov are to be mentioned, which lead to spectral energy distributions, assuming highly hypothetically, that turbulence is statistically isotropical and homogeneous. Between them there is a wide range of models with physically not well founded hypotheses. Overall, this leads to the statement of Feynman cited at the beginning, whereupon not much has changed since then.

This situation is characterized in recent treatises as for example by Trinh, Khanh Tuoc [10] in the following way:

" *the study of turbulence is immediately hampered by the surprising lack of a clear and concise definition of the physical process. Tsinober (2001) has published a long list of attempts at a definition by some of the most noted researchers in turbulence. The most common descriptions are vague: "a motion in which an irregular fluctuation (mixing, or eddying motion) is superimposed on the main stream" (Schlichting 1960), "a fluid motion of complex and irregular character" (Bayly, Orszag, Herbert, 1988)* *or negative as in the breakdown of laminar flow (Reynolds' experiment 1883). Some of the definitions are quite controversial like Saffman's (1981) "One of the best definition of turbulence is that it is a field of random chaotic vorticity" because the words random and chaotic would imply that a formal mathematical solution, which is necessarily deterministic, does not exist. Perhaps the most accurate definition can be attributed to Bradshaw (1971) "The only short but satisfactory answer to the question "what is turbulence" is that it is the general-solution of the Navier-Stokes equation". This definition cannot be argued with but it is singularly unhelpful since no general solution of the NS yet exists 160 years after they were formulated.*"

Fluctuation elements of the presented theory always form a dense point set, i.e. a definition of a continuum of such fluctuation elements is important deducing equations of motion in form of partial differential equations. On the other hand a concept of a stochastic theory of a fluctuating continuum within the meaning of an ensemble theory is deduced. Fundametal principles of this treatise as well as in the whole classical physics are **locality, causality and deterministics**. In this treatise particular emphasis is placed on specially defined **natural causality**, which in contrary to **Newtonian causality** of point mechanics only knows finite velocities. Discussed stochastics arises from statistics with an in thought experiment supposed unlimited ensemble of locally equivalent deterministic processes.[1]

The following transport equations are found :

1. *-transport equation of molecular self-diffusion as partial differential equation as well as integral equation. The transition probabillity of velocities is calculated, explicitly.*
2. *-transport equation of a passive particle transport by longitudinal continuum-fluktuations as partial differential equation as well as integral equation. The transition probabillity of velocities is calculated, explicitly.*
3. *-transport equation of a passive particle transport by turbulent continuum-fluktuations as partial differential equation as well as integral equation. The transition probabillity of velocities is calculated, explicitly.*

[1]If one wants, it can be seen as a many-world-theory of classical physics. However this is done creating beforehand unknown equations of the deterministic processes (in contrary to Everett's many-world-theory of quantum mechanics).

Part I.

Prerequisites

2. Definition of a Turbulent Fluid Continuum

2.1. Introduction

A proper definition of turbulence, which is based on a fluctuating, dense point set, does not exist. But this is neccessary establishing equations of movement in form of partial differential equations. The known Navier-Stokes equations are only providing sufficient solutions for laminar problems. Below a fluctuating fluid is defined, which is associated uniquely to a dense set of space points of the considered time. This definition is the prerequisite developing stochastic theories of turbulent transport of continously moved particles within the meaning of an ensemble theory, a deterministic theory and the connection of stochastic and a deterministic turbulence.

2.2. Definition of moved fluid-elements

At every time, space points ($\vec{\boldsymbol{x}}$) are assigned to fluid elements in a unique correspondence. As this applies to every space point ($\vec{\mathbf{x}}$) of the fluid field, the set of fluid elements is seen as a continuum. A Continuum of fluid element points (simply called fluid elements) is considered, where a fluid environment of non infinitesimal size is uniquely allocated to every fluid element point. Two infinitesimally neighboring fluid elements differ apart from their distance by their velocities and not quite identical material distributions of their neighborhoods. The neighborhoods of two nearby fluid elements overlap. A fluid element is shifted moving the material of its neighborhood. Though the material of such a fluid element may have changed marginally after an infinitesimal time interval t_ε, it can be identified principally by its prior material status. As every molecule possesses its own identity, there has to be at least an infinitesimally greater difference of material distribution to the neighborhoods of other fluid elements.
The neighborhoods exchange material with neighborhoods of adjacent fluid elements and vary their thermodynamic state (a local thermodynamic state does not necessarily exist). Their size is not infinitesimal, because a local thermodynamic state (if

physically existent) has to be detectable at least in thought experiment. The open neighborhoods have equally sized spherical shapes, generally. Near a solid border they are descibed by parts of spheres. Infinitesimally adjacent fluid elements possess overlapping neighborhoods. In an ε-surrounding they move in parallel. So one obtains a fluid, which is assumed to be a dense fluctuating point set, though there is no continuous matter distribution in Space-Time. That means it is possible to follow theoretically the history of every fluid element, though it has exchanged a lot of its initial material altering its local thermodynamic state.
Recapitulated:
Every space point ($\vec{\mathbf{x}}$) of the open point set of a considered fluid area is at every time in unique correspondence to a fluid element. The fluid is an abstract, dense set of fluctuating fluid elements, which do not generally correspond to material points.

2.3. Laminar moved fluids

A continuum of moved fluid elements is considered each uniquely assigned to a neighborhood and a velocity.

$$\vec{\mathbf{v}}_{t_\epsilon} = \frac{\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1}{t_\epsilon} \tag{2.1}$$

The fluid elements move along sufficiently often continuously differentiable trajectories. The accuracies of the considered motion quantities are determined by t_ε-measurement processes t_ε characterising the accuracy. Deriving the transport equation of turbulent particle transport a limes consideration ($\lim \boldsymbol{t_\epsilon} \to 0$) is subjected. The whole of the velocities create a velocity vector field having $\boldsymbol{rot}(\vec{\mathbf{v}}) \neq \mathbf{0}$ generally.[1] Though $\boldsymbol{rot}(\vec{\mathbf{v}})$ has dimension $[\mathbf{1/sec}]$, it does not refer to a rotation of laminar flow. In an infinitesimally surrounding area of a space-time-point $(\vec{\mathbf{x}}_0, t_0)$ a fluid flow can be defined locally [2] by parallelly moved fluid elements. Considering without loss of generality a fluid movement of velocity $\vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0)$ in a space point $\vec{\mathbf{x}}_0$ in cartesian coordinates, the velocity is described in an ϵ-neighborhood and parallel to the x-coordinate as follows:

$$\vec{\mathbf{v}}(\vec{\mathbf{x}}) = \begin{pmatrix} \mathbf{v}_x(\vec{\mathbf{x}}) \\ \mathbf{v}_y(\vec{\mathbf{x}}) \\ \mathbf{v}_z(\vec{\mathbf{x}}) \end{pmatrix} = \begin{pmatrix} \mathbf{v}_x(\vec{\mathbf{x}}_0) + & \frac{\partial \mathbf{v}_x}{\partial x}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta x + \frac{\partial \mathbf{v}_x}{\partial y}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta y + \frac{\partial \mathbf{v}_x}{\partial z}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta z + ... \\ & \frac{\partial \mathbf{v}_y}{\partial x}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta x + \frac{\partial \mathbf{v}_y}{\partial y}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta y + \frac{\partial \mathbf{v}_y}{\partial z}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta z + ... \\ & \frac{\partial \mathbf{v}_z}{\partial x}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta x + \frac{\partial \mathbf{v}_z}{\partial y}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta y + \frac{\partial \mathbf{v}_z}{\partial z}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta z + ... \end{pmatrix}$$

[1] in english literature $\boldsymbol{curl}(\vec{\mathbf{v}}) \neq \mathbf{0}$ is used but in turbulence the name $\boldsymbol{rot}$ is more adapted as will be seen

[2] except in stagnation points

The velocity components $\mathbf{v}_y(\vec{\mathbf{x}})$ and $\mathbf{v}_z(\vec{\mathbf{x}})$ **osculate** at the velocity $\vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0)$ spatially approaching (constant time t_0),

$$\begin{array}{c} \mathbf{v}_y(x_0, y, z_0) \longrightarrow \mathbf{v}_y(x_0, y_0, z_0) = \mathbf{0} \\ \mathbf{v}_z(x_0, y_0, z) \longrightarrow \mathbf{v}_z(x_0, y_0, z_0) = \mathbf{0} \end{array}.$$

That means especially, that all the partial derivations by y- or z-coordinate of 1. order of $\mathbf{v}_y(\vec{\mathbf{x}})$ and $\mathbf{v}_z(\vec{\mathbf{x}})$ disappear in the point (x_0, y_0, z_0).

$$\begin{array}{c} \lim\limits_{z \to z_0} \dfrac{\Delta \mathbf{v}_y}{\Delta z}\Big|_{\vec{\mathbf{x}}_0} = \lim\limits_{y \to y_0} \dfrac{\Delta \mathbf{v}_z}{\Delta y}\Big|_{\vec{\mathbf{x}}_0} = \mathbf{0} \\ \vec{\mathbf{x}}_0 = (x_0, y_0, z_0) \end{array}. \tag{2.2}$$

Applying the differential quotients in the $\vec{\boldsymbol{\nabla}}\times$ -operator expresssed in cartesian coordinates gives for the fluid velocity

$$(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}})_{|\vec{\mathbf{x}}_0} = \left(\begin{array}{c} 0 \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{array} \right)_{|\vec{\mathbf{x}}_0} , \vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0) \tag{2.3}$$

The orthogonality of $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \perp \vec{\mathbf{v}}$ is a fundamental quality [34] and a necessary condition for continuous fluid flow.

In this orthogonality velocity vector fields differ from deformation vector fields.

2.4. Turbulently moved fluids

Trying to identify the state of movement of a fluid element in turbulent fluids by a velocity $\vec{\mathbf{v}}_{t_\epsilon}$ it should be recognized, that the state of movement is not yet determined, as the path in every space point (except in turning points) is uniquely adapted by an infinitesimal circle segment. In the infinitesimal neighborhood of a path point the velocity is identified by an instantaneous axis of rotation $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and a radius vector $\vec{\mathbf{r}}_{t_\epsilon}$.[5]

$$\boxed{\vec{\mathbf{v}}_{t_\epsilon} = \vec{\boldsymbol{\omega}}_{t_\epsilon} \times \vec{\mathbf{r}}_{t_\epsilon}} \tag{2.4}$$

[3]this relation can not be found in literature.
[4]This is one reason why the known millenium prize question does not lead to a solution of the turbulence problem. However the validity problem of the Navier-Stokes-equations is more fatal.
[5]That is why turbulence can not be uniquely identified by experiments of local velocity statistics.

The considered vectorial motion quantities $\vec{\boldsymbol{\omega}}_{\boldsymbol{t_\epsilon}}$ and $\vec{\mathbf{r}}_{\boldsymbol{t_\epsilon}}$ are determined by t_ε-measurement processes, which are calculated later on by a limes process $\lim \boldsymbol{t_\epsilon} \to 0$. A fluid element originating from the point $\vec{\mathbf{x}}_0$ crossing $\vec{\mathbf{x}}_1$ after the time t_ε reaches $\vec{\mathbf{x}}_2$ after a further time t_ε.

$$\vec{\mathbf{x}}_0 \xrightarrow{\mathbf{t}_\epsilon} \vec{\mathbf{x}}_1 \xrightarrow{\mathbf{t}_\epsilon} \vec{\mathbf{x}}_2$$

By these 3 points a circle segment is uniquely drawn crossing point $\vec{\mathbf{x}}_1$ with radius vector $\vec{\mathbf{r}}_{t_\varepsilon}$ and speed of rotation $\vec{\boldsymbol{\omega}}_{t_\varepsilon}$. The local state of motion can not be described by velocity only, neither statistically nor deterministically. [6]
Thus the fluid element in the space-time-point $(\vec{\mathbf{x}}, t)$ is identified principally by the contents of the matter of its neighborhood and state of movement expressed by $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and $\vec{\mathbf{r}}_{t_\epsilon}$. In that way defined fluid elements move on sufficiently often continuously differentiable trajectories. They lead considering a continuum of fluctuating fluid elements to multiply continuously differentiable vector fields of motion. The continuum of moved fluid elements represent the turbulently collectiv movement of a discontinuously spaced Matter.
The field of turbulence is described by the two vector fields $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and $\vec{\mathbf{b}}_{t_\epsilon}$,

$$\vec{\mathbf{b}}_{t_\epsilon} = \vec{\mathbf{r}}_{t_\epsilon}/r^2_{t_\epsilon} \qquad \text{-curvature vector field.} \tag{2.5}$$

In addition, the results show that

$$\vec{\boldsymbol{\omega}}_{t_\epsilon} = \frac{1}{2}\mathbf{rot}(\vec{\mathbf{v}}_{t_\epsilon}). \tag{2.6}$$

$\mathbf{rot}(\vec{\mathbf{v}})$ has the meaning of a local rotation in the frame of turbulence. An infinitesimal disturbance of stationary pipe flow leads to an change of the significance of $\mathbf{rot}(\vec{\mathbf{v}})$, where $\mathbf{rot}(\vec{\mathbf{v}})$ does not correspond to a rotation initially. Whether starting motions of turbulence are suppressed, depends on an existent viscosity. These decelerations are generally weak. The beginning of turbulent movements avoid Newtonian friction as well as pressure gradients by means of hereto orthogonal motions.

Vortex fields in turbulence (local rotation fields will be identified with vortex fields) and radius fields may have turning points $(\vec{\boldsymbol{x}}, t)$ along the paths of the fluid elements, which means $\vec{\boldsymbol{\omega}} = \mathbf{0}$ and $\vec{\boldsymbol{r}} = \boldsymbol{\infty}$. [7] In this case the velocities are to be calculated by interpolation or extrapolation of the neighborhood, for example. The fluid elements are accompanied by a moving frame of $\vec{\boldsymbol{\omega}}$,$\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{v}}$ along their paths.
Deterministic considerations are found via stochastic descriptions, which could be designated as Lagrangian. Nevertheless, Lagrangian paths are calculated only after the deterministic turbulence field is determined.

Fluid motions can always be described

[6] This statement contadicts that of [16]

[7] The temporal and spatial neighborhood of a turning point does not have such singular properties.

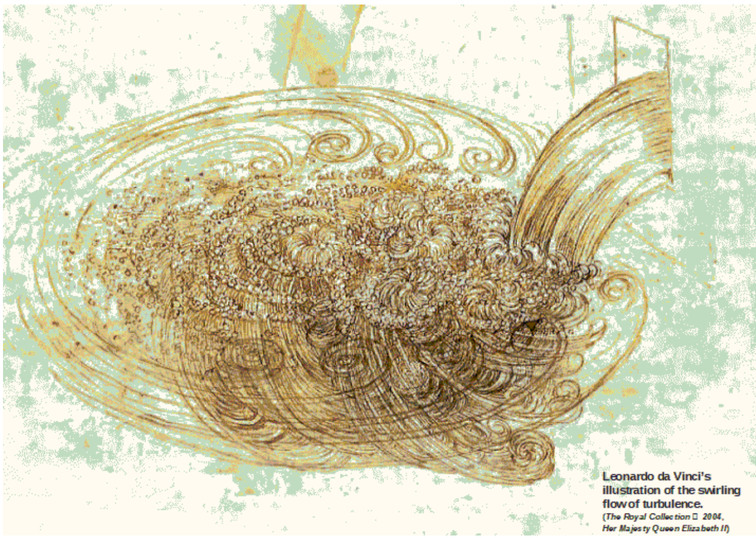


Figure 2.1.: Turbulences understood by Leonardo da Vinci (Such a picture of turbulence trajectories is in reality not possible)

by moved continua ! That is why considerations of stochastic particle transports are only possible in the frame of ensemble theories.

3. Distribution functions

3.1. Introduction

Stochastically physical processes generally refer to random transports of physical quantities from $\vec{\boldsymbol{A}} = (\vec{\mathbf{x}}_1, t_1)$ to $\vec{\boldsymbol{B}} = (\vec{\mathbf{x}}_2, t_2)$, where a diffusion equation results at the end of all the discussions as can be seen in the well known treatise of Chandrasekhar [1]. This takes place in accordance with the Langevin equation, all known attempts characterizing Brownian motion and applying Fokker-Planck-equation, too. The diffusion equation is subjected to a Newtonian causality, that means the related propagation speed is unlimited. This is not the case in nonrelativistic physics beyond Newtonian mechanics, generally, as shown in the further course of this treatise.
In this context the Boltzmann Equation, which is only applicable for extensively diluted gases, constitutes a particularity. Despite surprising successes the importance of this equation is obviously not appropriately appreciated. In first approximation the Navier-Stokes equations are derived from this equation. A linear version can be classified as key-equation of nuclear reactor physics and is used for radioactive shielding problems in its stationary formulation.[1]These equations are based on a 6-dimensional phase-space with the apparent disadvantage, that using distribution functions $f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}})$ a small but not infinitesimal phase space volume element $\triangle x \cdot \triangle y \cdot \triangle x \cdot \triangle v_x \cdot \triangle v_y \cdot \triangle v_z = \triangle \boldsymbol{V}$ is to be believed surrounding the phase space point $(\vec{\mathbf{x}}, \vec{\boldsymbol{v}})$. This situation is mathematically dissatisfying, as only a finite number of molecules can be existent inside this Volume, and executing $\lim \triangle \boldsymbol{V} \rightarrow 0$ there remain no molecules representing $f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}})$. Despite this contrariness the Boltzmann Equation, in general or linear form, is successful considering the results.[2]
Such a situation exists in other fields of physics, too. For example, no mathematically satisfying definition of a continuum is existent justifying partial differential equations like the Navier-Stokes-equations. Nevertheless they have performed satisfactorily in the case of laminar fluid dynamics, but extending to the general case of turbulence the known equations of laminar fluid dynamics fail. Deficient mathematical justification is sometimes balanced by experiments, not always. In the special case of

[1] 1968, associated with this the author has developed in his diploma thesis a numerical method (Monte-Carlo) for solving the linearely stationary Boltzmann Equation, without knowing this equation, simply by simulating the stochastic elementary processes.

[2] The Boltzmann Equation is the single equation describing the transition from a nonexisting local thermodynamic balance to a local thermodynamic balance.

kinetics, which is used for describing molecular self-diffusion, an ensemble theory is applied, which could be used for developing the Boltzmann Equation, too, avoiding the stated contrariness. On the other side the detailed mathematical formulations and their results do not alter by such a modified interpretation. But as phase space considerations are not possible for stochastically interpreted deterministic continuum fluctuations an equivalent treatment of ensemble theory will be used for all discussed problems.
The used distribution functions $f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}})$ are not functions of the 6-dimensional phase space as usually applied in statiscal mechanics but regular functions of space time with a probability density distrubution of motion quantities in every point $(\vec{\mathbf{x}}, t)$ obtained by an unlimited ensemble of parallelly equivalent systems. In the case of the stationary linear radiation transport equation[3] very different elementary particles like neutrons, electrons, $\alpha - particles$, $\gamma - particles$ etc. are simultaneously calculated by this equation.

A suitable ensemble-consideration is helpful to avoid Newtonian causality (see section 3.5), and to get rid of the mathematical inconsistency of ΔV of limited size with limited number of included particles. So mathematically not justified applications of related partial differential equations are avoided. This interpretation not altering the mathematical formulations in connection with gas kinetics the turbulence is lead to new relations.

3.2. Ensemble consideration of molecular self-diffusion

The specifically used construction may appear somehow artificially, but it is supposed to illustrate the classification of the usual diffusion equation as an approximate equation of a primary, with natural causality endowed transport equation. The particle density distributions are gained in thought experiment by an unlimited number of ensemble systems, which exist simultaneously. Their functions are sufficiently often, continuously differentiable in space and time. Regarding the quantities of motion the continuity condition is sufficient. This situation may be generated as follows:

An ensemble of parallel, extensively diluted monomolecular systems is considered to be in local thermodynamic ballance. They are all seen as statistically equivalent. They generally differ locally in an $\boldsymbol{\epsilon}$- neighborhood. Permitting for some

[3] This is a slight modification of the linear Boltzmann Equation with a streaming function $\frac{1}{v} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}})$ instead of distribution function $f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}})$), and the different velocities before and after molecular collisions are considered in the differential cross sections of the collision integral only [15].

time equivalent molecules to enter in all systems by equally distributed sources such that the additional quantity of gas is insignificant in relation to the original quantity of the gas, the additional part of the gas will be in the same statistical balance in all the systems of the ensemble after a short time, though it has not reached a homogenous distribution. While the added part of gas consists in every single system of a limited number of molecules only, it is possible to formulate a sufficiently often continuously differentiable particle density distribution $f(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t)$ for the added gas part, as the statistics of the velocities relates to the whole ensemble.

The expectation value of a suitable particle density may be constructed as follows. Around the point $(\vec{\mathbf{x}}, t)$ an equal-sized volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ is chosen out of all representatives μ of the ensemble in which a subset $\Delta\mathbf{N}_\mu$ of molecules is located. μ identifies a single representative of the ensemble. μ passing all values from $\mathbf{1}$ to $\boldsymbol{\infty}$ an unlimited number of ensemble-representatives is taken into account. The expectation value of a particle density of this ensemble-consideration in point $(\vec{\mathbf{x}}, t)$ in the small volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ of the neighborhood of $(\vec{\mathbf{x}}, t)$ results in

$$< \rho(\vec{\mathbf{x}}, t) > |_{\boldsymbol{\Delta V}} = \lim_{\boldsymbol{n \to \infty}} \frac{1}{n} \sum_{\mu=1}^{n} \frac{\Delta\mathbf{N}_\mu}{\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}}. \tag{3.1}$$

Contracting the volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ to the point $(\vec{\mathbf{x}}, t)$ one has

$$< \rho(\vec{\mathbf{x}}, t) >= \lim_{\boldsymbol{\Delta V}_{(\vec{\mathbf{x}},t)} \to \mathbf{0}} < \rho(\vec{\mathbf{x}}, t) > |_{\boldsymbol{\Delta V}}. \tag{3.2}$$

This function of expectation values is sufficiently often continuously differentiable in its depending variables, especially in space and time. In accordance with the velocities, consisting of amount and direction of motion, their distribution density is separated in these quantities as follows [4]

$$\boxed{\begin{aligned} &< \rho(\vec{\mathbf{x}}, t) > = \int_0^\infty \int_{4\pi} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}}) d\vec{\boldsymbol{\Omega}} dv = \int_{4\pi} h(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) d\vec{\boldsymbol{\Omega}} \\ &f(\vec{\mathbf{x}}, t, \vec{\mathbf{v}}) = f(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}) = h(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) \mathbf{g}(v) \\ &\int_0^\infty \mathbf{g}(v) dv = \mathbf{1}, \qquad \bar{v} = \int_0^\infty \mathbf{g}(v) v dv \\ &\vec{\boldsymbol{v}} = \boldsymbol{v} \cdot \vec{\boldsymbol{\Omega}}. \end{aligned}} \tag{3.3}$$

So the necessary connection is given by

[4] A separation ansatz $f(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}) = h(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) g(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}})$ is generally to be chosen.

$$h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) = \int_0^\infty f(\vec{\mathbf{x}}, t, \boldsymbol{v}\vec{\mathbf{\Omega}}) dv. \tag{3.4}$$

This enables the derivation of a transport equation of $h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}})$ the amounts of the velocities occuring as constant coefficients only.

At this the distribution of the velocity amounts $\mathbf{g}(\mathbf{v})$ is separated from the direction distribution $h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}})$. The diffusing particles possess a gaussian distribution and the equipartition law is applied. This is the prerequisite for deriving a suitable transport equation below and afterwards in less than first approximation a diffusion equation with constant diffusion coefficients.
This special distribution of the velocity amounts (equipartition law) corresponds to the assumed situation of molecular self-diffusion of chapter 5. The gained expectation value of the density does not exactly equal the value measured in an ensemble representative performed in a small volume. Thus one has

$$< \rho(\vec{\mathbf{x}}, t) > \approx \rho(\vec{\mathbf{x}}, t). \tag{3.5}$$

An exact measurement (this has nothing to do with a measurement of the macroscopic state quantity density) would result in

$$< \rho(\vec{\mathbf{x}}, t) > \neq \rho(\vec{\mathbf{x}}, t) = \begin{cases} \mathbf{1} & \text{one particle existent in point} \quad (\vec{\mathbf{x}}, t) \\ \mathbf{0} & \text{else} \end{cases}. \tag{3.6}$$

3.3. Ensemble consideration of stochastic particle transport in a continuum of longitudinal fluctuations

Similarily, further considerations occur to section 3.2. Around the point $(\vec{\mathbf{x}}, t)$ an equal-sized volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ is chosen out of all representatives μ of the ensemble in which a subset $\Delta\mathbf{N}_\mu$ of particles [5] is located. μ identifies a single representative of the ensemble. The expectation value of a particle density results in

$$< \rho(\vec{\mathbf{x}}, t) > = \lim_{\Delta\boldsymbol{V}_{(\vec{\mathbf{x}},t)} \to \mathbf{0}} \left(\lim_{\boldsymbol{n} \to \boldsymbol{\infty}} \frac{1}{n} \sum_{\mu=1}^{n} \frac{\Delta\mathbf{N}_\mu}{\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}} \right). \tag{3.7}$$

[5] the particles have to be of a size adapting the identical movement of present fluid element of the fluctuating continuum

This function of expectations is sufficiently often continuously differentiable in its variables of space and time. By the separation of the velocity in amount and direction the distribution density is described as follows

$$\boxed{\begin{aligned} < \rho(\vec{\mathbf{x}}, t) > &= \int_0^\infty \int_{4\pi} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}}) d\vec{\boldsymbol{\Omega}} dv = \int_{4\pi} \overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) d\vec{\boldsymbol{\Omega}} \\ f(\vec{\mathbf{x}}, t, \vec{\mathbf{v}}) &= G(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}) \overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) \\ \int_0^\infty G(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}) d\boldsymbol{v} &= \mathbf{1}, \qquad \overline{v}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) = \int_0^\infty G(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}) v d\boldsymbol{v} \\ \vec{\boldsymbol{v}} &= \boldsymbol{v} \cdot \vec{\boldsymbol{\Omega}}. \end{aligned}} \tag{3.8}$$

So the necessary combination

$$\overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) = \int_0^\infty f(\vec{\mathbf{x}}, t, \boldsymbol{v}\vec{\boldsymbol{\Omega}}) dv. \tag{3.9}$$

is achieved deriving a transport equation for $\overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}})$. In this transport equation the velocities occur as coefficients of the averaged velocity amounts in dependence of space, time and direction.

The gained expectation value of the density fails to comply with the measured ensemble representative, that is

$$< \rho(\vec{\mathbf{x}}, t) > \neq \rho(\vec{\mathbf{x}}, t) \tag{3.10}$$

For a single ensemble representative the distribution function f degenerates to a delta-function

$$f \rightarrow \delta(\vec{\mathbf{v}}_{(\vec{\boldsymbol{x}}, t)}, \vec{\mathbf{v}}) \tag{3.11}$$

with

$$\begin{aligned} \int_{\vec{\mathbf{v}}} \delta(\vec{\mathbf{v}}_{(\vec{\boldsymbol{x}}, t)}, \vec{\mathbf{v}}) d\vec{\mathbf{v}} &= \mathbf{1} \\ \int_{\vec{\mathbf{v}}} \delta(\vec{\mathbf{v}}_{(\vec{\boldsymbol{x}}, t)}, \vec{\mathbf{v}}) \vec{\mathbf{v}} d\vec{\mathbf{v}} &= \vec{\mathbf{v}}_{(\vec{\boldsymbol{x}}, t)}. \end{aligned} \tag{3.12}$$

3.4. Ensemble consideration of stochastic particle transport in turbulently moved continua

The fluid fluctuations of different ensemble-representatives may arise at the same time by equivalent macroscopic-physical processes with different infinitesimal perturbations. That is why different fluent motions are created in $(\vec{\mathbf{x}}, t)$ in every of the

parallel-sytems. The simultaneous release of passive particels retracing uniquely the motions of fluid movements may have taken place by a distribution of similar point sources in all parallel systems. So an own particle distribution is developed in every individual system in space-time. The statistical recording running over the whole ensemble leads to continuously differentiable distribution functions of a limited number of particles [6] in a single system.
Further considerations follow analogously to section 3.2. Around the point $(\vec{\mathbf{x}}, t)$ an equal-sized volume $\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}$ is chosen out of all representatives μ of the ensemble in which a subset $\Delta\mathbf{N}_\mu$ of particles is located. μ identifies a single representative of the ensemble. The expectation value of a particle density results in

$$< \rho(\vec{\mathbf{x}}, t) >= \lim_{\Delta\boldsymbol{V}_{(\vec{\mathbf{x}},t)}\to\mathbf{0}} \left(\lim_{\boldsymbol{n}\to\boldsymbol{\infty}} \frac{1}{n} \sum_{\mu=1}^{n} \frac{\Delta\mathbf{N}_\mu}{\Delta\mathbf{V}_{(\vec{\mathbf{x}},t)}} \right). \tag{3.13}$$

This function of expectation values arises out of a distribution function f of motion quantities

$$\begin{aligned} \vec{\omega} =& \vec{\omega}(\vec{\mathbf{x}}, t) \qquad \text{rotation speed} \\ \vec{\mathbf{r}} =& \vec{\mathbf{r}}(\vec{\mathbf{x}}, t) \qquad \text{radius vector} \\ \vec{\mathbf{v}} =& \vec{\omega}(\vec{\mathbf{x}}, t) \times \vec{\mathbf{r}}(\vec{\mathbf{x}}, t) \qquad \text{velocity vector} \end{aligned} \tag{3.14}$$

that means

$$f = f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}). \tag{3.15}$$

A separation results in

$$\boxed{\begin{gathered} < \rho(\vec{\mathbf{x}}, t) >= \int_{\vec{\mathbf{r}}} \int_{\vec{\boldsymbol{\omega}}} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) d\vec{\boldsymbol{\omega}} d\vec{\boldsymbol{r}} = \int_{2\pi} \int_{4\pi} \overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) d\vec{\boldsymbol{\Omega}} d\vec{\boldsymbol{\Theta}} \\ f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) = G(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ \int_0^\infty \int_0^\infty G(\vec{\mathbf{x}}, t, \boldsymbol{\omega}\vec{\boldsymbol{\Omega}}, r\vec{\boldsymbol{\Theta}}) d\omega dr = 1, \quad \int_0^\infty \int_0^\infty G(\vec{\mathbf{x}}, t, \boldsymbol{\omega}\vec{\boldsymbol{\Omega}}, r\vec{\boldsymbol{\Theta}}) \omega r d\omega dr = \overline{v}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ \int_0^\infty \int_0^\infty G(\vec{\mathbf{x}}, t, \boldsymbol{\omega}\vec{\boldsymbol{\Omega}}, r\vec{\boldsymbol{\Theta}}) r d\omega dr = \overline{r}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}), \quad \int_0^\infty \int_0^\infty G(\vec{\mathbf{x}}, t, \boldsymbol{\omega}\vec{\boldsymbol{\Omega}}, r\vec{\boldsymbol{\Theta}}) \omega d\omega dr = \overline{\omega}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ \overline{v}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \overline{\omega}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \cdot \overline{r}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ \vec{\mathbf{v}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \vec{\omega}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \times \vec{\mathbf{r}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}). \end{gathered}} \tag{3.16}$$

Such the necessary combination is given by

[6] the particles have to be of a size adapting the identical movement of present fluid element of the fluctuating continuum

$$\overline{f}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}) = \int_0^\infty \int_0^\infty f(\vec{\mathbf{x}}, t, \boldsymbol{\omega} \cdot \vec{\mathbf{\Omega}}, \mathbf{r} \cdot \vec{\mathbf{\Theta}}) d\omega dr. \tag{3.17}$$

This enables a transport equation of $\overline{f}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}})$ with the averaged amounts of rotation velocities and radius-vectors as coefficients.

The resulting expectation value of the density does not equal the measured value of a single ensemble representative. That is

$$< \rho(\vec{\mathbf{x}}, t) > \neq \rho(\vec{\mathbf{x}}, t) \tag{3.18}$$

Limiting to one system of the ensemble the distribution function degenerates to a delta-function

$$f \rightarrow \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \tag{3.19}$$

with

$$\begin{aligned}
&\int_{\vec{\mathbf{r}}} \int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}} = \mathbf{1} \\
&\int_{\vec{\mathbf{r}}} \int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \vec{\boldsymbol{\omega}} d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}}' = \vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)} \\
&\int_{\vec{\mathbf{r}}} \int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \vec{\mathbf{r}} d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}}' = \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}.
\end{aligned} \tag{3.20}$$

3.5. Definition of Markov Processes with natural causality

The probabilistic theory is related to random distributions of velocities $\vec{\boldsymbol{\pi}}$ moving from $(\vec{\mathbf{x}}, t)$ to $(\vec{\mathbf{x}} + \vec{\boldsymbol{\pi}} t_\epsilon, t + t_\epsilon)$. These velocity distributions may get together of vortex and curvature vector fields

$$\vec{\boldsymbol{\pi}} = \vec{\boldsymbol{\omega}} \times \frac{\vec{\boldsymbol{b}}}{b^2}.$$

The transport from $(\vec{\mathbf{x}} - t_\varepsilon \vec{\boldsymbol{\pi}}', \boldsymbol{t} - \boldsymbol{t_\epsilon})$ to $(\vec{\mathbf{x}}, t)$ is addionally controlled by transition probabilities

$$W_{\boldsymbol{t_\epsilon}} = W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\pi}}'),$$

resulting in

$$\boxed{f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\pi}) = \int_{\vec{\pi}'} W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\pi}}')f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon\vec{\boldsymbol{\pi}}', t - \boldsymbol{t_\epsilon}, \vec{\pi}')d\vec{\boldsymbol{\pi}}'}.$$

Such a relation we call a Markov Process of natural causality. According to Sen [12] there is a so called Newtonian causality in nonrelativistic physics implying the possibility of unlimited velocities. However Newtonian causality is restricted to Newtonian mechanics and stochastic processes of physics ending with diffusion equations when applied practically. [7] This applies not for formulations of the general or linear Boltzmann Equation. In electrodynamics the velocity of light is the limiting velocity. In this treatise one essential statement is: classical physics is generally not Newtonian. Further on

1. is shown, that diffusion equations can only be approximations of an exact description. The diffusion equation is related to an unlimited propagation speed. The diffusion coefficient is correlated with the velocity of sound. Exact descriptions lead via Boltzmannlike formulations.
2. is shown, that the second Newtonian law applies to fluid dynamics in limiting cases only. In field theories as fluid dynamics not force- but accelleration fields are expressed. These are generally not free of rot (equivalently curl) in contrary to a Newtonian force field. That is why it is reasonable to distinguish conservative from non conservative accelleration fields. In classical physics one has normally non conservative fields.(Though for students a contrary impression may occur.)

The Newtonian causality proves to be a limiting case of non relativistic classical physics. Subsequently a causal Markov Process is continuously used or derived. Overarching master equations can not exist, physically. The transition probabilities $W_{\boldsymbol{t_\epsilon}}$ depend on a time quantity $\boldsymbol{t_\epsilon}$ related to continuum fluctuations of measurement accuracy according to vectorial motion quantities. For $\boldsymbol{t_\epsilon} \rightarrow \mathbf{0}$ (exact motion quantities) the transition probabillity $W_{\boldsymbol{t_\epsilon}}$ degenerates to a δ-function.

Simultaneous details of space and momentum are not possible in the context of quantum mechanics. The Schrödinger Equation for free particles

$$i\hbar\frac{\partial\psi(\vec{\mathbf{x}}, t)}{\partial t} = -\frac{\hbar^2}{2\mu}\vec{\nabla}^2\psi(\vec{\mathbf{x}}, t) \tag{3.21}$$

can be transformed into a linear homogenuous integral eqution [7] [9]

[7] This statement applies to the Fokker-Planck and Langevin equation. See, for example, Chandrasekhar[1]

$$\psi(\vec{\mathbf{x}},t) = i\int G(\vec{\mathbf{x}},t;\vec{\mathbf{x}'},t')\psi(\vec{\mathbf{x}'},t')d\vec{\mathbf{x}'}. \tag{3.22}$$

The Green function

$$G(\vec{\mathbf{x}},t;\vec{\mathbf{x}}',t') = \left\langle \vec{\mathbf{x}} \left| \boldsymbol{exp}\Big(-\frac{\mathbf{i}}{\hbar}(t-t')\mathbf{H}\Big) \right| \vec{\mathbf{x}}' \right\rangle \tag{3.23}$$

is called Feynman kernel, too.

In the case of the diffusion equation

$$\frac{\partial\rho(\vec{\mathbf{x}},t)}{\partial t} = D\vec{\nabla}^2\rho(\vec{\mathbf{x}},t) \tag{3.24}$$

an equivalent integral equation the Green function understood as transition probabillity from $(\vec{\mathbf{x}}',t')$ to $(\vec{\mathbf{x}},t)$ exists with

$$\rho(\vec{\mathbf{x}},t) = \int_{V'} G(\vec{\mathbf{x}},t;\vec{\mathbf{x}}',t')\rho(\vec{\mathbf{x}}',t')d\vec{\mathbf{x}}' \tag{3.25}$$

and the Green function

$$G(\vec{\mathbf{x}},t;\vec{\mathbf{x}}',t') = \left(\frac{1}{4\pi D(t-t')}\right)^{\frac{3}{2}} \boldsymbol{e}^{-\frac{(\vec{\mathbf{x}}-\vec{\mathbf{x}}')^2}{4\pi D(t-t')}} \quad . \tag{3.26}$$

Equations based on a "heat-kernel"-structure are not exact in classical physics (as well as the Newtonian mechanics).

In quantum mechanics and quantum field theory natural causality is not possible because of the uncertainty principle. In Relativity there is the maximal possible velocity, the velocity of light. A geometrodynamic equation system of turbulence found further down does not contain such limiting velocities, explicitly. Velocity fields are calculated uniquely by an initial field giving to GR compatible results after mapping from Einstein Space into a suitable observer-space. Using other initial conditions higher velocities are possible.

Part II.

Stochastically continuous transport of passive scalar particles within the meaning of an ensemble-theory

4. Introduction

$$\frac{\partial \overline{f}}{\partial t} + \overline{\omega} \cdot \overline{r} \vec{\mathbf{\Omega}} \times \vec{\mathbf{\Theta}} \cdot \nabla \overline{f} = \frac{-1}{t_E} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \frac{l(l+1)}{2} \overline{f}_{lmk}(\vec{\mathbf{x}}, t) P_{lm}(\vec{\mathbf{\Omega}}) H_k(\vec{\mathbf{\Theta}})$$

$$\Updownarrow$$

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}, t) = \int_{2\pi} \int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}, \vec{\mathbf{\Omega}}', \vec{\mathbf{\Theta}}') \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \cdot \overline{v}' \vec{\mathbf{\Omega}}' \times \vec{\mathbf{\Theta}}', \vec{\mathbf{\Omega}}', \vec{\mathbf{\Theta}}'), t - t_{\boldsymbol{\epsilon}}) d\vec{\mathbf{\Omega}}' d\vec{\mathbf{\Theta}}'$$

The aim is the derivation of a stochastic transport equation of turbulent, passively moved scalar particles being based on particle balancing. Stochastics is always understood as the randomness of motions. From point $\vec{\mathbf{A}}$ originated a point $\vec{\mathbf{B}}$ is approached in consequence of a random velocity. The particle motions are totally adjusted to the fluid motions of the fluctuating continuum and reproduce single fluid motions in detail.[1] The used stochastics is based for one thing on an ensemble-consideration and on the other hand on a locally formulated motion process. So an equation is achieved owning local coefficients depending on space and time as well as the states of movement. The field of coefficients can be determined in principal in every desired level of detail by the deterministic turbulence theory. The transport equation is a partial differential equation shown to be equivalent to a derived integral equation. The respective stochastic process is immediately recognized as Markovian of natural causality. We call it causal Markov Process.

Chapter 5: Within the framework of kinetic theory a physical situation is selected handling the linear Boltzmann Equation. This equation is extensively studied in nuclear reactor physics called neutron Boltzmann Equation[15].[2] Using the above described ensemble consideration a statistical particle balance is formulated by local velocities and their unsteady changes by local cross sections. The resulting mathematical ties help the developments in further chapters as guideline and answer the question, which analogies exist between kinetic theory and turbulent stochastic continuum transport.

Chapter 6: The motion of passive scalar particles by longitudinal continuum fluctuations is examined. In the centre of the consideration is the development of transition

[1]The particles are assumed to have a suitable weight

[2]The insights of this theory had little impact on similar fields of physics.

probability densities of velocities. They depend as well as the velocities and their particle density distributions on the accuracy of a measuring process indexed by t_ϵ. $\lim \boldsymbol{t_\epsilon} \to 0$ means exact measurements and the transition probabilities result into $\delta-$functions. They have the property of test functions of the distribution theory with immediate physical meaning. Calculating them a transport equation in form of a partial differential equation as well as an equivalent integral equation is derived.

Chapter 7: In analogy to chapter 6 the motion by turbulent continuum fluctuations of passive particles is examined. The fluctuation directions are expressed by Eulerian angles and the distribution functions are developed by generalized spherical harmonics (we call them turbulence functions). A pair of equations is created consisting of a partial differential equation and an equivalent integral equation as in the cases of molecular self diffusion and the longitudinal (1+3)-dimensional continuum fluctuations. The three physical situations can be compared all the more as in the three cases the transition probabilities are explicitely formulated.

5. Brownian motion as molecular self-diffusion

$$\frac{\partial}{\partial t}h+\overline{v}\vec{\mathbf{\Omega}}\cdot\nabla h=\frac{1}{\tau}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\mathbf{\Omega}})$$

$$h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\boldsymbol{v_{t_\epsilon}}\vec{\mathbf{\Omega}},t)=\int_{4\pi}\int_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}},v'\vec{\mathbf{\Omega}}')h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-v'\vec{\mathbf{\Omega}}'t_\varepsilon,v'\vec{\mathbf{\Omega}}',t-\boldsymbol{t_\epsilon})dv'd\vec{\mathbf{\Omega}}'$$

5.1. Introduction

Brownian motion is understood as a disordered thermic motion of molecules in gases or fluids creating a disordered motion of suspended, sufficiently small particles. This Brownian motion is all the more livelier the smaller the particle quantity is. With increasing particle sizes the detailed molecular influence on the particle movement disappears and having suitable sizes the particles reproduce the turbulent fluid motions. The phenomenon of small paricles was first examined by Einstein and Smoluchowski.
Subsequently, the case of very small particles that is the statistical development of the molecular distribution of a gas is evaluated. In the treatises of Einstein[4] and Smoluchowski[13] the considerations lead in each case to a diffusion equation, which contains two fundamental deficiencies, though beeing sufficient for the purpose at that time. The propagation speed concerning a diffusion equation is unlimited. Immediately after switching a point particle source on there is at least an infinitesimal influence in arbitrary distance. In close proximity to a point source the solution of a diffusion equation shows a $\sim \frac{1}{r}$-behaviour. But it should be $\sim \frac{1}{r^2}$.

5.2. Transport equation of molecular self-diffusion

Examining the molecular self-diffusion in a highly diluted gas in thermodynamic equilibrium the linear Boltzmann equation will be derived. It is a linear integro-differential equation statistically describing the transport of diffusing particles by cross sections of the interacting particles. The whole gas medium is regarded as devided into two parts, a main part and an additional very small part. The diffusing of the small part in the main part without changing the statistical properties of the main part is considered. (See section 3.2) Due to the low density of the diffusing molecules a relevant self interaction within the small part can be excluded. Regarding the spatiotemporal development the velocity distribution density $\mathbf{g}(\mathbf{v})$ is normalised to 1.

$$\int_0^\infty \mathbf{g}(\mathbf{v}) d\boldsymbol{v} = \mathbf{1} \tag{5.1}$$

The diffusing part is depicted by

$$f(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t) = f(\vec{\mathbf{x}}, t, v\vec{\mathbf{\Omega}}) = h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}})\mathbf{g}(\mathbf{v}) \tag{5.2}$$

I.e. the velocity distribution is independent of space-time $(\vec{\mathbf{x}}, t)$ and direction $(\vec{\mathbf{\Omega}})$ (equipartition theorem).

This allows to talk about an expectation value for every space-time point of the particle density

$$< \Phi(\vec{\mathbf{x}}, t) >= \int_0^\infty \int_{4\pi} f(\vec{\mathbf{x}}, t, \vec{\boldsymbol{v}}) d\vec{\mathbf{\Omega}} dv = \int_{4\pi} h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) d\vec{\mathbf{\Omega}}. \tag{5.3}$$

However the measured value of the density $\Phi(\vec{\mathbf{x}}, t)$ is only a good approximation of the expectation value

$$< \Phi(\vec{\mathbf{x}}, t) > \approx \Phi(\vec{\mathbf{x}}, t). \tag{5.4}$$

This ceases to apply for the particle transport by fluctuating continua.

The total derivative of the distribution function f in direction of the velocity $v\vec{\mathbf{\Omega}}$ results in

$$\frac{d}{dt} f(\vec{\mathbf{x}}, t, v\vec{\mathbf{\Omega}}) = \frac{\partial}{\partial t} f + v\vec{\mathbf{\Omega}} \nabla f. \tag{5.5}$$

The change of the particle density distribution for the velocity $\vec{\mathbf{v}} = v\vec{\mathbf{\Omega}}$ is balanced by collisions of molecules modifying the velocities with a certain probability expressed by differential cross sections. Defining $\frac{1}{v} f(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t)$ as particle stream[1] the following

[1]Hereby no stream in the meaning of deterministic fluid dynamics is defined!

balance equation can be noted

$$\frac{1}{v}\frac{d}{dt}f(\vec{\mathbf{x}},t,v\vec{\mathbf{\Omega}}) = \frac{1}{v}\frac{\partial}{\partial t}f + \vec{\mathbf{\Omega}}\nabla f = I_+ - I_-. \tag{5.6}$$

I_+ corresponds to molecules coming from other directions $\vec{\Omega}'$.
I_- corresponds to molecules leaving direction $\vec{\Omega}$.

The particle distribution density varying in space this expression has to be different from zero. Otherwise the particle distribution density remains constant. So the assumed initial distribution is variously dispersed in space.

The momentum exchange is determined on one side by the cross sections of the impact partners and on the other side by the number of particles arriving at location $\vec{\mathbf{x}}$ and time t per unit area with the velocity $\vec{\mathbf{v}'}$ pivoting into the velocity $\vec{\mathbf{v}}$. This growth of the number of particles per time and unit-area is

$$I_+ = \rho\int_0^\infty\int_{4\pi}\sigma(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}')\cdot f(\vec{\mathbf{x}},t,v'\vec{\mathbf{\Omega}'})dv'd\vec{\mathbf{\Omega}'} \tag{5.7}$$

with

$$\rho = \text{constant density of the main part of the gas}$$
$$\sigma(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}') = \text{differential cross section, symmetrical in } \vec{\mathbf{v}} \text{ and } \vec{\mathbf{v}}'.$$

The particles simultaneously changing their velocity $\vec{\mathbf{v}}$ into another $\vec{\mathbf{v}'}$ the appropriate decrease of particle number per time- and area-unit is expressed by

$$I_- = \rho\int_0^\infty\int_{4\pi}\sigma(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}')\cdot f(\vec{\mathbf{x}},t,v\vec{\mathbf{\Omega}})dv'd\vec{\mathbf{\Omega}'} = \Sigma(v)f(\vec{\mathbf{x}},t,v\vec{\mathbf{\Omega}}) \tag{5.8}$$

due to

$$\Sigma(v) = \rho\int_0^\infty\int_{4\pi}\sigma(\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}')dv'd\vec{\mathbf{\Omega}''} \quad [\boldsymbol{m^{-1}}]. \tag{5.9}$$

$\Sigma(v)$ represents the total macroscopic cross section and the transport equation results in the molecular self diffusion equation

$$\boxed{\frac{1}{v}\frac{\partial}{\partial t}f + \vec{\mathbf{\Omega}} \cdot \nabla f = \rho \int_0^\infty \int_{4\pi} \sigma(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}') \cdot f(\vec{\mathbf{x}}, t, v'\vec{\mathbf{\Omega}}')dv'd\vec{\mathbf{\Omega}}' - \Sigma(v)f}. \tag{5.10}$$

Further on, considering the molecules being in statistical balance throughout the whole gas, i.e. the same Boltzmann Distribution $\mathbf{g}(\mathbf{v})$ is existing everywhere, an integration of $\int_0^\infty (5.10)dv$ results in a manageable equation as follows.

Defining

$$\overline{\Sigma} = \int_0^\infty \Sigma(v)g(v)dv \tag{5.11}$$

$$\overline{\sigma}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') = \int_0^\infty \int_0^\infty \sigma(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}')g(v')dvdv' \tag{5.12}$$

$$\overline{v} = \int_0^\infty vg(v)dv \tag{5.13}$$

gives[2]

$$\frac{\partial}{\partial t}h + \overline{v}\vec{\mathbf{\Omega}} \cdot \nabla h = \overline{v}\rho \int_{4\pi} \overline{\sigma}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') \cdot h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}')d\vec{\mathbf{\Omega}}' - \overline{v}\overline{\Sigma} \cdot h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}). \tag{5.14}$$

Developing by spherical harmonics (see appendix 8.2) yield in

$$\overline{\sigma}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') = \sum_{l=0}^{+\infty} \sigma_l P_l(cos(\alpha)) = \sum_{l=0}^{+\infty} \sigma_l \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}) P^*_{lm}(\vec{\mathbf{\Omega}}') \tag{5.15}$$

and

$$\begin{aligned} h(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) &= \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} h_{lm}(\mathbf{x}, t) P_{lm}(\vec{\mathbf{\Omega}}) \\ &= \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} h_{lm}(\mathbf{x}, t) P^*_{lm}(\vec{\mathbf{\Omega}}). \end{aligned} \tag{5.16}$$

[2]Such an equation corresponds in nuclear reactor physics to the one group neutron-transport-equation regardless of absorbtions-and fission effects.

These developments inserted into (5.14) and executing the respective integrations lead to

$$\begin{aligned}\frac{\partial}{\partial t}h + \overline{v}\vec{\boldsymbol{\Omega}}\cdot\nabla h =& \rho\overline{v}\sum_{l=1}^{+\infty}\sigma_l\frac{4\pi}{2l+1}\sum_{m=-l}^{m=+l}h_{lm}P^*_{lm}(\vec{\boldsymbol{\Omega}}) - \overline{v}\overline{\Sigma}\cdot\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}})\\ =& \sum_{l=1}^{+\infty}\overline{v}\{\rho\sigma_l\frac{4\pi}{2l+1} - \overline{\Sigma}\}\cdot\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}})\end{aligned} \tag{5.17}$$

It holds

$$\overline{\Sigma} = 4\pi\rho\sigma_0 = \rho\int_{4\pi}\overline{\sigma}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}')d\vec{\boldsymbol{\Omega}}' \quad [m^{-1}] \tag{5.18}$$

and defining

$$\begin{aligned}\boldsymbol{\tau^{-1}} &= \overline{v}\overline{\Sigma} \quad [\boldsymbol{sec^{-1}}]\\ \boldsymbol{\gamma_l} &= \left(\frac{\sigma_l}{\sigma_0}\frac{1}{2l+1} - 1\right) < 0 \quad \text{für} \quad l \geq 1 \quad [/]\end{aligned} \tag{5.19}$$

$$\Longrightarrow \boldsymbol{\gamma_0 = 0} \tag{5.20}$$

one gets

$$\boxed{\frac{\partial}{\partial t}h + \overline{v}\vec{\boldsymbol{\Omega}}\cdot\nabla h = \frac{1}{\tau}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}})}. \tag{5.21}$$

As $\boldsymbol{\gamma_l} \leq 0$, the development components of order l are the more rapidly decaying the l becoming greater. The total derivation in the direction of the velocity $\overline{v}\vec{\boldsymbol{\Omega}}$ leads to

$$\frac{d}{dt}h(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}}) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l}\frac{d}{dt}h_{lm}(\mathbf{x},t)P_{lm}(\vec{\boldsymbol{\Omega}}) = \frac{1}{\tau}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}h_{lm}(\vec{\mathbf{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}}). \tag{5.22}$$

That is why the time behaviour of the single development components result in

$$\frac{d}{dt}h_{lm}(t) = \frac{\gamma_l}{\tau}h_{lm} \tag{5.23}$$

and

$$h_{lm}(t) \sim \mathbf{exp}(\frac{\gamma_l}{\tau} \cdot t). \tag{5.24}$$

So approximations of first order turn out to approach exact solutions, asymptotically.[3]

5.3. Brownian motion as Markov Process with natural causality

Defining the transition probability density of directions

$$\boxed{\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') = \sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} e^{+\Upsilon_l \cdot \frac{t_\epsilon}{\tau}} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\mathbf{\Omega}}) P_{lm}(\vec{\mathbf{\Omega}}')} \tag{5.25}$$

and determining the following relationships

$$\epsilon = \frac{t_\epsilon}{\tau}, \qquad \frac{1}{\tau} = \overline{v} \cdot \overline{\Sigma} = \overline{v} \cdot 4\pi\rho\sigma_0 = const \tag{5.26}$$

$$\overline{\sigma}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') = \sum_{l=0}^{+\infty} \sigma_l P_l(cos(\alpha)) = \sum_{l=0}^{+\infty} \sigma_l \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}) P^*_{lm}(\vec{\mathbf{\Omega}}') \tag{5.27}$$

$$\Upsilon_l = \left(\frac{\sigma_l}{\sigma_0} \cdot \frac{1}{2l+1} - 1 \right) \tag{5.28}$$

an integral equation of self-diffusion results in dependence of directions of motions, cross sections and locally averaged absolute values of velocities as coefficients,

$$\boxed{h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) = \int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}') \cdot h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - \overline{v}\vec{\mathbf{\Omega}}'\boldsymbol{t_\epsilon}, t - \boldsymbol{t_\epsilon}, \vec{\mathbf{\Omega}}') d\vec{\mathbf{\Omega}}'} \tag{5.29}$$

from which equation

$$\frac{\partial}{\partial t} h + \overline{v}\vec{\mathbf{\Omega}} \cdot \nabla h = \frac{1}{\tau} \sum_{l=1}^{+\infty} \Upsilon_l \cdot \sum_{m=-l}^{+l} h_{lm}(\vec{\mathbf{x}}, t) P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{5.21}$$

[3] This is correct in the case of lacking absorbtion processes. We are only regarding scattering.

may be reconstructed.

Proof:

$h_{\boldsymbol{t_\epsilon}}$ developed around $\vec{\mathbf{x}}$ and t until first order one gets

$$h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - \overline{v}'_{t_\epsilon}\vec{\boldsymbol{\Omega}}' \cdot t_\varepsilon, \vec{\boldsymbol{\Omega}}', t - t_{\boldsymbol{\epsilon}}) = h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}', t) - \tau \cdot \epsilon \cdot [\frac{\partial h'_{\boldsymbol{t_\epsilon}}}{\partial t} + \vec{\boldsymbol{v}}'_{t_\epsilon} \cdot \vec{\boldsymbol{\nabla}} h'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] \quad (5.30)$$

with $t_\epsilon = \tau \cdot \epsilon$. Inserted into (5.29) this leads to

$$h_{\boldsymbol{t_\epsilon}} = \int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} h'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \cdot \tau \cdot \epsilon \cdot [\frac{\partial h'_{\boldsymbol{t_\epsilon}}}{\partial t} + \vec{\overline{\boldsymbol{v}}}'_{t_\epsilon} \cdot \vec{\boldsymbol{\nabla}} h'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] d\vec{\boldsymbol{\Omega}}' \quad (5.31)$$

and simple conversions give

$$\frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} h'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - h_{\boldsymbol{t_\epsilon}}}{\epsilon} = \int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \cdot \tau [\cdot \frac{\partial h'_{\boldsymbol{t_\epsilon}}}{\partial t} + \vec{\overline{\boldsymbol{v}}}'_{t_\epsilon} \cdot \vec{\boldsymbol{\nabla}} h'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] d\vec{\boldsymbol{\Omega}}'. \quad (5.32)$$

Executing the limiting process $t_\epsilon \to 0$ the transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ results in a δ-function and the particle density distribution $h_{\boldsymbol{t_\epsilon}}$ achieves the limiting function h.

$$\begin{aligned}
&\lim_{\boldsymbol{t_\epsilon} \to 0} \overline{W}_{\boldsymbol{t_\epsilon}} = \delta(\vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \quad \text{:delta-Function} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} h_{\boldsymbol{t_\epsilon}} = h \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \vec{\overline{\boldsymbol{v}}}_{t_\epsilon} = \vec{\overline{\boldsymbol{v}}} = \overline{\boldsymbol{v}} \cdot \vec{\boldsymbol{\Omega}} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \vec{\boldsymbol{\Omega}}_{t_\epsilon} = \vec{\boldsymbol{\Omega}}
\end{aligned} \quad (5.33)$$

Executing the limiting process $\boldsymbol{t_\epsilon} \to \mathbf{0}$ on equation (5.32) the $\boldsymbol{t_\epsilon}$-indexing disappears in accordance with the distribution functions.

Before the limes process is carried out the following integrations lead to

$$\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}')h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}')d\vec{\boldsymbol{\Omega}}' =$$
$$\int_{4\pi}\Bigg[\sum_{l=0}^{+\infty}\frac{2l+1}{4\pi}e^{+\Upsilon_l\cdot\frac{t_\epsilon}{\tau}} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}})P_{lm}(\vec{\boldsymbol{\Omega}}')\Bigg]h_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}')d\vec{\boldsymbol{\Omega}}'$$
$$= \int_{4\pi}\Bigg[\sum_{l=0}^{+\infty}\frac{2l+1}{4\pi}e^{+\Upsilon_l\cdot\frac{t_\epsilon}{\tau}} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}})P_{lm}(\vec{\boldsymbol{\Omega}}')\Bigg]\sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l} h_{\boldsymbol{t_\epsilon}lm}(\mathbf{x}, t)P^*_{lm}(\vec{\boldsymbol{\Omega}}')d\vec{\boldsymbol{\Omega}}' \tag{5.34}$$
$$= \sum_{l=0}^{+\infty} e^{+\Upsilon_l\cdot\frac{t_\epsilon}{\tau}}\sum_{m=-l}^{m=+l} h_{\boldsymbol{t_\epsilon}lm}(\mathbf{x}, t)P^*_{lm}(\vec{\boldsymbol{\Omega}}).$$

Thus one gets

$$\frac{\int_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}h'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}' - h_{\boldsymbol{t_\epsilon}}}{\epsilon\cdot\tau}$$
$$= \frac{\int_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}h'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}' - \sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l} h_{\boldsymbol{t_\epsilon}lm}(\mathbf{x}, t)P^*_{lm}(\vec{\boldsymbol{\Omega}})}{\epsilon\cdot\tau} \tag{5.35}$$
$$= \frac{\sum_{l=0}^{+\infty}(e^{+\Upsilon_l\cdot\frac{t_\epsilon}{\tau}} - 1)\sum_{m=-l}^{m=+l} h_{\boldsymbol{t_\epsilon}lm}(\mathbf{x}, t)P^*_{lm}(\vec{\boldsymbol{\Omega}})}{\epsilon\cdot\tau}.$$

Setting

$$\Upsilon_l = \lim_{\boldsymbol{t_\epsilon}\to 0}\frac{e^{+\Upsilon_l\cdot\frac{t_\epsilon}{\tau}} - 1}{\epsilon}, \qquad t_\epsilon = \epsilon\cdot\tau \tag{5.36}$$

creates

$$\lim_{\boldsymbol{\epsilon}\to 0}\frac{\int_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}h'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}' - h_{\boldsymbol{t_\epsilon}}}{\epsilon\cdot\tau} = \frac{1}{\tau}\sum_{l=1}^{+\infty}\Upsilon_l\cdot\sum_{m=-l}^{+l} h_{lm}(\vec{\mathbf{x}}, t)P^*_{lm}(\vec{\boldsymbol{\Omega}}) \tag{5.37}$$

and

$$\lim_{\boldsymbol{\epsilon}\to 0}\frac{\int_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}h'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}' - h_{\boldsymbol{t_\epsilon}}}{\epsilon\cdot\tau} = \frac{\partial h}{\partial t} + \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{\nabla}}h, \tag{5.38}$$

which results in

$$\boxed{\frac{\partial h}{\partial t} + \overline{v}\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\nabla}} h = \frac{1}{\tau} \sum_{l=1}^{+\infty} \Upsilon_l \cdot \sum_{m=-l}^{+l} h_{lm}(\vec{\mathbf{x}}, t) P^*_{lm}(\vec{\mathbf{\Omega}})}. \tag{5.39}$$

q.e.d.

Extending the transition probability density $\overline{W}$ by the velocity distribution $g(v')$

$$W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}, v'\vec{\mathbf{\Omega}}') = g(v')\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}'), \tag{5.40}$$

one gets the Brownian molecular motion under the terms of the described model in the most general form.

$$\boxed{f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \boldsymbol{v_{t_\epsilon}}\vec{\mathbf{\Omega}}, t) = \int_{4\pi} \int_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}, v'\vec{\mathbf{\Omega}}') f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - v'\vec{\mathbf{\Omega}}' t_\varepsilon, v'\vec{\mathbf{\Omega}}', t - t_{\boldsymbol{\epsilon}}) dv' d\vec{\mathbf{\Omega}}'} \tag{5.41}$$

The transition probabilities are not symmetric in contrary to the differential cross section!

5.4. Approximation formula

An approximation formula of 1. order of the equation

$$\frac{\partial}{\partial t} h + \overline{v}\vec{\mathbf{\Omega}} \cdot \nabla h = \frac{1}{\tau} \sum_{l=1}^{\infty} \boldsymbol{\gamma_l} \cdot \sum_{m=-l}^{+l} h_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\mathbf{\Omega}}) \tag{5.42}$$

is beeing looked for. The approach accounts for the methods of the transport theory of nuclear reactor physics [15]. In cartesian coordinates this leads to

$$\frac{\partial}{\partial t} h + \overline{v} \cdot \left(\mathbf{\Omega}_x \frac{\partial}{\partial x} h + \mathbf{\Omega}_y \frac{\partial}{\partial y} h + \mathbf{\Omega}_z \frac{\partial}{\partial z} h \right) = \frac{1}{\tau} \sum_{l=1}^{\infty} \boldsymbol{\gamma_l} \cdot \sum_{m=-l}^{+l} h_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\mathbf{\Omega}}). \tag{5.43}$$

$$\begin{aligned} \boldsymbol{\Omega_x} &= sin(\vartheta)cos(\varphi) \\ \boldsymbol{\Omega_y} &= sin(\vartheta)sin(\varphi) \\ \boldsymbol{\Omega_z} &= cos(\vartheta) \end{aligned} \tag{5.44}$$

The spherical harmonics of 0th and 1st Order are

$$\begin{aligned} &P_{00} = 1 \qquad P_{1-1} = 2^{-\frac{1}{2}} e^{-i\varphi} sin\vartheta \qquad P_{10} = cos\vartheta \qquad P_{11} = -2^{-\frac{1}{2}} e^{i\varphi} sin\vartheta \\ &P^*_{00} = 1 \qquad P^*_{1-1} = 2^{-\frac{1}{2}} e^{+i\varphi} sin\vartheta \qquad P^*_{10} = cos\vartheta \qquad P^*_{11} = -2^{-\frac{1}{2}} e^{-i\varphi} sin\vartheta. \end{aligned} \tag{5.45}$$

In cartesian coordinates until 1st order this leads to

$$\frac{\partial}{\partial t} h + \overline{v} \cdot \left(\boldsymbol{\Omega}_x \frac{\partial}{\partial x} h + \boldsymbol{\Omega}_y \frac{\partial}{\partial y} h + \boldsymbol{\Omega}_z \frac{\partial}{\partial z} h \right) = \frac{1}{\tau} \boldsymbol{\gamma_1} \cdot \sum_{m=-1}^{+1} h_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\boldsymbol{\Omega}}) \tag{5.46}$$

The direction vectors in cartesian coordinates expressed by spherical harmonics are written

$$\begin{aligned} \boldsymbol{\Omega}_x &= 2^{-\frac{1}{2}} [P_{1-1} - P_{11}] \\ \boldsymbol{\Omega_y} &= -i2^{-\frac{1}{2}} [P_{1-1} + P_{11}] \\ \boldsymbol{\Omega_z} &= P_{10}. \end{aligned} \tag{5.47}$$

The transport equation in 1st approximation is reduced to

$$\begin{aligned} &\frac{\partial}{\partial t}(h_{00}P_{00} + h_{1-1}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}) \\ &+ \overline{v}\Bigg[\cdot 2^{-\frac{1}{2}} [P_{1-1} - P_{11}] \frac{\partial}{\partial x}(h_{00}P_{00} + h_{1-1}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}) \\ &- i2^{-\frac{1}{2}} [P_{1-1} + P_{11}] \frac{\partial}{\partial y}(h_{00}P_{00} + h_{1-1}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}) \\ &+ P_{10} \frac{\partial}{\partial z}(h_{00}P_{00} + h_{1-1}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}) \Bigg] \\ &= \frac{1}{\tau} \boldsymbol{\gamma_1} \cdot (h_{1-10}P_{1-1} + h_{10}P_{10} + h_{11}P_{11}). \end{aligned} \tag{5.48}$$

After integrating $\int(5.48 P^*_{lm}(\vec{\boldsymbol{\Omega}}) d\vec{\boldsymbol{\Omega}}$ for $l = 0, 1$ the evolution equation set until 1st order

$$\frac{\partial h_{00}}{\partial t} + \frac{\overline{v}}{3}\left[2^{-\frac{1}{2}}\left(-\frac{\partial h_{11}}{\partial x} + \frac{\partial h_{1-1}}{\partial x}\right) - i2^{-\frac{1}{2}}\left(\frac{\partial h_{11}}{\partial y} + \frac{\partial h_{1-1}}{\partial y}\right) + \frac{\partial h_{10}}{\partial z}\right] = 0 \tag{5.49}$$

$$\frac{\partial h_{10}}{\partial t} + \overline{v}\frac{\partial h_{00}}{\partial z} - \frac{\Upsilon_1}{\tau} h_{10} = 0 \tag{5.50}$$

$$\frac{\partial h_{1-1}}{\partial t} + \overline{v}2^{-\frac{1}{2}}\left(\frac{\partial h_{00}}{\partial x} + i\frac{\partial h_{00}}{\partial y}\right) - \frac{\Upsilon_1}{\tau} h_{1-1} = 0 \tag{5.51}$$

$$\frac{\partial h_{11}}{\partial t} + \overline{v}2^{-\frac{1}{2}}\left(-\frac{\partial h_{00}}{\partial x} + i\frac{\partial h_{00}}{\partial y}\right) - \frac{\Upsilon_1}{\tau} h_{11} = 0 \tag{5.52}$$

is approached.

Now we define a vector field $\vec{\boldsymbol{J}}$.

$$\begin{aligned} J_x =& \frac{4\pi}{3} 2^{-\frac{1}{2}} h(_{1-1} - h_{11}) \\ J_y =& -i\frac{4\pi}{3} 2^{-\frac{1}{2}} (h_{1-1} + h_{11}) \\ J_z =& \frac{4\pi}{3} h_{10} \\ \boldsymbol{\Phi} =& 4\pi h_{00} \end{aligned} \tag{5.53}$$

Insertion (5.53) into (5.49) gives

$$\frac{\partial \boldsymbol{\Phi}}{\partial t} + \overline{v} \cdot \vec{\nabla} \cdot \vec{\boldsymbol{J}} = 0. \tag{5.54}$$

$\boldsymbol{\Phi}$ is the particle density of an in a thought experiment assumed small part of the molecular set. Inserting (5.53) into (5.50) until (5.52) leads to

$$\vec{\boldsymbol{J}} = \frac{\tau}{\Upsilon_1}\left[\frac{\overline{v}}{3}\vec{\nabla}\Phi + \frac{\partial \vec{\boldsymbol{J}}}{\partial t}\right] \tag{5.55}$$

with $\Upsilon_1 = \left(\frac{1}{3}\frac{\sigma_1}{\sigma_0} - 1\right) = -\eta$

So a telegrapher's equation arises[4]

$$\frac{\tau}{\eta}\frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial \Phi}{\partial t} = \frac{\tau}{\eta}\overline{v}\vec{\nabla} \cdot \frac{\overline{v}}{3}\vec{\nabla}\Phi. \tag{5.56}$$

[4]To derive telegrapher's equation relativistic considerations are not necessary as is stated in [3]. The propagation speed is closely connected with the speed of sound.

As η, τ and $\overline{v}$ represent constants, the telegrapher's equation is written

$$\boxed{\begin{aligned} \frac{\tau}{\eta}\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial\Phi}{\partial t} &= \boldsymbol{D}\Delta\boldsymbol{\Phi} \\ \textit{with} \qquad \boldsymbol{D} &= \frac{\tau}{\eta}\frac{\overline{v}^2}{3} \end{aligned}} \tag{5.57}$$

D=diffusion coefficient, η dimensionless, $\tau = (\overline{v} \cdot \overline{\Sigma})^{-1}$= mean free collision time
$\overline{v}$=mean amount of velocity
Compared to the 1st derivation the term with temporal derivation of 2nd order can normally be neglected.

The dependence of the diffusion coefficient from macroscopic state variables of an ideal gas may happen as follows:

The equation of state of the ideal gas becomes

$$\boldsymbol{p} = \boldsymbol{\rho R T}. \tag{5.58}$$

The mean quadratic velocity of a Maxwellian velocity distribution of particles with mass m is [2]

$$\overline{\boldsymbol{v^2}} = \frac{3kT}{m} \tag{5.59}$$

$\Longrightarrow$

$$\overline{\boldsymbol{v}} = \sqrt{\frac{8}{\pi}\frac{kT}{m}} = \sqrt{\frac{8}{\pi}\frac{p}{\rho}} \tag{5.60}$$

m means the mass of a molecule.
k is the Boltzmann constant.

To get a comparison with the speed of sound at a Gaussian velocity distribution

$$\boldsymbol{c} = \sqrt{\left.\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\rho}}\right|_{\boldsymbol{T}}} \tag{5.61}$$

one obtains

$$\overline{\boldsymbol{v}} = \sqrt{\frac{8}{\pi}}\boldsymbol{c} \tag{5.62}$$

and

$$\boxed{\boldsymbol{D} = \frac{1}{\eta\overline{\Sigma}}\frac{8}{3\pi}\boldsymbol{c^2}} \ . \tag{5.63}$$

The propagation speed for Brownian molecular motion is $\overline{\boldsymbol{v}}$. This in particular becomes apparent by equation (5.29). In connection with the diffusion approximation an unlimited propagation speed is assigned. This leads to solutions approaching asymptotically to those of the linear Boltzmann Equation. In close proximity to point sources (less than 3 average free lengths afar[5]) one obtains the following characteristics of the exact and the diffusional solution.

$$\Phi \sim \frac{1}{r^2} \quad \text{solution of the transport equation in the proximity of a point source} \tag{5.64}$$

Anticipating this result from an exact theory appears directly plausible.

$$\Phi \sim \frac{1}{r} \quad \text{solution of the diffusion approximation in the proximity of a point source} \tag{5.65}$$

Avoiding such deficiencies it is neccessary to take a stochastic velocity distribution into account as root of the diffusion process. Analyzing turbulent particle transport this does not satisfy.

5.5. Appendix: equations for the spherical hamonics components

The general equations arise out of

$$\int (5.43) P^*_{lm}(\vec{\boldsymbol{\Omega}}) d\vec{\boldsymbol{\Omega}} \tag{5.66}$$

$\Longrightarrow$

[5] An experience of the neutron transport theory [15]

$$\begin{aligned}\frac{\partial h_{lm}(\vec{\mathbf{x}},t)}{\partial t} = -\overline{v}\Bigg[&\frac{\sqrt{(l+2+m)(l+1+m)}}{2l+3}\left(-\frac{1}{2}\frac{\partial h_{l+1,m+1}}{\partial x}-\frac{i}{2}\frac{\partial h_{l+1,m+1}}{\partial y}\right)\\ &+\frac{\sqrt{(l+1-m)(l+2-m)}}{2l+3}\left(\frac{1}{2}\frac{\partial h_{l+1,m-1}}{\partial x}-\frac{i}{2}\frac{\partial h_{l+1,m-1}}{\partial y}\right)\\ &+\frac{\sqrt{(l-1-m)(l-m)}}{2l-1}\left(\frac{1}{2}\frac{\partial h_{l-1,m+1}}{\partial x}+\frac{i}{2}\frac{\partial h_{l-1,m+1}}{\partial y}\right)\\ &+\frac{\sqrt{(l+m)(l+m-1)}}{2l-1}\left(-\frac{1}{2}\frac{\partial h_{l-1,m-1}}{\partial x}+\frac{i}{2}\frac{\partial h_{l-1,m-1}}{\partial y}\right)\\ &+\frac{\sqrt{(l+1+m)(l-m+1)}}{2l+3}\frac{\partial h_{l+1,m}}{\partial z}+\frac{\sqrt{(l+m)(l-m)}}{2l-1}\frac{\partial h_{l-1,m}}{\partial z}\Bigg]-\frac{\Upsilon_l}{\tau}h_{lm}\end{aligned} \tag{5.67}$$

6. Stochastic transport by longitudinal fluctuations of a continuum

$$\boxed{\begin{array}{c}\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\mathbf{\Omega}},t) = \int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\mathbf{\Omega}},\vec{\mathbf{\Omega}}')\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \cdot \overline{v}'\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}', t - t_{\boldsymbol{\epsilon}})d\vec{\mathbf{\Omega}}' \\ \Updownarrow \\ \frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\mathbf{\Omega}} \cdot \vec{\boldsymbol{\nabla}}\overline{f} = -\frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\frac{l(l+1)}{2}\overline{f}_{l,m}(\vec{\mathbf{x}},t)P_{l,m}(\vec{\mathbf{\Omega}})\end{array}}$$

6.1. Introduction

The motion of passive particles by longitudinal continuum fluctuations is examined. The particles are moved in this field without interaction.[1] In accordance with section 3.3 they perform detailed motions of single fluid elements of fluid continua. The considered velocities of the particles are determined by measure processes. The particles coming from point x_1 and moving further for a time t_ε are detected in x_2. So the velocity $\vec{\mathbf{v}}_{t_\epsilon}$ may be assigned to

$$\vec{\mathbf{v}}_{t_\epsilon} = \frac{\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1}{t_\epsilon} = v_{t_\epsilon}\vec{\mathbf{\Omega}}_{t_\epsilon}. \tag{6.1}$$

This corresponds to $(\vec{\mathbf{x}}_1, t) \longrightarrow (\vec{\mathbf{x}}_2, t + t_\varepsilon) = (\vec{\mathbf{x}}_1 + \vec{\mathbf{v}}_{t_\epsilon} \cdot t_\varepsilon, t + t_\varepsilon)$.

According to an ensemble consideration (see chapter 3) for every point $(\vec{\boldsymbol{x}}, t)$ a continuously differentiable particle density distribution of velocities $\vec{\mathbf{v}}_{t_\epsilon}$ is assigned in accordance with

$$f_{\boldsymbol{t_\epsilon}} = f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\mathbf{v}}_{t_\epsilon}, t). \tag{6.2}$$

[1] Such conditions generally lead to linear equations.

The functions indexed with $\boldsymbol{t_\epsilon}$ or $\boldsymbol{\epsilon}$ enclose motion quantities $\vec{\mathbf{v}}_{t_\epsilon}$ or their motion directions $\vec{\boldsymbol{\Omega}}_{\boldsymbol{t_\epsilon}}$ as variables subordinated to an understanding of measurement accuracy. The indexing of motion quantities with $\boldsymbol{t_\epsilon}$ or $\boldsymbol{\epsilon}$ may be dropped if their functions are indexed. Executing a limiting process, for instance

$$\lim_{\boldsymbol{t_\epsilon}\to 0} f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t) = f(\vec{\mathbf{x}}, \vec{\mathbf{v}}, t) \tag{6.3}$$

f and $\vec{\mathbf{v}}$ are literally understood as results of exact measurement processes. [2] Integrating the particle density distribution over the velocity one obtains an expectation value of a particle density not generally coinciding with the actually measured value ρ.

$$< \boldsymbol{\rho_{t_\epsilon}(\vec{x}, t)} >= \int_{4\pi}\int_0^\infty f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, v\vec{\boldsymbol{\Omega}}, t)dvd\vec{\boldsymbol{\Omega}} \neq \boldsymbol{\rho_{t_\epsilon}}(\vec{\mathbf{x}}, t) \tag{6.4}$$

This is contradicting the molecular self-diffusion beeing an inherent stochastic process.
It results into a rigorously derived partial differential equation calculating particle density distributions in dependence on space-time and motion directions. The initially unlimited number of unknown coefficients is reduced to one, a local time-scaling. The initially abstractly formulated transition probabilities obtain their precise functional dependencies alternativly generating an integral equation. For numerical solutions there are always suitable Monte-Carlo methods possible.

Equations of 1st approximation substantially differ from usual diffusion equations.

6.2. Transport by Markov Processes with natural causality

The probability particles at location $\vec{x}$ and time t changing their velocity from $\vec{\boldsymbol{v}}'_{t_\epsilon} = v'\vec{\boldsymbol{\Omega}}'$ to $\vec{\boldsymbol{v}}_{t_\epsilon} = v\vec{\boldsymbol{\Omega}}$ is given by the transition probability

$$W_{\boldsymbol{t_\epsilon}} = W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}, v'\vec{\boldsymbol{\Omega}}') \tag{6.5}$$

with

$$\int_{4\pi}\int_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}, v'\vec{\boldsymbol{\Omega}}')dv'd\vec{\boldsymbol{\Omega}}' = 1. \tag{6.6}$$

[2]With the indexing $\boldsymbol{t_\epsilon}$ an assigned measurement process is always understood according to accuracy.

So the following Markov Process is defined by

$$\boxed{f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, v\vec{\boldsymbol{\Omega}}, t) = \int_{4\pi}\int_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, v\vec{\boldsymbol{\Omega}}, v'\vec{\boldsymbol{\Omega}}') f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \cdot v'\vec{\boldsymbol{\Omega}}', v'\vec{\boldsymbol{\Omega}}', t - t_{\boldsymbol{\epsilon}}) dv' d\vec{\boldsymbol{\Omega}}'} \tag{6.7}$$

$$\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} - \vec{\mathbf{x_1}} = v'_{t_\epsilon}\vec{\boldsymbol{\Omega}}'_{t_\epsilon} \cdot t_{\boldsymbol{\epsilon}} \tag{6.8}$$

For the transition probability $W_{\boldsymbol{t_\epsilon}}$ merely steadiness is reqired regarding all variables. The sequence of the velocities $\vec{\boldsymbol{v}}'_{t_\epsilon}, \vec{\boldsymbol{v}}_{t_\epsilon}$ means a motion

$$(\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}'_{t_\epsilon} \cdot t_{\boldsymbol{\epsilon}}, t - t_\epsilon, \vec{\boldsymbol{v}}'_{t_\epsilon}) \longrightarrow (\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{v}}_{t_\epsilon}). \tag{6.9}$$

At the process $t_\varepsilon \to 0$ the transition probabilities W_{t_ϵ} prove to be physical realisations of test functions of distribution theory.
The passive particles have to reproduce the motions of the fluctuation field, exactly. For the particle density distribution $f_{t_\varepsilon}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{v}})$ a separation approach is formulated without restriction of generality:

$$\begin{aligned} f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - v\vec{\boldsymbol{\Omega}} \cdot t_\varepsilon, v\vec{\boldsymbol{\Omega}}, t) =& G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - v\vec{\boldsymbol{\Omega}} \cdot t_\varepsilon, v\vec{\boldsymbol{\Omega}}, t)\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - \overline{v}\vec{\boldsymbol{\Omega}} \cdot t_\varepsilon, \vec{\boldsymbol{\Omega}}, t) \\ &\int_0^\infty G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, v\vec{\boldsymbol{\Omega}}, t) dv = \mathbf{1} \end{aligned} \tag{6.10}$$

$\Longrightarrow$

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - \overline{v}\vec{\boldsymbol{\Omega}} \cdot t_\varepsilon, \vec{\boldsymbol{\Omega}}, t) = \int_0^\infty f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - v\vec{\boldsymbol{\Omega}} \cdot t_\varepsilon, v\vec{\boldsymbol{\Omega}}, t) dv \tag{6.11}$$

This results in

$$\overline{\mathbf{v}} = \overline{\mathbf{v}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) = \int_0^\infty G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, v\vec{\boldsymbol{\Omega}}, t) \cdot \mathbf{v} dv. \tag{6.12}$$

I.e. $\overline{\mathbf{v}}$ is dependent on $(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}})$.

A transition probability only in dependence on the directions and space-time $\overline{W}_{\boldsymbol{t_\epsilon}}$ is obtained by integration of $W_{\boldsymbol{t_\epsilon}}$ over the velocity amounts v'_{t_ϵ} and v_{t_ϵ}

$$\boxed{\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Omega}}',t)=\int\limits_0^\infty\int\limits_0^\infty W_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},t,v\vec{\boldsymbol{\Omega}},v'\vec{\boldsymbol{\Omega}}')G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-v'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,v'\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})dv'dv}. \tag{6.13}$$

Now an integration of $\int_0^\infty(6.7)dv$ is leading to

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}},t)=\int\limits_0^\infty\int\limits_0^\infty\int\limits_{4\pi} W_{\boldsymbol{t_\epsilon}}G_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-v'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,v'\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-\overline{v}'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})dv'dvd\vec{\boldsymbol{\Omega}}' \tag{6.14}$$

respectively

$$\boxed{\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}},t)=\int\limits_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Omega}}')\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-\overline{v}'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})d\vec{\boldsymbol{\Omega}}'}. \tag{6.15}$$

$\overline{f}_{\boldsymbol{t_\epsilon}}$ in the integrand is developed about $\vec{\mathbf{x}}$ and t until 1st order and one obtains

$$\begin{aligned}\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-\overline{v}'\vec{\boldsymbol{\Omega}}'\cdot t_\varepsilon,\vec{\boldsymbol{\Omega}}',t-t_{\boldsymbol{\epsilon}})&=\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',t)-\tau_E\cdot\epsilon\cdot[\frac{\partial\overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t}+\overline{v}'_{\boldsymbol{t_\epsilon}}\vec{\boldsymbol{\Omega}}'_{\boldsymbol{t_\epsilon}}\cdot\vec{\boldsymbol{\nabla}}\overline{f}'_{\boldsymbol{t_\epsilon}}+O(\epsilon^2)]\\ \overline{f}'_{\boldsymbol{t_\epsilon}}&=\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',t)\end{aligned} \tag{6.16}$$

with $t_\epsilon=\tau_E\cdot\epsilon$ and $\tau_E=const.$ Inserted in (6.15) this leads to

$$\overline{f}_{\boldsymbol{t_\epsilon}}=\int_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'-\int\limits_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}\cdot\tau_E\cdot\epsilon\cdot[\frac{\partial\overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t}+\vec{\overline{\boldsymbol{v}}}'_{\boldsymbol{t_\epsilon}}\cdot\vec{\boldsymbol{\nabla}}\overline{f}'_{\boldsymbol{t_\epsilon}}+O(\epsilon^2)]d\vec{\boldsymbol{\Omega}}'. \tag{6.17}$$

and simple conversions give

$$\frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon} = \int\limits_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \cdot \tau_E [\cdot \frac{\partial \overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t} + \vec{\overline{\boldsymbol{v}}}'_{t_\epsilon} \cdot \vec{\boldsymbol{\nabla}} \overline{f}'_{\boldsymbol{t_\epsilon}} + O(\epsilon)] d\vec{\boldsymbol{\Omega}}'. \tag{6.18}$$

The process $t_\epsilon \to 0$ applied to the transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ ceates a δ-function and the particle density distribution $\overline{f}_{\boldsymbol{t_\epsilon}}$ results in $\overline{f}$.

$$\begin{aligned}
&\lim_{\boldsymbol{t_\epsilon} \to 0} \overline{W}_{\boldsymbol{t_\epsilon}} = \delta(\vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \quad \text{:delta-Function} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \overline{f}_{\boldsymbol{t_\epsilon}} = \overline{f} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \vec{\overline{\boldsymbol{v}}}_{t_\epsilon} = \vec{\overline{\boldsymbol{v}}} = \overline{\boldsymbol{v}} \cdot \vec{\boldsymbol{\Omega}} \\
&\lim_{\boldsymbol{t_\epsilon} \to 0} \vec{\boldsymbol{\Omega}}_{t_\epsilon} = \vec{\boldsymbol{\Omega}}
\end{aligned} \tag{6.19}$$

These relations applied to equation (6.18) give

$$\boxed{\lim_{\epsilon \to 0} \frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \boldsymbol{\tau_E}} = \frac{\partial \overline{f}}{\partial t} + \vec{\overline{\boldsymbol{v}}} \cdot \vec{\boldsymbol{\nabla}} \overline{f}} \tag{6.20}$$

and

$$\lim_{\epsilon \to 0} \frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \boldsymbol{\tau_E}} \tag{6.21}$$

subsequently called exchange-term.

6.3. Calculation of the exchange-term

The dependencies of the transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ on the initially uncorrelated movement directions $\vec{\boldsymbol{\Omega}}$ and $\vec{\boldsymbol{\Omega}}'$ may be expressed by the scalar product of the movement directions $\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}'$ and the simultaneous interchange of the constant time scaling $\boldsymbol{\tau_E}$ by a time scaling depending on location, time and direction $\boldsymbol{t_E}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, t)$, i.e.

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{\int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \boldsymbol{\tau_E}} &= \lim_{\epsilon \to 0} \frac{\int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}} (\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \boldsymbol{t_E}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, t)} \\
\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') &\longrightarrow \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') \\
\boldsymbol{\tau_E} &\longrightarrow \boldsymbol{t_E}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, t).
\end{aligned} \tag{6.22}$$

The direction distribution of the particles is developed by complex spherical harmonics P_{lm}, the transition probability by legendre polynomials P_l.

$$\overline{f}(\vec{\mathbf{x}}, \vec{\mathbf{\Omega}}, t) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \overline{f}_{lm}(\vec{\mathbf{x}}, t) P_{lm}(\vec{\mathbf{\Omega}}) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \overline{f}_{lm}(\vec{\mathbf{x}}, t) P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{6.23}$$

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}' \cdot \vec{\mathbf{\Omega}}) = \sum_{l=0}^{+\infty} \widetilde{W}_{\boldsymbol{t_\epsilon} l} P_l(cos(\alpha)) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} \widetilde{W}_{\boldsymbol{t_\epsilon} l} P_{lm}(\vec{\mathbf{\Omega}}') P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{6.24}$$

The spherical harmonics P_{lm} are

$$P_{lm}(\vec{\mathbf{\Omega}}) = e^{im\varphi} \frac{(-sin(\vartheta))^m}{l!2^l} \cdot \left(\frac{(l-m)!}{(l+m)!} \right)^{\frac{1}{2}} \frac{d^{l+m}(cos^2\vartheta - 1)^l}{(dcos\vartheta)^{l+m}} \tag{6.25}$$

The normalisation holds:

$$\int_{4\pi} P_{lm} P^*_{lm} d\vec{\mathbf{\Omega}} = \begin{cases} \frac{4\pi}{2l+1} & \text{l} = \text{l' and m=m'} \\ 0 & \text{else} \end{cases} \tag{6.26}$$

There is the relation between spherical harmonics P_{lm} and Legendre polynomials P_l:

$$P_l(\vec{\mathbf{\Omega}}' \cdot \vec{\mathbf{\Omega}}) = P_l(cos(\alpha)) = \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}') P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{6.27}$$

Thus one has

$$\begin{aligned} \int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{\epsilon}} d\vec{\mathbf{\Omega}}' &= \int_{4\pi} \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} \widetilde{W}_{\boldsymbol{t_\epsilon} l} P_{lm}(\vec{\mathbf{\Omega}}') P^*_{lm}(\vec{\mathbf{\Omega}}) \cdot \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \overline{f}_{\boldsymbol{t_\epsilon} lm}(\vec{\mathbf{x}}, t) P^*_{lm}(\vec{\mathbf{\Omega}}') d\vec{\mathbf{\Omega}}' \\ &= \sum_{l=0}^{+\infty} \widetilde{W}_{\boldsymbol{t_\epsilon} l} \frac{4\pi}{2l+1} \sum_{m=-l}^{+l} P^*_{lm}(\vec{\mathbf{\Omega}}) \overline{f}_{\boldsymbol{t_\epsilon} lm}(\vec{\mathbf{x}}, t). \end{aligned} \tag{6.28}$$

The left side of equation (6.20) results in

$$\lim_{\epsilon\to 0}\frac{\int_{4\pi}\widetilde{W}_{\boldsymbol{t}_\epsilon}\overline{f}'_{\boldsymbol{t}_\epsilon}d\vec{\boldsymbol{\Omega}}'-\overline{f}_{\boldsymbol{t}_\epsilon}}{t_E\cdot\epsilon}=\lim_{\epsilon\to 0}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\frac{(\widetilde{W}_{\boldsymbol{t}_\epsilon l}\frac{4\pi}{2l+1}-1)}{t_E\cdot\epsilon}\overline{f}_{\boldsymbol{t}_\epsilon lm}(\vec{\mathbf{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}})$$
$$=\frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\Upsilon_l\overline{f}_{\boldsymbol{t}_\epsilon lm}(\vec{\mathbf{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}}) \tag{6.29}$$

with

$$\Upsilon_l=\lim_{\epsilon\to 0}\frac{(\widetilde{W}_{\boldsymbol{t}_\epsilon l}\frac{4\pi}{2l+1}-1)}{\boldsymbol{\epsilon}} \tag{6.30}$$

as exchange coefficient.

Now equation (6.20) yields

$$\boxed{\frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\Upsilon_l\overline{f}_{lm}(\vec{\mathbf{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})=\frac{\partial\overline{f}}{\partial t}+\overline{v}\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\nabla}}\overline{f}}. \tag{6.31}$$

6.4. Calculation of the exchange-coefficients Υ_l

The transition probability is outlined by Legendre-polynomials respectively spherical harmonics:

$$\widetilde{W}_{\boldsymbol{t}_\epsilon}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}')=\sum_{l=0}^{+\infty}\widetilde{W}_{\boldsymbol{t}_\epsilon l}P_l(cos(\vartheta))=\sum_{l=0}^{+\infty}\widetilde{W}_{\boldsymbol{t}_\epsilon l}\sum_{m=-l}^{m=+l}P_{lm}(\vec{\boldsymbol{\Omega}})P^*_{lm}(\vec{\boldsymbol{\Omega}}')$$
$$cos(\vartheta)=\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}'=\mu. \tag{6.32}$$

On the other hand is

$$\lim_{\boldsymbol{t_\epsilon}\to 0} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') = \delta(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}')$$
$$\delta(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') = \sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} \sum_{m=-l}^{m=+l} P_{lm}(\vec{\boldsymbol{\Omega}}) P^*_{lm}(\vec{\boldsymbol{\Omega}}') = \sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} \boldsymbol{P_l} \qquad \text{see(8.20).} \tag{6.33}$$

$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\mu) \geqslant 0$ is only in the range $\mu \in [1-\varepsilon, 1]$ essentially different from 0. So the Legendre polynomials are approximated by

$$\begin{aligned}
&P_l(\mu) = 1 - \frac{dP_l}{d\mu}|_1 \cdot \varepsilon + O(\varepsilon^2) \qquad \varepsilon = 1-\mu \\
&\frac{dP_l}{d\mu}|_1 = \frac{l(l+1)}{2} \quad \text{ see (8.1) } \boldsymbol{P_0} = 1, \boldsymbol{P_1} = \mu \\
&\Longrightarrow \\
&P_l(\mu) = P_0 - (P_0 - P_1)\frac{l(l+1)}{2} + O(\varepsilon^2).
\end{aligned} \tag{6.34}$$

Using

$$\int_{-1}^{+1} P_l P_{l'} d\mu = \delta_{ll'} \frac{2}{2l+1} \tag{6.35}$$

follows

$$\int_{-1}^{+1} \widetilde{W}_{\boldsymbol{t_\epsilon}} P_l d\mu = 2\widetilde{W}_{\boldsymbol{t_\epsilon}0} - l(l+1)\widetilde{W}_{\boldsymbol{t_\epsilon}0} + \frac{l(l+1)}{3}\widetilde{W}_{\boldsymbol{t_\epsilon}1} = \frac{2}{2l+1}\widetilde{W}_{\boldsymbol{t_\epsilon}l}. \tag{6.36}$$

Furthermore is

$$\begin{aligned}
&\int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') d\vec{\boldsymbol{\Omega}}' = \int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}0} d\vec{\boldsymbol{\Omega}}' = \boldsymbol{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}0} = \boldsymbol{1} \\
&\quad \Longrightarrow \qquad \widetilde{W}_{\boldsymbol{t_\epsilon}0} = \frac{1}{4\pi},
\end{aligned} \tag{6.37}$$

as $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ for $\boldsymbol{t_\epsilon} \to \boldsymbol{0}$ degenerates to a δ-function. That is why the $\widetilde{W}_{\boldsymbol{t_\epsilon}l}$ are expressed by $\widetilde{W}_{\boldsymbol{t_\epsilon}1}$ and the determination of $\widetilde{W}_{\boldsymbol{t_\epsilon}1}$ remains to be calculated. We set

$$\lim_{\boldsymbol{\epsilon}\to 0} \frac{(\widetilde{W}_{\boldsymbol{t_\epsilon} 1}\frac{4\pi}{3} - 1)}{\boldsymbol{\epsilon}} = \boldsymbol{\zeta}. \tag{6.38}$$

Multiplying equation (6.36) with 2π leads to

$$\frac{4\pi}{2l+1}\widetilde{W}_{\boldsymbol{t_\epsilon} l} = 4\pi\widetilde{W}_{\boldsymbol{t_\epsilon} 0} - (4\pi)\frac{l(l+1)}{2}\widetilde{W}_{\boldsymbol{t_\epsilon} 0} + \frac{4\pi}{3}\frac{l(l+1)}{2}\widetilde{W}_{\boldsymbol{t_\epsilon} 1}. \tag{6.39}$$

I.e.

$$\begin{aligned} \frac{4\pi}{2l+1}\widetilde{W}_{\boldsymbol{t_\epsilon} l} - \mathbf{1} = \frac{l(l+1)}{2}(\frac{4\pi}{3}\widetilde{W}_{\boldsymbol{t_\epsilon} 1} - 1) = -\frac{l(l+1)}{2}\boldsymbol{\zeta} + O(\epsilon^2) = \boldsymbol{\Upsilon_l} + O(\epsilon^2) \\ \boldsymbol{\Upsilon_l} = -\frac{l(l+1)}{2}\boldsymbol{\zeta} \end{aligned} \tag{6.40}$$

$\Longrightarrow$

$$\boxed{\frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\nabla}}\overline{f} = -\frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\frac{l(l+1)}{2}\overline{f}_{l,m}(\vec{\mathbf{x}},t)P_{l,m}(\vec{\boldsymbol{\Omega}})} \tag{6.41}$$

This equation only contains the unknown coefficients t_E and $\overline{v}$ principally depending upon the space-time-point $(\vec{\mathbf{x}}, t)$ and the fluctuation direction $\vec{\boldsymbol{\Omega}}$

$$\begin{aligned} t_E =& t_E(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) \\ \overline{v} =& \overline{v}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}). \end{aligned} \tag{6.42}$$

The total derivation of $\overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}})$ with respect to t in direction of $\vec{\boldsymbol{\Omega}}$ leads to

$$\frac{d}{dt}\overline{f}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\Omega}}) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l}\frac{d}{dt}\overline{f}_{lm}(\mathbf{x}, t)P_{lm}(\vec{\boldsymbol{\Omega}}) = \frac{1}{t_E}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}\overline{f}_{lm}(\vec{\mathbf{x}}, t)P_{lm}(\vec{\boldsymbol{\Omega}}). \tag{6.43}$$

The time behavior of the spherical harmonic components is described by the equations

$$\frac{d}{dt}\overline{f}_{lm}(t) = \frac{\gamma_l}{t_E}\overline{f}_{lm} \tag{6.44}$$

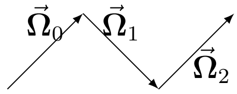
and result in

$$\overline{f}_{lm}(t) \sim \mathbf{exp}(\frac{\gamma_l}{t_E} \cdot t). \tag{6.45}$$

The greater the order l the more powerful is its temporal decay.

6.5. Reconstruction of the transition probabilities $\overline{W}_{\boldsymbol{t_\epsilon}}$

The Transition probability $\widetilde{W}_{\boldsymbol{t_\epsilon} 0\rightarrow 1\rightarrow 2}$, changing the movement direction $\vec{\boldsymbol{\Omega}}$ at the times t_0, t_1, t_2 from $\vec{\boldsymbol{\Omega}}_0$ via $\vec{\boldsymbol{\Omega}}_1$ to $\vec{\boldsymbol{\Omega}}_2$



is the product of the single transition probabilities.

$$\widetilde{W}_{\boldsymbol{t_\epsilon},0\rightarrow 1\rightarrow 2} = \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_1) \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\boldsymbol{\Omega}}_1 \cdot \vec{\boldsymbol{\Omega}}_2). \tag{6.46}$$

On the other side

$$\widetilde{W}_{\boldsymbol{t_\epsilon} l} = (1 + \Upsilon_l \boldsymbol{\epsilon})\frac{2l+1}{4\pi} + O(\boldsymbol{\epsilon}^2) \tag{6.47}$$

holds and thus arises

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') = \sum_{l=0}^{+\infty}(1 + \Upsilon_l \boldsymbol{\epsilon})\frac{2l+1}{4\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}})P_{lm}(\vec{\boldsymbol{\Omega}}') + O(\boldsymbol{\epsilon}^2). \tag{6.48}$$

The probability, that a particle changes the direction after an infinitesimal time interval $\epsilon \cdot t_E$ from $\vec{\boldsymbol{\Omega}}_0$ to $\vec{\boldsymbol{\Omega}}_2$ is given by

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_2) = \int_{4\pi} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_1) \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\boldsymbol{\Omega}}_1 \cdot \vec{\boldsymbol{\Omega}}_2) d\vec{\boldsymbol{\Omega}}_1 \tag{6.49}$$

and

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_2) = \sum_{l=0}^{+\infty}\left(1 + \Upsilon_l \frac{\boldsymbol{\epsilon}}{\boldsymbol{2}}\right)^2 \frac{2l+1}{4\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}}_0)P_{lm}(\vec{\boldsymbol{\Omega}}_2) + O(\boldsymbol{\epsilon}^2). \tag{6.50}$$

Using n intermediate steps $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ is expressed by an integral over the product of the single transition probabilities.

$$\widetilde{W}_{\boldsymbol{t_\epsilon},0\to 1\ldots\to n} = \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_1) \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_1 \cdot \vec{\boldsymbol{\Omega}}_2) \ldots . \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_{n-1} \cdot \vec{\boldsymbol{\Omega}}_n) \tag{6.51}$$

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_n) = \int_{4\pi}\int_{4\pi} \ldots . \int_{4\pi} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}} \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}} \ldots . \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}} d\vec{\boldsymbol{\Omega}}_1 \ldots . d\vec{\boldsymbol{\Omega}}_{n-1} \tag{6.52}$$

For $n \to \infty$ this results in:

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') = \lim_{\boldsymbol{n}\to\infty} \sum_{l=0}^{+\infty} \left\{ 1 + \frac{\epsilon \Upsilon_l}{n} \right\}^n \frac{2l+1}{4\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}}) P_{lm}(\vec{\boldsymbol{\Omega}}') + O(\boldsymbol{\epsilon}^2) \tag{6.53}$$

and finally

$$\boxed{\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}') = \sum_{l=0}^{+\infty} e^{\Upsilon_l \cdot \epsilon} \, \frac{2l+1}{4\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}}) P_{lm}(\vec{\boldsymbol{\Omega}}') + O(\boldsymbol{\epsilon}^2)}. \tag{6.54}$$

Selecting $\boldsymbol{\varepsilon} = \frac{\boldsymbol{t_\varepsilon}}{\boldsymbol{t_E(\vec{x},t,\vec{\Omega})}}$ the exchange function $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ may be understood in the dependencies

$$\widetilde{W}_{\boldsymbol{t_\epsilon}} = \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}'). \tag{6.55}$$

Therefore

$$\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \approx \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \tag{6.56}$$

is calculated, too.$\Longrightarrow$

$$\boxed{\begin{aligned} \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}, t) &= \int_{4\pi} \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Omega}}') \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \cdot \overline{v}' \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}', t - t_{\boldsymbol{\epsilon}}) d\vec{\boldsymbol{\Omega}}' \\ \overline{v}' &= \overline{v}'(\vec{\mathbf{x}}, \vec{\boldsymbol{\Omega}}', t) \end{aligned}} \tag{6.57}$$

The Transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ is unsymmetrical in the direction quantities on account of $\boldsymbol{\varepsilon} = \frac{\boldsymbol{t_\varepsilon}}{\boldsymbol{t_E(\vec{x},t,\vec{\Omega})}}$.

6.6. Approximation formula

In 1st approximation a telegrapher's equations is derived out of the linear Boltzmann Equation leading to the known diffusion equation without taking into account the second time derivation. In this case the diffusion equation is proved to be usefull. Subsequent considerations are displaying which relation exists between the 1st approximation of the particle transport by longitudinal continuum fluctuations and the known diffusion equation.

Assuming the simplification

$$\frac{1}{t_E} = \tau(\vec{\mathbf{x}}, \vec{\mathbf{\Omega}}, t) = \tau_0 = const \tag{6.58}$$

the transport equation described in 1st approximation is

$$\frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\mathbf{\Omega}} \cdot \nabla \overline{f} = -\frac{1}{t_E} \cdot \sum_{m=-1}^{+1} \overline{f}_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\mathbf{\Omega}}). \tag{6.59}$$

In cartesian coordinates one gets

$$\frac{\partial \overline{f}}{\partial t} + \overline{v}_x \mathbf{\Omega}_x \cdot \frac{\partial \overline{f}}{\partial x} + \overline{v}_y \mathbf{\Omega}_y \cdot \frac{\partial \overline{f}}{\partial y} + \overline{v}_z \mathbf{\Omega}_z \cdot \frac{\partial \overline{f}}{\partial z} = -\frac{1}{t_E} \cdot \sum_{m=-1}^{+1} \overline{f}_{1m}(\vec{\mathbf{x}}, t) P_{1m}(\vec{\mathbf{\Omega}}) \tag{6.60}$$

with

$$\overline{v}_x = \overline{v}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) \quad \overline{v}_y = \overline{v}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}) \quad \overline{v}_z = \overline{v}(\vec{\mathbf{x}}, t, \vec{\mathbf{\Omega}}). \tag{6.61}$$

Subsequently we confine us on

$$\overline{v}_x = \overline{v}_y = \overline{v}_z = \overline{v}(\vec{\mathbf{x}}), \tag{6.62}$$

suggesting an isotropy of fluctuation motions in the statistical ensemble. The conditions are selected such that the further derivations analogous to 5.4 follow until to a telegrapher's equation.

$$t_E \frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial \Phi}{\partial t} = \boldsymbol{t_E} \overline{v} \vec{\nabla} \cdot \frac{\overline{v}}{3} \vec{\nabla} \Phi \tag{6.63}$$

The usual diffusion coefficient normally contained in $\vec{\nabla} \cdot \boldsymbol{D} \vec{\nabla} \Phi$ cannot be found. In the equation above $\boldsymbol{D} = \boldsymbol{t_E} \cdot \frac{\overline{v}^2}{3}$ is contained partly outside partly between the $\vec{\nabla}-$operators. This has consequences in inhomogeneous media. Such problems arise unrecognized using the Bousinesque approach . I.e. in an inhomogenuous medium this approach may be fatal. The term of second derivation by time has nothing to do with relativistic theory. Because of the small size of t_E it may generally be neglected.

7. Stochastic transport by turbulent continuum-fluctuations

$$\frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\mathbf{\Omega}} \times \vec{\mathbf{\Theta}} \cdot \nabla \overline{f} = \frac{-1}{t_E} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \sum_{k=-\infty}^{+\infty} \frac{l(l+1)}{2} \overline{f}_{lmk}(\vec{\mathbf{x}}, t) P_{lm}(\vec{\mathbf{\Omega}}) H_k(\vec{\mathbf{\Theta}})$$

$$\Updownarrow$$

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}, t) = \int_{2\pi} \int_{4\pi} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}, \vec{\mathbf{\Omega}}', \vec{\mathbf{\Theta}}') \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}} - t_\varepsilon \cdot \overline{v}' \vec{\mathbf{\Omega}}' \times \vec{\mathbf{\Theta}}', \vec{\mathbf{\Omega}}', \vec{\mathbf{\Theta}}'), t - \boldsymbol{t_\epsilon}) d\vec{\mathbf{\Omega}}' d\vec{\mathbf{\Theta}}'$$

7.1. Introduction

The motion of passive particles by turbulent continuum fluctuations is examined. The particles are moved not affecting this field. Their trajectories correspond in every $\varepsilon-$neiborhood of a point to a circle segment passed with the velocity

$$\vec{\mathbf{v}}_{\boldsymbol{t_\epsilon}} = \vec{\boldsymbol{\omega}}_{\boldsymbol{t_\epsilon}} \times \vec{\boldsymbol{r}}_{\boldsymbol{t_\epsilon}}. \tag{7.1}$$

The considered motion quantities $\vec{\boldsymbol{\omega}}_{\boldsymbol{t_\epsilon}}$ and $\vec{\boldsymbol{r}}_{\boldsymbol{t_\epsilon}}$ are determined by successively detecting a single particle originating from a point $\vec{\boldsymbol{x}}_0$ after a time t_ε moving to $\vec{\boldsymbol{x}}_1$ and after a further time t_ε to $\vec{\boldsymbol{x}}_2$. By these 3 points a circle segment is uniquely defined for the point $\vec{\boldsymbol{x}}_1$ with radius vector $\vec{\boldsymbol{r}}_{t_\varepsilon}$ and a rotation speed $\vec{\boldsymbol{\omega}}_{t_\varepsilon}$.

$$\begin{aligned} \vec{\boldsymbol{r}}_{t_\varepsilon} &= \boldsymbol{r}_{t_\varepsilon} \cdot \vec{\mathbf{\Theta}}_{t_\varepsilon} \\ \vec{\boldsymbol{\omega}}_{t_\varepsilon} &= \boldsymbol{\omega}_{t_\varepsilon} \cdot \vec{\mathbf{\Omega}}_{t_\varepsilon} \end{aligned} \tag{7.2}$$

In the special case $\vec{\boldsymbol{\omega}}_{t_\varepsilon} \to \mathbf{0}$ and $\vec{\boldsymbol{r}} \to +\infty$ the velocity $\vec{\mathbf{v}}_{\boldsymbol{t_\epsilon}}$ is revealed out of its neighborhood.[1] The particle density distributions are received in a thought experiment by an unlimited number of deterministic ensemble-systems (see chapter 3). In every

[1] Applying the deterministic theory this problem must be treated numerically.

point $(\vec{x}, t)$ a continuously differentiable particle density distribution of the motion quantities $\vec{\omega}_{t_\epsilon}$ and $\vec{r}_{t_\epsilon}$ is assigned in accordance with

$$f_{t_\epsilon} = f_{t_\epsilon}(\vec{x}, t, \vec{\omega}, \vec{r}). \tag{7.3}$$

The with t_ϵ indexed functions are automatically assumed to contain motion quantities of corresponding measurement accuracies. The indexing of the motion quantities can be omitted if the functions are indexed. After execution of a limiting process for example

$$\lim_{t_\epsilon \to 0} f_{t_\epsilon}(\vec{x}, t, \vec{\omega}, \vec{r}) = f(\vec{x}, t, \vec{\omega}, \vec{r}) \tag{7.4}$$

f and $(\vec{\omega}, \vec{r})$ are understood according to an exact measuring process. Integrating the particle density distribution over the motion quantities one obtains expectation values of a particle density not conforming with the actual particle density ρ.

$$< \rho_{t_\epsilon}(\vec{x}, t) > = \int_{2\pi} \int_{4\pi} \int_0^\infty \int_0^\infty f_{t_\epsilon}(\vec{x}, t, \omega \cdot \vec{\Omega}, r \cdot \vec{\Theta}) d\omega dr d\vec{\Omega} d\vec{\Theta} \neq \rho_{t_\epsilon}(\vec{x}, t) \tag{7.5}$$

A strictly deduced partial differential equation is obtained calculating the development of spatio-temporal particle density distributions. The incipiently unlimited number of unknown coefficients is reduced to a local time-scaling related to the vortex calculation of an associated deterministic theory discussed in further chapters. The initially abstractly formulated transition probabilities get concrete functional dependencies. There are always found suitable Monte-Carlo methods treating them with the help of the deterministic theory described in further chapters.

7.2. The transport as Markov Process with natural causality

A particle at location $\vec{x}$ and time t changing its velocity from $\vec{v}' = (\vec{\omega}' \times \vec{r}')$ to $\vec{v} = (\vec{\omega} \times \vec{r})$ is given by the transition probability

$$W_{t_\epsilon} = W_{t_\epsilon}(\vec{x}, t; \vec{\omega}, \vec{r}; \vec{\omega}', \vec{r}') \tag{7.6}$$

with

$$\int_0^\infty \int_0^\infty \int_{4\pi} \int_{2\pi} W_{t_\epsilon}(\vec{x}, t; \vec{\omega}, \vec{r}; \vec{\omega}', \vec{r}') d\omega' dr' d\Omega' d\Theta' = 1. \tag{7.7}$$

$\Longrightarrow$

$$\boxed{\begin{array}{c} f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) = \\ \int\limits_0^\infty \int\limits_0^\infty \int\limits_{4\pi} \int\limits_{2\pi} W_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}') f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}}' \times \vec{\boldsymbol{r}}' \cdot t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}', t - t_{\boldsymbol{\epsilon}}) d\boldsymbol{\omega}' dr' d\boldsymbol{\Omega}' d\boldsymbol{\Theta}' \end{array}} \tag{7.8}$$

Continuity is required respectively of all variables of the transition probability $W_{\boldsymbol{t_\epsilon}}$. The sequence of velocities $\vec{\boldsymbol{v}}'_{t_\epsilon}, \vec{\boldsymbol{v}}_{t_\epsilon}$ means a motion from

$$(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}}'_{t_\epsilon} \times \vec{\boldsymbol{r}}'_{t_\epsilon} \cdot t_{\boldsymbol{\epsilon}}, t - t_\epsilon, \vec{\boldsymbol{\omega}}'_{t_\epsilon} \times \vec{\boldsymbol{r}}'_{t_\epsilon}) \quad \text{to} \quad (\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}_{t_\epsilon} \times \vec{\boldsymbol{r}}_{t_\epsilon}). \tag{7.9}$$

For the limiting process $t_\varepsilon \to 0$ the transition probabilities W_{t_ϵ} prove to be physical realizations of test functions of the distribution theory.

$$\lim_{\boldsymbol{t_\epsilon} \to 0} W_{\boldsymbol{t_\epsilon}} = \delta(\vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}; \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}'). \tag{7.10}$$

The passive scalar particles precisely reproduce the motions of the fluctuation field. For the particle density distribution $f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}})$ the following separation aproach is used without loss of generality:

$$f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \cdot t_{\boldsymbol{\epsilon}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) = G_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \cdot t_{\boldsymbol{\epsilon}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \overline{v} \vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot t_{\boldsymbol{\epsilon}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \tag{7.11}$$

with

$$\begin{aligned} &\int_0^\infty \int_0^\infty G_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \boldsymbol{\omega} \vec{\boldsymbol{\Omega}}, r \vec{\boldsymbol{\Theta}}) d\omega dr = 1 \\ &\int_0^\infty \int_0^\infty G_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \boldsymbol{\omega} \vec{\boldsymbol{\Omega}}, r \vec{\boldsymbol{\Theta}}) \omega r d\omega dr = \overline{v}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \\ &\overline{v}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \overline{\omega}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \cdot \overline{r}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \end{aligned} \tag{7.12}$$

$\Longrightarrow$

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \overline{v} \vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot t_\epsilon, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \int_0^\infty \int_0^\infty f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \cdot t_{\boldsymbol{\epsilon}}, t, \boldsymbol{\omega} \cdot \vec{\boldsymbol{\Omega}}, \boldsymbol{r} \cdot \vec{\boldsymbol{\Theta}}) d\omega dr \tag{7.13}$$

One obtains a transition probability $\overline{W}_{\boldsymbol{t_\epsilon}}$ only depending on the directions by integrating $W_{\boldsymbol{t_\epsilon}}$ over the amounts ω', r', ω, r.

$$\boxed{\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') = \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty W_{\boldsymbol{t_\epsilon}} G_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}}' \times \vec{\boldsymbol{r}}' \cdot t_\epsilon, t - t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}') d\omega' dr' d\omega dr} \tag{7.14}$$

The integration

$$\int_0^\infty \int_0^\infty (7.8) d\omega dr \tag{7.15}$$

gives

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \int\limits_0^\infty \int\limits_{4\pi} \int\limits_{2\pi} W_{\boldsymbol{t_\epsilon}} f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\omega}}' \times \vec{\boldsymbol{r}}' \cdot t_\epsilon, t - t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}') d\omega' dr' d\omega dr d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}' \tag{7.16}$$

$$\Longrightarrow \boxed{\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) = \int\limits_{4\pi} \int\limits_{2\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \overline{v}' \vec{\boldsymbol{\Omega}}' \times \vec{\boldsymbol{\Theta}}' \cdot t_\epsilon, t - t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}'} \tag{7.17}$$

In the integrand $\overline{f}_{\boldsymbol{t_\epsilon}}$ is developed around $\vec{\boldsymbol{x}}$ and t:

$$\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_{\boldsymbol{\epsilon}}, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') = \overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') - \tau_E \cdot \epsilon \cdot [\frac{\partial \overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t} + \overline{v}' \vec{\boldsymbol{\Omega}}' \times \vec{\boldsymbol{\Theta}}' \cdot \nabla \overline{f}'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] \tag{7.18}$$

This leads to

$$\frac{\int_{4\pi} \int_{2\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon} =$$
$$\int\limits_{4\pi} \int\limits_{2\pi} \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}', \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}) \cdot \tau_E [\cdot \frac{\partial \overline{f}'_{\boldsymbol{t_\epsilon}}}{\partial t} + \overline{v}' \vec{\boldsymbol{\Omega}}' \times \vec{\boldsymbol{\Theta}}' \cdot t_{\boldsymbol{\epsilon}} \cdot \nabla \overline{f}'_{\boldsymbol{t_\epsilon}} + O(\epsilon^2)] d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}'. \tag{7.19}$$

As

$$\lim_{\boldsymbol{t_\epsilon} \to 0} \overline{W}_{\boldsymbol{t_\epsilon}} = \delta(\vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}; \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') \tag{7.20}$$

$\Longrightarrow$

$$\lim_{\boldsymbol{\epsilon} \to 0} \frac{\int_{4\pi} \int_{2\pi} \overline{W}_{\boldsymbol{t_\epsilon}} \overline{f}'_{\boldsymbol{t_\epsilon}} d\vec{\boldsymbol{\Omega}}' d\vec{\boldsymbol{\Theta}}' - \overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon \cdot \tau_E} = \frac{\partial \overline{f}}{\partial t} + \overline{v} \vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot \nabla \overline{f} \ . \tag{7.21}$$

Furtheron

$$\boxed{\lim_{\epsilon\to 0}\frac{\int_{4\pi}\int_{2\pi}\overline{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'-\overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon\cdot\tau_E}} \tag{7.22}$$

is called exchange-term.

7.3. Calculation of the exchange-term

Exchange term dependencies of scalar products $\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}'$ and $\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}'$ are taken into account istead of individually depending directions $\vec{\boldsymbol{\Omega}}$, $\vec{\boldsymbol{\Omega}}'$ and $\vec{\boldsymbol{\Theta}}$, $\vec{\boldsymbol{\Theta}}$ demanding the following relation

$$\lim_{\epsilon\to 0}\frac{\int_{2\pi}\int_{4\pi}\overline{W}_{\boldsymbol{t_\epsilon}}\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'-\overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon\cdot\tau_E}=\lim_{\epsilon\to 0}\frac{\int_{2\pi}\int_{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}')\overline{f}'_{\boldsymbol{t_\epsilon}}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'-\overline{f}_{\boldsymbol{t_\epsilon}}}{\epsilon\cdot t_E}. \tag{7.23}$$

The following transitions

$$\begin{aligned}\tau_E = const \qquad &\longrightarrow \qquad t_E=t_E(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})\\ \overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}};\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}') \qquad &\longrightarrow \qquad \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}')\end{aligned} \tag{7.24}$$

are regarded. Moreover, a separation of $\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}'$ and $\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}'$ is asumed:

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}')=V_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}\cdot\vec{\boldsymbol{\Omega}}')\cdot M_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Theta}}\cdot\vec{\boldsymbol{\Theta}}'). \tag{7.25}$$

Functions of the unit vectors $\vec{\boldsymbol{\Omega}}$ and $\vec{\boldsymbol{\Theta}}$ are presented by a complete orthogonal function system representing an extension of the spherical harmonics called turbulence functions.

$$\begin{aligned}\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}}) &= \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\overline{f}_{\boldsymbol{t_\epsilon}lmk}(\vec{\boldsymbol{x}},t)Q_{lmk}(\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})\\ &= \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\overline{f}_{\boldsymbol{t_\epsilon}lmk}(\vec{\boldsymbol{x}},t)Q^*_{lmk}(\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})\end{aligned} \tag{7.26}$$

$$\int_{2\pi}\int_{4\pi}Q_{lmk}(\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})Q^*_{lmk}(\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}')d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'=\begin{cases}\frac{8\pi^2}{2l+1} & \text{for l = l' and m=m'}\\ 0 & \text{else}\end{cases} \tag{7.27}$$

with

$$
\begin{aligned}
Q_{lmk}(\vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}) =& P_{lm}(\vec{\mathbf{\Omega}}) H_k(\vec{\mathbf{\Theta}}) \\
\int_{2\pi} H_{k'}(\vec{\mathbf{\Theta}}) H_k^*(\vec{\mathbf{\Theta}}) d\vec{\mathbf{\Theta}} =& \begin{cases} 2\pi & \text{for k'=k} \\ 0 & \text{else} \end{cases} \\
H_k(\vec{\mathbf{\Theta}}) =& e^{ik\theta}
\end{aligned}
\tag{7.28}
$$

The product $\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}'$ in the separated exchange function $V_{\boldsymbol{t_\epsilon}}$ is developed by spherical harmonics.

$$
\begin{aligned}
V_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}' \cdot \vec{\mathbf{\Omega}}) = \sum_{l=0}^{+\infty} V_{\boldsymbol{t_\epsilon} l} P_l(cos(\alpha)) =& \sum_{l=0}^{+\infty} V_{\boldsymbol{t_\epsilon} l} \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}') P_{lm}^*(\vec{\mathbf{\Omega}}) \\
&\text{mit} \\
\lim_{\boldsymbol{t\epsilon} \to 0} V_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}' \cdot \vec{\mathbf{\Omega}}) =& \delta(\vec{\mathbf{\Omega}}, \vec{\mathbf{\Omega}}')
\end{aligned}
\tag{7.29}
$$

The product $\vec{\mathbf{\Theta}} \cdot \vec{\mathbf{\Theta}}'$ in the separated exchange function $M_{\boldsymbol{t_\epsilon}}$ is developed by functions H_k.

$$
M_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Theta}}' \cdot \vec{\mathbf{\Theta}}) = \sum_{k=0}^{+\infty} M_{\boldsymbol{t_\epsilon} k} cos(k\beta) = \frac{1}{2} \sum_{k=0}^{k=+\infty} M_{\boldsymbol{t_\epsilon} k} [H_k(\vec{\mathbf{\Theta}}') H_k^*(\vec{\mathbf{\Theta}}) + H_{-k}(\vec{\mathbf{\Theta}}') H_{-k}^*(\vec{\mathbf{\Theta}})]
\tag{7.30}
$$

with

$$
\begin{aligned}
cos(k\beta) =& \frac{1}{2}[H_k(\vec{\mathbf{\Theta}}') H_k^*(\vec{\mathbf{\Theta}}) + H_{-k}(\vec{\mathbf{\Theta}}') H_{-k}^*(\vec{\mathbf{\Theta}})] = \frac{1}{2}[e^{ik(\theta'-\theta)} + e^{-ik(\theta'-\theta)}] \\
\vec{\mathbf{\Theta}}' \cdot \vec{\mathbf{\Theta}} = cos(\beta) = cos(\theta' - \theta) =& \frac{1}{2}[H_1(\vec{\mathbf{\Theta}}') H_1^*(\vec{\mathbf{\Theta}}) + H_{-1}(\vec{\mathbf{\Theta}}') H_{-1}^*(\vec{\mathbf{\Theta}})] = \frac{1}{2}[e^{i(\theta'-\theta)} + e^{-i(\theta'-\theta)}] \\
\lim_{t_\epsilon \to 0} M_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Theta}}' \cdot \vec{\mathbf{\Theta}}) =& \delta(\vec{\mathbf{\Theta}}, \vec{\mathbf{\Theta}}')
\end{aligned}
\tag{7.31}
$$

$\Longrightarrow$

$$\int_{4\pi}\int_{2\pi}\widetilde{W}_{\boldsymbol{t}_\epsilon}\overline{f}'_{\boldsymbol{t}_\epsilon}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}} = \int_{4\pi}\int_{2\pi}V_{\boldsymbol{t}_\epsilon}(\vec{\boldsymbol{\Omega}}'\cdot\vec{\boldsymbol{\Omega}})\cdot M_{\boldsymbol{t}_\epsilon}(\vec{\boldsymbol{\Theta}}'\cdot\vec{\boldsymbol{\Theta}})\overline{f}'_{\boldsymbol{t}_\epsilon}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'$$
$$= \int_{4\pi}\int_{2\pi}\Bigg[\{\sum_{l=0}^{+\infty}V_{\boldsymbol{t}_\epsilon l}\sum_{m=-l}^{m=+l}P_{lm}(\vec{\boldsymbol{\Omega}}')P^*_{lm}(\vec{\boldsymbol{\Omega}})\cdot\frac{1}{2}\sum_{k=0}^{k=+\infty}M_{\boldsymbol{t}_\epsilon k}[H_k(\vec{\boldsymbol{\Theta}}')H^*_k(\vec{\boldsymbol{\Theta}})+H_{-k}(\vec{\boldsymbol{\Theta}}')H^*_{-k}(\vec{\boldsymbol{\Theta}})]\}$$
$$\cdot\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\overline{f}_{\boldsymbol{t}_\epsilon lmk}(\vec{\boldsymbol{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}}')H^*_k(\vec{\boldsymbol{\Theta}}')\Bigg]d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'$$
$$= \sum_{l=0}^{+\infty}V_{\boldsymbol{t}_\epsilon l}\frac{4\pi}{2l+1}\sum_{m=-l}^{+l}P^*_{lm}(\vec{\boldsymbol{\Omega}})\sum_{k=0}^{+\infty}M_{\boldsymbol{t}_\epsilon k}2\pi\overline{f}_{\boldsymbol{t}_\epsilon lmk}(\vec{\boldsymbol{x}},t)H^*_k(\vec{\boldsymbol{\Theta}}). \tag{7.32}$$

Finally the exchange term results in

$$\lim_{\boldsymbol{\epsilon}\to 0}\frac{\int_{4\pi}\int_{2\pi}\widetilde{W}_{\boldsymbol{t}_\epsilon}\overline{f}'_{\boldsymbol{t}_\epsilon}d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'-\overline{f}_{\boldsymbol{t}_\epsilon}}{\epsilon}$$
$$= \lim_{\boldsymbol{\epsilon}\to 0}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=0}^{+\infty}\frac{(V_{\boldsymbol{t}_\epsilon l}\frac{4\pi}{2l+1}M_{\boldsymbol{t}_\epsilon k}2\pi-1)}{\boldsymbol{\epsilon}}\overline{f}_{\boldsymbol{t}_\epsilon lmk}(\vec{\boldsymbol{x}},t)P^*_{lm}(\vec{\boldsymbol{\Omega}})H^*_k(\vec{\boldsymbol{\Theta}}) \tag{7.33}$$
$$= \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=0}^{+\infty}\Upsilon_{lk}\overline{f}_{lmk}(\vec{\boldsymbol{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})H_k(\vec{\boldsymbol{\Theta}}).$$

With the exchange coefficients

$$\Upsilon_{lk} = \lim_{\boldsymbol{\epsilon}\to 0}\frac{(V_{\boldsymbol{t}_\epsilon l}\frac{4\pi}{2l+1}M_{\boldsymbol{t}_\epsilon k}2\pi-1)}{\boldsymbol{\epsilon}} \tag{7.34}$$

the transport equation

$$\boxed{\frac{\partial\overline{f}}{\partial t}+\overline{v}\vec{\boldsymbol{\Omega}}\times\vec{\boldsymbol{\Theta}}\cdot\nabla\overline{f} = \frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\Upsilon_{lk}\overline{f}_{lmk}(\vec{\boldsymbol{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})H_k(\vec{\boldsymbol{\Theta}})} \tag{7.35}$$

is achieved. Further on it is shown that in Υ_{lk} the index k may be skipped.

7.4. Calculation of the exchange-coefficients Υ_l

Considering an overall closed volume range V the particle number in the entire volume remains constant if no absorbtion is assumed.

$$\text{total number of particles} = \int_{\mathbb{V}} \int_{4\pi} \int_{2\pi} \overline{f} d\vec{\boldsymbol{\Omega}} d\vec{\boldsymbol{\Theta}} d\mathbb{V} = const. \tag{7.36}$$

$\Longrightarrow$

$$\frac{d}{dt} \int_{\mathbb{V}} \int_{4\pi} \int_{2\pi} \overline{f} d\vec{\boldsymbol{\Omega}} d\vec{\boldsymbol{\Theta}} d\mathbb{V} = \int_{\mathbb{V}} \int_{4\pi} \int_{2\pi} [\frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{\Theta}} \cdot \nabla \overline{f}] d\vec{\boldsymbol{\Omega}} d\vec{\boldsymbol{\Theta}} d\mathbb{V} = \Upsilon_{0,0} \cdot \boldsymbol{V} = \boldsymbol{0} \tag{7.37}$$

and thus

$$\boxed{\Upsilon_{0,0} = 0}. \tag{7.38}$$

Getting an overview over the exchange function $M_{\boldsymbol{t}_\epsilon}$ the essential relations are presented again with the following equations:

$$M_{\boldsymbol{t}_\epsilon}(\vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}}) = \sum_{k=0}^{+\infty} M_{\boldsymbol{t}_\epsilon k} cos(k\beta) = \frac{1}{2} \sum_{k=0}^{k=+\infty} M_{\boldsymbol{t}_\epsilon k}[H_k(\vec{\boldsymbol{\Theta}}')H_k^*(\vec{\boldsymbol{\Theta}}) + H_{-k}(\vec{\boldsymbol{\Theta}}')H_{-k}^*(\vec{\boldsymbol{\Theta}})]$$
$$cos(k\beta) = \frac{1}{2}[H_k(\vec{\boldsymbol{\Theta}}')H_k^*(\vec{\boldsymbol{\Theta}}) + H_{-k}(\vec{\boldsymbol{\Theta}}')H_{-k}^*(\vec{\boldsymbol{\Theta}})] = \frac{1}{2}[e^{ik(\theta'-\theta)} + e^{-ik(\theta'-\theta)}]$$
$$\vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}} = cos(\beta) = cos(\theta' - \theta) = \frac{1}{2}[H_1(\vec{\boldsymbol{\Theta}}')H_1^*(\vec{\boldsymbol{\Theta}}) + H_{-1}(\vec{\boldsymbol{\Theta}}')H_{-1}^*(\vec{\boldsymbol{\Theta}})] = \frac{1}{2}[e^{i(\theta'-\theta)} + e^{-i(\theta'-\theta)}]$$
$$\lim_{\epsilon \to 0} M_{\boldsymbol{t}_\epsilon}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}}) = \delta(\vec{\boldsymbol{\Theta}}, \vec{\boldsymbol{\Theta}}')$$
$$\int_{2\pi} H_{k'}(\vec{\boldsymbol{\Theta}})H_k^*(\vec{\boldsymbol{\Theta}}) d\vec{\boldsymbol{\Theta}} = \begin{cases} 2\pi & \text{für k'=k} \\ 0 & \text{else} \end{cases}$$

$M_{\boldsymbol{t}_\epsilon}(\vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}}) = \sum_{k=0}^{+\infty} M_{\boldsymbol{t}_\epsilon k} cos(k\beta)$ only takes values essentially different from 0 in an ε-neighborhood of $\beta = 0$, such that $\vec{\boldsymbol{\Theta}}' \cdot \vec{\boldsymbol{\Theta}} = cos(\beta) = 1 - O(\varepsilon^2)$ is sufficient. $\Longrightarrow$

$$2\pi \cdot M_{\boldsymbol{t}_\epsilon k} = \int_{-\pi}^{+\pi} M_{\boldsymbol{t}_\epsilon} cos(k\beta) d\beta = \int_{-\pi}^{+\pi} M_{\boldsymbol{t}_\epsilon}(1 - O(\epsilon)) d\beta = 2\pi \cdot M_{\boldsymbol{t}_\epsilon 0} - O(\epsilon^2). \tag{7.39}$$

On the other hand

$$\int_{2\pi} M_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Theta}} \cdot \vec{\mathbf{\Theta}})' d\vec{\mathbf{\Theta}}' =$$
$$\frac{1}{2}\int_{2\pi} \sum_{k=0}^{k=+\infty} M_{\boldsymbol{t_\epsilon} k}[H_k(\vec{\mathbf{\Theta}}')H_k^*(\vec{\mathbf{\Theta}}) + H_{-k}(\vec{\mathbf{\Theta}}')H_{-k}^*(\vec{\mathbf{\Theta}})]d\vec{\mathbf{\Theta}}' = 2\pi \cdot M_{\boldsymbol{t_\epsilon}0} = 1. \tag{7.40}$$

is valid. $\Longrightarrow$

$$M_{\boldsymbol{t_\epsilon} k} = M_{\boldsymbol{t_\epsilon} 0} = \frac{1}{2\pi}. \tag{7.41}$$

The calculation of the exchange coefficients is not influenced by $M_{\boldsymbol{t_\epsilon}}$ the Υ-values given by

$$\Upsilon_l = \lim_{t_\epsilon \to 0} \frac{(V_{\boldsymbol{t_\epsilon} l}\frac{4\pi}{2l+1} - 1)}{\boldsymbol{t_\epsilon}}. \tag{7.42}$$

Further calculation of the Υ_l analogously happen to section 6.4 with the result

$$\Upsilon_l = -\frac{l(l+1)}{2}\zeta \qquad \zeta = const. \tag{7.43}$$

Now the equation of turbulent particle transport is written

$$\boxed{\frac{\partial \overline{f}}{\partial t} + \overline{v}\vec{\mathbf{\Omega}} \times \vec{\mathbf{\Theta}} \cdot \nabla \overline{f} = -\frac{1}{t_E}\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty} \frac{l(l+1)}{2}\overline{f}_{lmk}(\vec{\boldsymbol{x}}, t)P_{lm}(\vec{\mathbf{\Omega}})H_k(\vec{\mathbf{\Theta}})} \tag{7.44}$$

the coefficient $\frac{\zeta}{t_E}$ replaced by $\frac{1}{t_E}$. A more complicated dependency of $t_E = t_E(\vec{\boldsymbol{x}}, t, \vec{\mathbf{\Omega}})$ possibly remains. Maybe, physically justified simplifications lead to practical solutions. The below presented theory of deterministic turbulence enables the calcultion of these coefficients by numerical evaluation.

Die total derivative with respect to time gives

$$\begin{aligned}\frac{d}{dt}\overline{f}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}}) &= \sum_{l=0}^{+\infty}\sum_{m=-l}^{m=+l}\sum_{k=-\infty}^{+\infty}\frac{d}{dt}\overline{f}_{lmk}(\vec{\boldsymbol{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})H_k(\vec{\boldsymbol{\Theta}}) \\ &= \frac{1}{t_E}\sum_{l=1}^{+\infty}\boldsymbol{\gamma_l}\cdot\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty}\overline{f}_{lmk}(\vec{\boldsymbol{x}},t)P_{lm}(\vec{\boldsymbol{\Omega}})H_k(\vec{\boldsymbol{\Theta}}).\end{aligned} \tag{7.45}$$

The time behavior of the single modes are obtained by

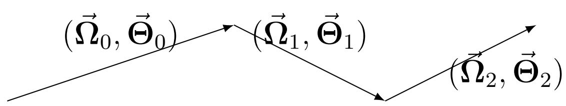
$$\frac{d}{dt}\overline{f}_{lmk}(t) = \frac{\gamma_l}{t_E}\overline{f}_{lmk} \tag{7.46}$$

$$\overline{f}_{lmk}(t) \sim \boldsymbol{exp}(\frac{\gamma_l}{t_E}\cdot t). \tag{7.47}$$

The greater the order l the more powerful is its temporal decay.

7.5. Reconstruction of the transition probabilities $\overline{W}_{\boldsymbol{t_\epsilon}}$

The transition probability $\widetilde{W}_{\boldsymbol{t_\epsilon},0\to1\to2}$, a particle changing its motion pair of directions $(\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}})$ at the times t_0,t_1,t_2 from $(\vec{\boldsymbol{\Omega}}_0,\vec{\boldsymbol{\Theta}}_0)$ via $(\vec{\boldsymbol{\Omega}}_1,\vec{\boldsymbol{\Theta}}_1)$ to $(\vec{\boldsymbol{\Omega}}_2,\vec{\boldsymbol{\Theta}}_2)$,



results out of the product of the single probabilities of the pairs of directions (vortex vector and radius vector direction of motion in a circle segment). The grafical presentation is meant symbolically because such a pair of directions does not compose to an overall direction. $\vec{\boldsymbol{\Omega}}_i$ is always orthogonal to $\vec{\boldsymbol{\Theta}}_i$. A vectorial overall direction of $\vec{\boldsymbol{\Omega}}_i$ and $\vec{\boldsymbol{\Theta}}_i$ has no physical meaning in the 3 dimensional space. [2]

$$\widetilde{W}_{\boldsymbol{t_\epsilon},0\to1\to2} = \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\boldsymbol{\Omega}}_0\cdot\vec{\boldsymbol{\Omega}}_1,\vec{\boldsymbol{\Theta}}_0\cdot\vec{\boldsymbol{\Theta}}_1)\cdot\widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\boldsymbol{\Omega}}_1\cdot\vec{\boldsymbol{\Omega}}_2,\vec{\boldsymbol{\Theta}}_1\cdot\vec{\boldsymbol{\Theta}}_2) \tag{7.48}$$

[2] $\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}}$ would make a single direction vector in a 4-dimensional space. The longitudinal fluctuations in the 4-dimensional space should accord to turbulence in the 3-dimensional space.

The probability , that a particle changes its pair of directions within a time $t_\varepsilon = \varepsilon \cdot t_E$ from $(\vec{\mathbf{\Omega}}_0, \vec{\mathbf{\Theta}}_0)$ to $(\vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_2)$, is obtained by

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_0 \cdot \vec{\mathbf{\Theta}}_2) = \int_{2\pi} \int_{4\pi} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_1, \vec{\mathbf{\Theta}}_0 \cdot \vec{\mathbf{\Theta}}_1) \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_1 \cdot \vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_1 \cdot \vec{\mathbf{\Theta}}_2) d\vec{\mathbf{\Omega}}_1 d\vec{\mathbf{\Theta}}_1. \tag{7.49}$$

The evolution coefficients of the transition probability are

$$\widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{\boldsymbol{2}} l} = \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \tag{7.50}$$

and therefore

$$\begin{aligned} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_1, \vec{\mathbf{\Theta}}_0 \cdot \vec{\mathbf{\Theta}}_1) = & \sum_{l=0}^{+\infty} \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \sum_{m=-l}^{+l} P_{lm}(\vec{\mathbf{\Omega}}_1) P^*_{lm}(\vec{\mathbf{\Omega}}_0) \\ & \cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} [H_k(\vec{\mathbf{\Theta}}_1) H^*_k(\vec{\mathbf{\Theta}}_0) + H_{-k}(\vec{\mathbf{\Theta}}_1) H^*_{-k}(\vec{\mathbf{\Theta}}_0)]. \end{aligned} \tag{7.51}$$

respectively

$$\begin{aligned} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{2}}(\vec{\mathbf{\Omega}}_1 \cdot \vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_1 \cdot \vec{\mathbf{\Theta}}_2) = & \sum_{l=0}^{+\infty} \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \sum_{m=-l}^{+l} P_{lm}(\vec{\mathbf{\Omega}}_2) P^*_{lm}(\vec{\mathbf{\Omega}}_1) \\ & \cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} [H_k(\vec{\mathbf{\Theta}}_2) H^*_k(\vec{\mathbf{\Theta}}_1) + H_{-k}(\vec{\mathbf{\Theta}}_2) H^*_{-k}(\vec{\mathbf{\Theta}}_1)]. \end{aligned} \tag{7.52}$$

Integrating (7.49) one obtains

$$\begin{aligned} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{\Omega}}_0 \cdot \vec{\mathbf{\Omega}}_2, \vec{\mathbf{\Theta}}_0 \cdot \vec{\mathbf{\Theta}}_2) = & \sum_{l=0}^{+\infty} \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\}^2 \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \sum_{m=-l}^{+l} P^*_{lm}(\vec{\mathbf{\Omega}}_0) P_{lm}(\vec{\mathbf{\Omega}}_2) \\ & \cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} [H_k(\vec{\mathbf{\Theta}}_0) H^*_k(\vec{\mathbf{\Theta}}_2) + H_{-k}(\vec{\mathbf{\Theta}}_0) H^*_{-k}(\vec{\mathbf{\Theta}}_2)]. \end{aligned} \tag{7.53}$$

Using n intermediate stages $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ is expressed by an integral over the product of the single transition probabilities.

$$\widetilde{W}_{\boldsymbol{t_\epsilon},0\to 1\ldots\to n} = \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_1, \vec{\boldsymbol{\Theta}}_0 \cdot \vec{\boldsymbol{\Theta}}_1) \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_1 \cdot \vec{\boldsymbol{\Omega}}_2, \vec{\boldsymbol{\Theta}}_1 \cdot \vec{\boldsymbol{\Theta}}_2) \ldots . \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}}(\vec{\boldsymbol{\Omega}}_{n-1} \cdot \vec{\boldsymbol{\Omega}}_n, \vec{\boldsymbol{\Theta}}_{n-1} \cdot \vec{\boldsymbol{\Theta}}_n) \tag{7.54}$$

$$\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}}_0 \cdot \vec{\boldsymbol{\Omega}}_n, \vec{\boldsymbol{\Theta}}_0 \cdot \vec{\boldsymbol{\Theta}}_n) = \int_{2\pi}\int_{4\pi}\int_{2\pi}\int_{4\pi} \ldots . \int_{2\pi}\int_{4\pi} \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}} \cdot \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}} \ldots . \widetilde{W}_{\frac{\boldsymbol{t_\epsilon}}{n}} d\vec{\boldsymbol{\Omega}}_1 d\vec{\boldsymbol{\Theta}}_1 \ldots . d\vec{\boldsymbol{\Omega}}_{n-1} d\vec{\boldsymbol{\Theta}}_{n-1} \tag{7.55}$$

$$\begin{aligned} \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}} \cdot \vec{\boldsymbol{\Theta}}') &= \lim_{\boldsymbol{n}\to\infty} \sum_{l=0}^{+\infty} \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\}^n \frac{2l+1}{4\pi} \\ \cdot \frac{1}{2\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}}) P_{lm}(\vec{\boldsymbol{\Omega}}') &\cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} [H_k(\vec{\boldsymbol{\Theta}}') H^*_k(\vec{\boldsymbol{\Theta}}) + H_{-k}(\vec{\boldsymbol{\Theta}}') H^*_{-k}(\vec{\boldsymbol{\Theta}})] \end{aligned} \tag{7.56}$$

For $n \to \infty$ arises

$$\lim_{\boldsymbol{n}\to\infty} \left\{1 + \frac{\epsilon \cdot \Upsilon_l}{2}\right\}^n = e^{\Upsilon_l \cdot \epsilon} \tag{7.57}$$

and using (7.55)

$\Longrightarrow$

$$\boxed{\begin{aligned} &\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}} \cdot \vec{\boldsymbol{\Theta}}') = \\ &\sum_{l=0}^{+\infty} e^{\Upsilon_l \cdot \epsilon} \frac{2l+1}{4\pi} \cdot \frac{1}{2\pi} \cdot \sum_{m=-l}^{+l} P^*_{lm}(\vec{\boldsymbol{\Omega}}) P_{lm}(\vec{\boldsymbol{\Omega}}') \cdot \frac{1}{2} \sum_{k=-\infty}^{+\infty} [H_k(\vec{\boldsymbol{\Theta}}') H^*_k(\vec{\boldsymbol{\Theta}}) + H_{-k}(\vec{\boldsymbol{\Theta}}') H^*_{-k}(\vec{\boldsymbol{\Theta}})] \end{aligned}} . \tag{7.58}$$

Choosing $\boldsymbol{\varepsilon} = \frac{\boldsymbol{t_\varepsilon}}{\boldsymbol{t_E(\vec{x},t,\vec{\Omega})}}$ the exchange function $\widetilde{W}_{\boldsymbol{t_\epsilon}}$ may be understood in the dependencies

$$\widetilde{W}_{\boldsymbol{t_\epsilon}} = \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}} \cdot \vec{\boldsymbol{\Theta}}') \tag{7.59}$$

and

$$\overline{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}}, \vec{\boldsymbol{\Theta}}, \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}}') \approx \widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\Omega}} \cdot \vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Theta}} \cdot \vec{\boldsymbol{\Theta}}') \tag{7.60}$$

is given, too. $\Longrightarrow$

$$\boxed{\begin{array}{c}\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}},t)=\displaystyle\int_{2\pi}\int_{4\pi}\widetilde{W}_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\boldsymbol{\Omega}},\vec{\boldsymbol{\Theta}},\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}')\overline{f}_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}-t_\varepsilon\cdot\overline{v}'\vec{\boldsymbol{\Omega}}'\times\vec{\boldsymbol{\Theta}}',\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}'),t-t_{\boldsymbol{\epsilon}})d\vec{\boldsymbol{\Omega}}'d\vec{\boldsymbol{\Theta}}'\\ \overline{v}'=\overline{v}'(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}',t)=\overline{\omega}'(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}',t)\cdot\overline{r}'(\vec{\mathbf{x}},\vec{\boldsymbol{\Omega}}',\vec{\boldsymbol{\Theta}}',t)\end{array}} \tag{7.61}$$

8. Appendix

8.1. Legendre-Polynomials

The Legendre-polynomials are defined within the interval $[-1, +1]$ by

$$P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, n \in N. \tag{8.1}$$

They represent a complete orthogonal function system with

$$\int_{-1}^{+1} P_n(x) P_m(x) dx = \begin{cases} \frac{2}{2m+1} & \text{for m} = n \\ 0 & \text{else.} \end{cases} \tag{8.2}$$

Every continuously differentiable function f(x) defined within [-1,+1] can be developed by Legendre-polynomials according to

$$f(x) = \sum_{l=0}^{\infty} f_l P_l(x). \tag{8.3}$$

The f_l are the evolution coefficients. A presentation of the $\delta - function$ by Legendre-polynomials is obtained by

$$\delta(x, x') = \sum_{l=0}^{\infty} \frac{2m+1}{2} P_l(x) P_l(x') \; . \tag{8.4}$$

Important recurrence equations are

$$\begin{aligned} (n+1) P_{n+1} &= (2n+1) x P_n(x) - n P_{n-1}(x) \\ P'_{n+1}(x) - x P'_n(x) &= (n+1) P_n(x) \; , n = 0, 1, 2, ... \\ (1 - x^2) P'_n(x) &= n P_{n-1}(x) - n x P_n(x). \end{aligned} \tag{8.5}$$

An integral representation of the Legendre-polynomials is obtained by

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}(x+\sqrt{x^2-1}cos(\varphi))^n d\varphi. \tag{8.6}$$

Owing to $|x+\sqrt{x^2-1}cos(\theta)| = |cos(\theta)+isin(\theta)cos(\theta)| \leqslant 1$

$$|P_n(x)| \leqslant 1 \tag{8.7}$$

follows. These polynomials have their maximum for $x = 1$, particularly

$$P_n(1) = 1. \tag{8.8}$$

$$\boxed{\frac{dP_l(x)}{dx}|_1 = \frac{l(l+1)}{2}} \tag{8.9}$$

is proved by complete induction.

Proof :

1.$P_0'(1) = 0$

Assumption:
2.$P_n'(1) = \frac{n(n+1)}{2}$

$\Longrightarrow$

3.$P_{n+1}'(1) = \frac{(n+2)(n+1)}{2}$ wegen (8.5) $P_{n+1}'(1) - P_n'(1) = (n+1)P_n(1)$
q.e.d.

8.2. Spherical harmonics

The Spherical harmonics [[15] page 224] represent a complete orthogonal, complex function system on the spherical surface

$$\begin{aligned} P_{lm}(\vec{\mathbf{\Omega}}) =& e^{im\varphi}\frac{(-sin(\vartheta))^m}{l!2^l}\cdot\left(\frac{(l-m)!}{(l+m)!}\right)^{\frac{1}{2}}\frac{d^{l+m}(cos^2\vartheta-1)^l}{(dcos\vartheta)^{l+m}} \\ =& e^{im\varphi}\frac{(sin(\vartheta))^{-m}}{l!2^l}\cdot\left(\frac{(l+m)!}{(l-m)!}\right)^{\frac{1}{2}}\frac{d^{l-m}(cos^2\vartheta-1)^l}{(dcos\vartheta)^{l-m}} \end{aligned} \tag{8.10}$$

with

$$P_{l,-m}(\vec{\mathbf{\Omega}}) = (-)^m P^*_{lm}(\vec{\mathbf{\Omega}}) \tag{8.11}$$

and

$$\int_{4\pi} d\vec{\mathbf{\Omega}} P_{l'm'}(\vec{\mathbf{\Omega}}) P^*_{lm}(\vec{\mathbf{\Omega}}) = \delta_{l'l}\delta_{m'm}\frac{4\pi}{2l+1}\ . \tag{8.12}$$

All continuously differentiable functions on the spherical surface $f(\Omega) = f(\theta, \phi)$ can be developed according to

$$f(\vec{\mathbf{\Omega}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l} f_{lm}P_{lm}(\vec{\mathbf{\Omega}}) \tag{8.13}$$

the f_{lm} representing the evolution coefficients. The $P^*_{lm}(\vec{\mathbf{\Omega}})$ being complex to $P_{lm}(\vec{\mathbf{\Omega}})$ $f(\vec{\mathbf{\Omega}})$ can be alternatively considered

$$f(\vec{\mathbf{\Omega}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=+l} f_{lm}P^*_{lm}(\vec{\mathbf{\Omega}}). \tag{8.14}$$

The spherical harmonics for $l = 0, 1$ are

$$\begin{aligned} P_{00} =& P^*_{00} = 1 \\ P_{1,-1}(\vec{\mathbf{\Omega}}) =& 2^{-\frac{1}{2}}e^{-i\varphi}sin(\vartheta),\ P^*_{1,-1} = 2^{-\frac{1}{2}}e^{i\varphi}sin(\vartheta) \\ P_{1,0}(\vec{\mathbf{\Omega}}) =& P^*_{1,0}(\vec{\mathbf{\Omega}}) = cos(\vartheta) = P_1(\vec{\mathbf{\Omega}}) \\ P_{1,1}(\vec{\mathbf{\Omega}}) =& -2^{-\frac{1}{2}}e^{i\varphi}sin(\vartheta),\ P^*_{1,1}(\vec{\mathbf{\Omega}}) = -2^{-\frac{1}{2}}e^{-i\varphi}sin(\vartheta)\ . \end{aligned} \tag{8.15}$$

The connection of spherical harmonics and Legendre-polynomials is obtained by

$$P_{l0} = P_{l0}^{*} = P_l. \tag{8.16}$$

Furthermore the addition theorem

$$P_l(cos(\vartheta)) = \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}}^{'})P_{lm}^{*}(\vec{\mathbf{\Omega}}) \tag{8.17}$$

matters with

$$cos(\vartheta) = \vec{\mathbf{\Omega}}^{'} \cdot \vec{\mathbf{\Omega}}. \tag{8.18}$$

The $\delta-$function depending on the spherical harmonics may be stated by

$$\delta(\vec{\mathbf{\Omega}}, \vec{\mathbf{\Omega}}^{'}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}})P_{lm}^{*}(\vec{\mathbf{\Omega}}^{'}) \tag{8.19}$$

and

$$\delta(\vec{\mathbf{\Omega}}, \vec{\mathbf{\Omega}}^{'}) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\vec{\mathbf{\Omega}} \cdot \vec{\mathbf{\Omega}}^{'}). \tag{8.20}$$

8.3. Turbulence-functions

Functions of the unit direction vectors $\vec{\mathbf{\Omega}}$ and $\vec{\mathbf{\Theta}}$ are represented by a complete orthogonal function system meaning an extension of the spherical harmonics. We call them turbulence functions.

$$\begin{aligned} Q_{lmk}(\vec{\mathbf{\Omega}}, \vec{\mathbf{\Theta}}) =& P_{lm}(\vec{\mathbf{\Omega}})H_k(\vec{\mathbf{\Theta}}) \\ & P_{lm}(\vec{\mathbf{\Omega}}) \qquad \text{spherical harmonics} \\ \int_{2\pi} H_{k'}(\vec{\mathbf{\Theta}})H_k^{*}(\vec{\mathbf{\Theta}})d\vec{\mathbf{\Theta}} =& \begin{cases} 2\pi & \text{for k'=k} \\ 0 & \text{else} \end{cases} \\ H_k(\vec{\mathbf{\Theta}}) =& e^{ik\theta} \end{aligned} \tag{8.21}$$

$$cos(\vartheta) = \vec{\mathbf{\Omega}}^{'} \cdot \vec{\mathbf{\Omega}}. \tag{8.22}$$

with

$$\int_{2\pi}\int_{4\pi} Q_{lmk}(\vec{\mathbf{\Omega}},\vec{\mathbf{\Theta}})Q^*_{lmk}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Theta}}')d\vec{\mathbf{\Omega}}'d\vec{\mathbf{\Theta}}' = \begin{cases} \frac{8\pi^2}{2l+1} & \text{for } l=l'; m=m'; k=k' \\ 0 & \text{else} \end{cases} \tag{8.23}$$

Such, suitable distribution functions are described by

$$\begin{aligned} f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}},t,\vec{\mathbf{\Omega}},\vec{\mathbf{\Theta}}) &= \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l}\sum_{k=-\infty}^{+\infty} f_{lmk}(\vec{\boldsymbol{x}},t)Q_{lmk}(\vec{\mathbf{\Omega}},\vec{\mathbf{\Theta}}) \\ f(\vec{\boldsymbol{x}},t,\vec{\mathbf{\Omega}},\mathbf{\Theta}) &= \sum_{l=0}^{+\infty}\sum_{m=-l}^{+l} P_{lm}(\vec{\mathbf{\Omega}})\sum_{k=-\infty}^{+\infty} f_{lmk}(\vec{\boldsymbol{x}},t)H_k(\vec{\mathbf{\Theta}}). \end{aligned} \tag{8.24}$$

Die $\delta-$function depending on the turbulence functions is expressed

$$\delta(\vec{\mathbf{\Omega}},\vec{\mathbf{\Omega}}';\vec{\mathbf{\Theta}},\vec{\mathbf{\Theta}}') = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\sum_{m=-l}^{m=+l} P_{lm}(\vec{\mathbf{\Omega}})P^*_{lm}(\vec{\mathbf{\Omega}}')\sum_{k=-\infty}^{+\infty}\frac{1}{2\pi}H_k(\vec{\mathbf{\Theta}})H^*_k(\vec{\mathbf{\Theta}}') \tag{8.25}$$

and such

$$\delta(\vec{\mathbf{\Omega}},\vec{\mathbf{\Omega}}';\vec{\mathbf{\Theta}},\vec{\mathbf{\Theta}}') = \frac{1}{8\pi^2}\sum_{l=0}^{\infty}(2l+1)P_l(\vec{\mathbf{\Omega}}\cdot\vec{\mathbf{\Omega}}')\sum_{k=-\infty}^{+\infty} exp(ik(\Theta-\Theta')). \tag{8.26}$$

8.4. Euler-angles as fluctuation properties of the turbulent particle transport

The angles respectively unit direction vectors $\vec{\mathbf{\Omega}}$ and $\vec{\mathbf{\Theta}}$ of turbulent motions are applied using the turbulence functions. The unit vector $\vec{\mathbf{\Omega}}\times\vec{\mathbf{\Theta}}$ with $\vec{\mathbf{\Omega}}\perp\vec{\mathbf{\Theta}}$ depending on the angles θ, φ and ϑ is determined. Initially, the direction vector $\vec{\mathbf{\Omega}}$

$$\vec{\mathbf{\Omega}}^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{8.27}$$

may be given before a rotation. The orthogonal direction vector $\vec{\mathbf{\Theta}}^0$ may be descripted in this starting situation by

$$\vec{\mathbf{\Theta}}^0 = \begin{pmatrix} sin\theta \\ cos\theta \\ 0 \end{pmatrix} \tag{8.28}$$

The rotation $\mathbf{T} = \mathbf{T}_2 \cdot \mathbf{T}_1$ with

$$\mathbf{T}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\vartheta & sin\vartheta \\ 0 & -sin\vartheta & cos\vartheta \end{pmatrix} \tag{8.29}$$

and

$$\mathbf{T}_2 = \begin{pmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.30}$$

results in

$$\mathbf{T} = \mathbf{T}_2 \cdot \mathbf{T}_1 = \begin{pmatrix} cos\varphi & sin\varphi cos\vartheta & sin\varphi sin\vartheta \\ -sin\varphi & cos\varphi cos\vartheta & cos\varphi sin\vartheta \\ 0 & -sin\vartheta & cos\vartheta \end{pmatrix} \tag{8.31}$$

with the unit vectors

$$\begin{aligned} \vec{\mathbf{\Theta}} = \mathbf{T} \cdot \vec{\mathbf{\Theta}}^0 &= \begin{pmatrix} cos\varphi sin\theta + sin\varphi cos\vartheta cos\theta \\ -sin\varphi sin\theta + cos\varphi cos\vartheta cos\theta \\ -sin\vartheta cos\theta \end{pmatrix} \\ \vec{\mathbf{\Omega}} = \mathbf{T} \cdot \vec{\mathbf{\Omega}}^0 &= \begin{pmatrix} sin\varphi sin\vartheta \\ cos\varphi sin\vartheta \\ cos\vartheta \end{pmatrix} \end{aligned} \tag{8.32}$$

and

$$\begin{aligned} \vec{\mathbf{\Omega}} \times \vec{\mathbf{\Theta}} &= \begin{pmatrix} sin\varphi sin\vartheta \\ cos\varphi sin\vartheta \\ cos\vartheta \end{pmatrix} \times \begin{pmatrix} cos\varphi sin\theta + sin\varphi cos\vartheta cos\theta \\ -sin\varphi sin\theta + cos\varphi cos\vartheta cos\theta \\ -sin\vartheta cos\theta \end{pmatrix} \\ &= \begin{pmatrix} -cos\varphi cos\theta + sin\varphi cos\vartheta sin\theta \\ cos\varphi cos\vartheta sin\theta + sin\varphi cos\theta \\ -sin\vartheta sin\theta \end{pmatrix}. \end{aligned} \tag{8.33}$$

Bibliography

[1] Chandrasekhar S.: Stochastic Problems in Physics and Astronomy Reviews of Modern Physics 15, no. l (1943): l-89

[2] Döring Werner: EINFÜHRUNG IN DIE THEORETISCHE PHYSIK V Statistische Mechanik Sammlung Göschen 1957 page 23

[3] Dunkel Jörn: Relativistic Brownian Motion and Diffusion Processes-Dissertation 2008 Universität Augsburg

[4] Einstein A.: Über die von der mokularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen-Annalen der Physik 17, 549 (1905)

[5] Feynman Richard P.: The Feynman LECTURES ON PHYSICS Volume I Addison-Wesley Publishing Company, Inc 1964 page 3-9

[6] Grad Harold: Principles of the Kinetic Theory of Gases , Handbuch der Physik,Flügge Band 12, 1958, page 205

[7] Greiner Walter/Reinhardt Joachim: Theoretische Physik Band 7: Quantenelektrodynamik Verlag Harri Deutsch . Thun und Frankfurt am Main 1984 page 18

[8] Gröbner W., Lesky P.: MATHEMATISCHE METHODEN DER PHYSIK I BI Hochschultaschebücher

[9] Grosche C. Steiner: Handbook of Feynman Path Integrals 1998 Springer page 2

[10] Khanh Tuoc,Trinh: A THEORY OF TURBULENCE-K.T.Trinh@massey.ac.nz page 43

[11] Marmier Prof. Dr.- KERNPHYSIK I: Vorlesung 1965 Verlag des Vereins der Mathematiker und Physiker an der ETH Zürich

[12] Sen R. N.: Causality, Measurement Theory and the Differentiable Structure of Space-Time CAMBRIDGE MONOGRAPHS ON MATHEMATICAL PHYSICS 2010 page 11

[13] Von Smoluchowski M.(1914): Studien über Molekularstatistik von Emulsionen und deren Zusammenhang mit der Brown'schen Bewegung. Sitzungsberichte. Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. CXXIII Band. X Heft. Dez. 1914. Abteilung II A. Wien. pp.2381-2405.

[14] Warnemünde, Rolf: Untersuchung von Anregungsfunktionen für Neutronenreaktionen nach der Aktivierungsmethode-Universität Hamburg 1969 Diplomarbeit

[15] Weinberg Alwin M., Wigner Eugine P.: THE PHYSICAl THEORY of NEUTRON CHAIN REACTORS The University of Chicago Press 1958

[16] Wilczek M.: Theory for the single-point velocity statistics of fully developed turbulence- A. Daitche and R. Friedrich February 2011