IMAGE RECONSTRUCTION WITH A NON–PARALLELISM CONSTRAINT

Antonio Boccuto, *Ivan Gerace*

Dipartimento di Matematica e Informatica,
Università degli Studi di Perugia

ABSTRACT

We consider the problem of restorating images from blur and noise. We find the minimum of the primal energy function, which has two terms, related to faithfulness to the data, and smoothness constraints, respectively. In general, we do not know and we have to estimate the discontinuities of the ideal image. We require that the obtained images are piecewise continuous and with thin edges. We associate with the primal energy function a dual energy function, which treats discontinuities implicitly. We determine a dual energy function, which is convex and takes into account non-parallelism constraints, in order to have thin edges. The proposed dual energy can be used as initial function in a GNC (Graduated Non-Convexity)-type algorithm, to obtain reconstructed images with Boolean discontinuities. In the experimental results, we show that the formation of parallel lines is inhibited.

Index Terms— Image denoising, image edge detection, image restoration, minimization methods.

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1. INTRODUCTION

In the literature, there have been several studies about the problem of reconstruction of images, which have various applications in different areas of science, for example medical diagnostic methods (see also [16, 17, 30]), civil and military engineering and thermography (see also [2, 12, 13, 14, 15, 24]). This problem, in general is ill–conditioned and/or ill–posed in the Hadamard sense (see also [5, 18, 21]). Using suitable regularization techniques (see also [3, 18, 19]), it is possible to bring it to a well–posed problem, whose solution is the minimum of the *primal energy function*. This function is formed by a term which deals with the faithfulness of the solution to the data and another term, related to regularity of the solution (see for instance [18, 20]). We deal with discontinuities in the intensity field (see also [20]), since in real images some discontinuities are present in correspondence with edges of different objects. To examine such discontinuities, we use some line variables (see also [20]). It is possible to minimize the primal energy function in these variables, in order to construct a *dual energy function* (see also [9, 11, 19]), which consider discontinuities implicitly.

To have better deblurred images, it is possible to consider some constraints in order to avoid close parallel discontinuities and inhibit thick boundaries between the smooth areas (see also [22]). Indeed, in [4, 9] it was shown that the images reconstructed by such techniques are more similar to the ideal images. In this paper we give a new duality theorem (see also [1, 3, 4, 6, 7, 8, 9, 19], in order to consider also these kinds of constraints, and construct a new potential function, which treats the non-parallelism constraints and is convex in a suitably large subset of a Euclidean space. To get better images, we deal with a dual energy function which considers implicitly Boolean line variables. It is possible to apply the given duality theorem also to such function. In general, the dual energy function is not convex. So, in order to find its minimum, we give a GNC (Graduated Non-Convexity)-type technique (see for instance [4, 9, 23, 25, 26, 27, 28]), using as first convex approximation the proposed convex dual energy function. Our experimental results confirm that, by means of such techniques, it is possible to obtain images not having parallel lines.

2. THE PROBLEM OF RECONSTRUCTION OF IMAGES

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{n}$ be vectors of dimension n^2 and A be a matrix of order $n^2 \times n^2$. We suppose that the intensity values of the associated pixels are in one column, following the lexicographic order. The formulation of the direct problem is $\mathbf{y} = A\mathbf{x} + \mathbf{n}$, where $\mathbf{x}$ and $\mathbf{y}$ indicate the original and the observed blurred image, respectively, and $\mathbf{n}$ is the noise on the image, which is supposed to be independent, identically distributed and Gaussian, and to have zero mean and known variance σ^2. The matrix A represents the blur acting on the image.

The problem of restoring images is to find a reconstruction $\mathbf{x}$ blurred image $\mathbf{y}$, the matrix A and the variance of the noise σ^2. In general, this problem is ill–conditioned and/or ill–posed in the Hadamard sense (see also [5, 18, 21, 29]).

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Observe that, as studied in [8, 9], if we want to have a satisfactory output both of costs and of exactness of the reconstructed images, it is convenient to consider second order finite differences. To this aim, we introduce the *cliques*, which are the sets of the points of a square grid on which finite differences of second order are defined. We denote by C the set of all cliques and by b_c the Boolean line variable related to the clique $c \in C$; in particular, the value zero corresponds to a discontinuity of the involved image in c. The vector $\mathbf{b}$ is the set of all line variables b_c. We denote the second order finite difference operator of the vector $\mathbf{x}$ associated with the clique c by $D_c\mathbf{x}$.

A regularized solution of the studied problem is the argument of the minimum of the *primal energy function*, defined by

$$E(\mathbf{x}, \mathbf{b}) = \|\mathbf{y} - A\mathbf{x}\|^2 + \sum_{c \in C} \left[\lambda^2 (D_c\mathbf{x})^2 b_c + \beta(b_c)\right], \tag{1}$$

where β is a non-increasing function (*balancing function*), and $\|\cdot\|$ is the Euclidean norm. The first term in the right hand of (1) is associated with confidence of the solution with the data and the last one is associated with a regularity condition on $\mathbf{x}$. The parameter λ^2 is related to faithfulness to the data and regularization of the solutions. When λ^2 is close to 0, we describe a wide confidence with the data, while when λ^2 tends to $+\infty$ we indicate a faithfulness to the a priori informations.

In order to minimize the function in (1), we first compute the minimum with respect to $\mathbf{b}$. So, the dual energy function $E_d(\mathbf{x})$ is given as

$$E_d(\mathbf{x}) = \inf_{\mathbf{b} \in B^{|C|}} E(\mathbf{x}, \mathbf{b}), \tag{2}$$

where $|C|$ denotes the cardinality of the set C (see, for instance, [6, 7, 9, 19]). Note that, by [10, Theorem 1], the quantity E_d is well-defined. We have

$$E_d(\mathbf{x}) = \|\mathbf{y} - A\mathbf{x}\|^2 + \sum_{c \in C} g(D_c\mathbf{x}), \tag{3}$$

where

$$g(t) = \inf_{b \in B} (\lambda^2 b t^2 + \beta(b)) \tag{4}$$

is the *potential function*, which to every value of the finite difference operator associates a suitable cost, and is independent of the involved clique (see also [19]).

3. THE CONSTRAINT OF NON-PARALLELISM

In order to inhibit the formation of parallel lines, which could be generated by the blur, we consider the possible relations between near cliques. So, in (1), we add a term $Q(\mathbf{b})$, which represents this kind of constraint (see for example [4, 9, 20, 22]). We define a partial order $\preceq$ on C, by

$$\{(i,j),(i-1,j),(i-2,j)\} \preceq \{(h,j),(h-1,j),(h-2,j)\} \Leftrightarrow i \leq h,$$

$$\{(i,j),(i,j-1),(i,j-2)\} \preceq \{(i,h),(i,h-1),(i,h-2)\} \Leftrightarrow j \leq h.$$

We indicate with $c-1$ and $c-2$ the cliques which precede immediately c and $c-1$, respectively (if they exist). Since each clique includes three near pixels, when there are two close areas with different second partial derivatives are present, them at least one line variable b_c and the corresponding one b_{c-1} take low values. We want that the further previous line variable b_{c-2} does not assume too low values, because in this case we do not have a completely deblurred reconstructed image. So, we assume that

$$Q(\mathbf{b}) = \sum_{c \in C} \rho(b_c, b_{c-2}), \tag{5}$$

where $\rho = \rho(u, v)$ is a suitable function, not constant and non-increasing in both variables. If $c-2$ is not defined, then we set $b_{c-2} = 0$ in in (5).

From (1) we get

$$\begin{aligned} E(\mathbf{x}, \mathbf{b}) &= \|\mathbf{y} - A\mathbf{x}\|^2 + \sum_{c \in C} \big(\lambda^2 (D_c\mathbf{x})^2 b_c \\ &+ \beta(b_c)\big) + Q(\mathbf{b}), \end{aligned} \tag{6}$$

where Q is as in (5). In general, it is difficult to calculate explicitly the dual energy function. Thus, as in [8] and [9], we consider an approximation $\xi_{c-2}(\mathbf{x})$ of b_{c-2}, given by

$$\begin{aligned} \xi_{c-2}(\mathbf{x}) &= \arg_{c-2} \min_{\mathbf{b} \in B^{|C|}} E(\mathbf{x}, \mathbf{b}) \\ &= \arg \min_{b \in B} (\lambda^2 (D_{c-2}(\mathbf{x}))^2 b + \beta(b)), \end{aligned} \tag{7}$$

where E is as in (1). As ξ_{c-2} depends only on $D_{c-2}(\mathbf{x})$, then $\xi_{c-2}(\mathbf{x})$ can be expressed as $\xi_{c-2}(\mathbf{x}) = \phi(D_{c-2}(\mathbf{x}))$, where ϕ is a non-increasing function (see also [8, 9]). So, the primal energy in (6) can be approximated by

$$\begin{aligned} E(\mathbf{x}, \mathbf{b}) &= \|\mathbf{y} - A\mathbf{x}\|^2 + \sum_{c \in C} \big[\lambda^2 (D_c\mathbf{x})^2 b_c \\ &+ \beta(b_c) + \rho(b_c, \phi(D_{c-2}(\mathbf{x})))\big], \end{aligned}$$

by assuming that, if $c-2$ does not exist, then $\rho(b_c, \phi(D_{c-2}(\mathbf{x})))$ is null. Set now

$$\gamma(b, t_2) = \beta(b) + \rho(b, \phi(t_2)), \quad b, t_2 \in \mathbb{R}. \tag{8}$$

Note that the function ρ is not constant and non-increasing, γ is not constant, and γ is even and non-decreasing on $\mathbb{R}_0^+$ with respect to the variable t_2. Moreover, to have a constraint of line variable continuation, we suppose that γ is not constant, and that γ is even and non-increasing on $\mathbb{R}_0^+$ with respect to t_2 (see also [8, 9]). Furthermore, we have

$$E_d(\mathbf{x}) = \|\mathbf{y} - A\mathbf{x}\|^2 + \sum_{c\in C} \varphi(D_c(\mathbf{x}), D_{c-2}(\mathbf{x})), \tag{9}$$

where

$$\varphi(u, v) = \inf_{b\in B} (\lambda^2 bu^2 + \beta(b) + \rho(b, \phi(v))). \tag{10}$$

Also in this case, the functions φ and γ in (10) and (8) are called *potential* and *balancing function*, respectively.

In [8] we proved the following duality theorem, giving a relation between primal and dual energies which includes the non-parallelism constraint.

Theorem 3.1 (a) *Let* $\lambda \in \mathbb{R} \setminus \{0\}$ *and* $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. *For every* $t_2 \in \mathbb{R}$, *set* $g_{t_2}(t_1) = \varphi(t_1, t_2)$, *suppose that* $g_{t_2}(0) \in \mathbb{R}$, $\sup_{t_1\in\mathbb{R}_0^+} g_{t_2}(t_1) > 0$ *and assume that*

3.1.1) g_{t_2} *is upper semicontinuous and even on* $\mathbb{R}$ *and non-decreasing on* $\mathbb{R}_0^+$;

3.1.2) *the function* $f_{t_2} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *defined by*

$$f_{t_2}(t_1) = \begin{cases} \varphi(\sqrt{t_1}, t_2), & \text{if } t_1 \geq 0, \\ -\infty, & \text{if } t_1 < 0, \end{cases}$$

is concave on $\mathbb{R}_0^+$, *and* $\lim_{t_1\to+\infty} \frac{f_{t_2}(t_1)}{t_1} = 0$ *for every* $t_2 \in \mathbb{R}$;

3.1.3) φ *is not constant, and the function* $t_2 \mapsto \varphi(t_1, t_2)$ *is even on* $\mathbb{R}$ *and non-decreasing on* $\mathbb{R}_0^+$ *for any* $t_1 \in \mathbb{R}_0^+$.

Then there is a function $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+ \cup \{+\infty\}$ *such that, if* $\beta_{t_2} : \mathbb{R} \to \mathbb{R}_0^+ \cup \{+\infty\}$ *is defined by* $\beta_{t_2}(b) = \gamma(b, t_2)$, b, $t_2 \in \mathbb{R}$, *then*

3.1.4) $\beta_{t_2}(b) < +\infty$ *for each* $b > 0$ *and* $t_2 \in \mathbb{R}$, *and* $\beta_{t_2}(b) = +\infty$ *for every* $b < 0$ *and* $t_2 \in \mathbb{R}$.

3.1.5) $\gamma(b, t_2) = \beta_{t_2}(b) = \sup_{t_1\in\mathbb{R}_0^+} (-\lambda^2 bt_1 + \varphi(\sqrt{t_1}, t_2))$ *for any* b, $t_2 \in \mathbb{R}$;

3.1.6) $\varphi(t_1, t_2) = \inf_{b\in\mathbb{R}} (\lambda^2 b\, t_1^2 + \gamma(b, t_2))$ *for each* t_1, $t_2 \in \mathbb{R}$;

for any $t_2 \in \mathbb{R}$, *we get*

3.1.7) β_{t_2} *is non-increasing on* $\mathbb{R}$;

3.1.8) β_{t_2} *is convex and lower semicontinuous on* $\mathbb{R}$;

3.1.9) $\lim_{b\to+\infty} \beta_{t_2}(b) \in \mathbb{R}$;

3.1.10) γ *is not constant, and the function* $t_2 \mapsto \gamma(b, t_2)$ *is even on* $\mathbb{R}$ *and non-decreasing on* $\mathbb{R}_0^+$ *for any* $b \in \mathbb{R}$.

(b) *Conversely, if* $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+ \cup \{+\infty\}$ *satisfies* 3.1.4), 3.1.7), 3.1.8), 3.1.9), 3.1.10), *then there is a function* $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, *with* $\varphi(0, t_2) \in \mathbb{R}$ *for every* $t_2 \in \mathbb{R}$, *such that the conditions* 3.1.1), 3.1.2), 3.1.3), 3.1.5), 3.1.6) *hold.*

4. THE POTENTIAL FUNCTIONS

Now we propose a potential function φ, according to (10), convex in a suitable domain of $\mathbb{R} \times \mathbb{R}$ and fulfilling the hypotheses of Theorem 3.1. So, the dual energy turns to be convex on a large enough subset of $\mathbb{R}^{n^2}$ (see also [8]). Fix λ^2, $\varepsilon > 0$ and $\delta \in (0, 1)$, and let $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(t_1, t_2) = \varphi_\delta(t_1, t_2) = \lambda^2(\varepsilon t_2^2 + 1)|t_1|^{2-\delta}. \tag{11}$$

Note that ε is a positive parameter related to the non-parallelism constraint.

In order to consider Boolean line variables for improving the results, we can take the following potential function

$$\widetilde{\varphi}(t_1, v) = \begin{cases} \widetilde{g}(t_1, 0), & \text{if } |v| < s = \frac{\sqrt{\alpha}}{\lambda}, \\ \\ \widetilde{g}(t_1, \varepsilon), & \text{otherwise}, \end{cases} \tag{12}$$

where $\alpha > 0$ is a suitable parameter related to an edge in the reconstructed image, s is a threshold for introducing a discontinuity, ε is associated with the constraint of non-parallelism, and $\widetilde{g} : \mathbb{R} \times (-\alpha, \infty) \to \mathbb{R}_0^+$ is given by

$$\widetilde{g}(t_1, v) = \begin{cases} \lambda^2 t_1^2, & \text{if } |t_1| < \frac{\sqrt{\alpha+v}}{\lambda}, \\ \\ \alpha + v, & \text{otherwise} \end{cases}$$

(see also [8, 9]). Observe that the function $\widetilde{\varphi}$ in (12) fulfils the hypotheses of Theorem 3.1.

To minimize the energy function

$$\widetilde{E_d}(\mathbf{x}) = \|\mathbf{y} - A\mathbf{x}\|^2 + \sum_{c\in C} \widetilde{\varphi}(D_c(\mathbf{x}), D_{c-2}(\mathbf{x})),$$

we use a GNC (*Graduated Non-Convexity*) technique, since $\widetilde{E}_d$ is not convex (see also [4, 8, 9, 23, 25, 26, 27, 28]). As first convex approximation, we use the dual energy in (9), where φ is as in (11). Note that, differently than in [9], the first convex approximation inhibits parallel lines, since our proposed function satisfies Theorem 3.1.

5. EXPERIMENTAL RESULTS

In this section we show, by means of experimental results, the role of the non-parallelism constraint in the used GNC algorithm in avoiding the formation of triple lines in the reconstruct image.

In Figure 1 (a) we present an ideal real image, while Figure 1 (b) contains the data deteriorated image. In Figure 1

(c) there is the reconstruction without the constraint of non-parallelism, and Figure 1 (d) contains the line elements of the reconstruction. Note that there are still several triple edges. In Figure 1 (e) we reconstruct the image taking into account the non-parallelism constraint. In Figure 1 (f), the triple edges are almost completely eliminated.

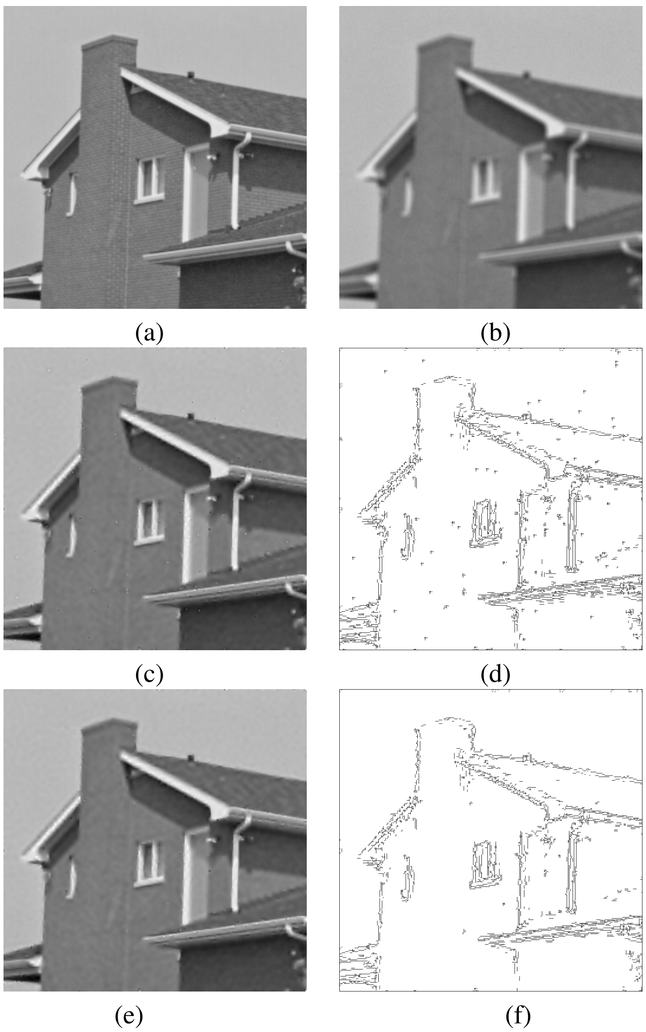

Fig. 1. (a) The original 256×256 image; (b) the image blurred by a uniform blur mask of dimension 5 corrupted by a Gaussian noise with variance $\sigma^2 = 9$; (c) restoration by the GNC algorithm using the parameters $\lambda = 0.3$, $\alpha = 3 \cdot 10^{-4}$, $\varepsilon = 0$ (mean square error equal to 6.4732217), and (d) its line elements; (e) restoration by the GNC algorithm using the parameters $\lambda = 0.3$, $\alpha = \varepsilon = 3 \cdot 10^{-4}$ (mean square error equal to 5.9848682), and (f) its line elements.

6. CONCLUSIONS

We dealt with the problem of reconstruction of images corrupted by blur and noise. We studied the problem of minimizing the primal energy function consisting in the sum of the terms, associated with faithfulness with the data and the smoothness constraints, respectively. The obtained images are requested to be piecewise continuous and with thin edges. We investigated some fundamental properties of the dual energy function, which treats discontinuities implicitly. We gave a Fenchel-type duality theorem and found a convex dual energy function, in connection with the non-parallelism constraint and in order to treat implicitly Boolean discontinuities. In order to minimize it, we presented a GNC-type technique which uses as first convex approximation the proposed convex energy function. In the experimental results, it is shown that by means of the proposed method it is possible to obtain reconstructed images, which do not present incorrect parallel edges.

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