ON THE DISTRIBUTION OF ADDITION CHAINS

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Abstract. In this paper we study the theory of addition chains producing any given number $n \geq 3$. With the goal of estimating the partial sums of an additive chain, we introduce the notion of the determiners and the regulators of an addition chain and prove the following identities

$$\sum_{j=2}^{\delta(n)+1} s_j = 2(n-1) + (\delta(n)-1) + \kappa(a_{\delta(n)}) - \varrho(r_{\delta(n)+1}) + \int_{2}^{\delta(n)-1} \sum_{2 \leq j \leq t} \varrho(r_j)dt$$

where

$$2, s_3 = \kappa(a_3) + \varrho(r_3), \ldots, s_{k-1} = \kappa(a_{k-1}) + \varrho(r_{k-1}), s_k = \kappa(a_k) + \varrho(r_k) = n$$

are the associated generators of the chain $1, 2, \ldots, s_{k-1}, s_k = n$ of length $\delta(n)$. Also we obtain the identity

$$\sum_{j=2}^{\delta(n)+1} \kappa(a_j) = (n-1) + (\delta(n)-1) + \kappa(a_{\delta(n)}) - \varrho(r_{\delta(n)+1}) + \int_{2}^{\delta(n)-1} \sum_{2 \leq j \leq t} \varrho(r_j)dt.$$

1. Introduction

The notion of an addition chain producing $n \geq 3$, introduced by Arnold Scholz, is a sequence of numbers of the form

$$1, 2, \ldots, s_{k-1}, s_k = n$$

where each term in the sequence is generated by adding two earlier terms and the terms are allowed to be homogeneous. The number of terms in the sequence determines the length of the chain. The length of the smallest such chain producing n is the shortest length of the addition chain. It is a well-known problem to determine the length of the shortest addition chain producing numbers $2^n - 1$ of special forms. More formally the conjecture states

Conjecture 1.1. Let $\iota(n)$ for $n \geq 3$ be the shortest addition chain producing n, then the inequality is valid

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

The conjecture was studied fairly soon afterwards by Alfred Brauer, who obtained some weaker bounds [1]. There had also been an amazing computational work in verifying the conjecture [2]. In this paper we study the partial sums of the addition chain producing $n \geq 3$. We obtain the following results in this paper

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Theorem 1.1. *Let $1,2,\ldots,s_{k-1},s_k$ be an additive chain producing $n\geq 3$ of length $\delta(n)$ with associated generators*

$$2,s_3=\kappa(a_3)+\varrho(r_3),\ldots,s_{k-1}=\kappa(a_{k-1})+\varrho(r_{k-1}),s_k=\kappa(a_k)+\varrho(r_k)=n.$$

Then we have

$$\sum_{j=2}^{\delta(n)+1}s_j=2(n-1)+(\delta(n)-1)+\kappa(a_{\delta(n)})-\varrho(r_{\delta(n)+1})+\int\limits_{2}^{\delta(n)-1}\sum_{2\leq j\leq t}\varrho(r_j)dt.$$

2. The regulators and determiners of an addition chain

In this section we recall the notion of an addition chain and introduce the notion of the generators of the chain and their accompanying determiners and regulators.

Definition 2.1. Let $n\geq 3$, then by the addition chain of length $k-1$ producing n we mean the sequence

$$1,2,\ldots,s_{k-1},s_k$$

with the corresponding sequence of partition

$$2=1+1,\ldots,s_{k-1}=\kappa(a_{k-1})+\varrho(r_{k-1}),s_k=\kappa(a_k)+\varrho(r_k)=n$$

with $\kappa(a_{i+1})=\kappa(a_i)+\varrho(r_i)$ for $2\leq i\leq k$. We call the partition $\kappa(a_i)+\varrho(r_i)$ the i th generator of the chain for $2\leq i\leq k$. We call $\kappa(a_i)$ the determinant and $\varrho(r_i)$ the regulator of the i th generator of the chain. We call the sequence $(\varrho(r_i))$ the regulators of the additive chain and $(\kappa(a_i))$ the determiners of the chain for $2\leq i\leq k$. We call the subsequence (s_{j_m}) for $2\leq j\leq k$ and $1\leq m\leq t\leq k$ a truncated additive chain producing n.

Theorem 2.2. *Let $1,2,\ldots,s_{k-1},s_k$ be an additive chain producing $n\geq 3$ with associated generators*

$$2=1+1,\ldots,s_{k-1}=\kappa(a_{k-1})+\varrho(r_{k-1}),s_k=\kappa(a_k)+\varrho(r_k)=n.$$

Then the following relation is valid

$$\sum_{j=2}^{k}\varrho(b_j)=n-1.$$

Proof. First we notice that $\varrho(r_k)=n-\kappa(a_k)$. It follows that

$$\begin{aligned}\varrho(r_k)+\iota(r_{k-1})&=n-\kappa(a_k)+\varrho(r_{k-1})\\&=n-(\kappa(a_{k-1})+\varrho(r_{k-1}))+\varrho(r_{k-1})\\&=n-\kappa(a_{k-1}).\end{aligned}$$

Again we obtain from the following iteration

$$\begin{aligned}\varrho(r_k)+\varrho(r_{k-1})+\varrho(r_{k-2})&=n-\kappa(a_{k-1})+\varrho(r_{k-2})\\&=n-(\kappa(a_{k-2})+\varrho(r_{k-2}))+\varrho(r_{k-2})\\&=n-\kappa(a_{k-2}).\end{aligned}$$

By iterating downwards in this manner the relation follows. □

Corollary 2.1. Let $1,2,\ldots,s_{k-1},s_k$ be an additive chain producing $n\geq 3$ with associated generators

$$2=1+1,\ldots,s_{k-1}=\kappa(a_{k-1})+\varrho(r_{k-1}),s_k=\kappa(a_k)+\varrho(r_k)=n.$$

If we denote the length of the chain by $\delta(n)$, then the following inequality remain valid

$$\frac{n-1}{\sup(\varrho(r_j))_{j=2}^{\delta(n)+1}}\leq \delta(n)\leq \frac{n-1}{\operatorname{Inf}(\varrho(r_j))_{j=2}^{\delta(n)+1}}.$$

Proof. The result follows by applying the relation in Theorem 2.2 and noting that the number of regulators $\varrho(r_j)$ in the chain with multiplicity counts as the length of the chain. □

Theorem 2.3. *Let $1,2,\ldots,s_{k-1},s_k$ be an additive chain producing $n\geq 3$ of length $\delta(n)$ with associated generators*

$$2,s_3=\kappa(a_3)+\varrho(r_3),\ldots,s_{k-1}=\kappa(a_{k-1})+\varrho(r_{k-1}),s_k=\kappa(a_k)+\varrho(r_k)=n.$$

Then the following relations hold

$$\sum_{j=2}^{\delta(n)}\kappa(a_j)=(\delta(n)-1)+\sum_{j=2}^{\delta(n)-1}(\delta(n)-j)\varrho(r_j).$$

Proof. This identity is easily established by induction on an iteration on the sum. We first notice that

$$\begin{aligned}\kappa(a_2)+\kappa(a_3)&=\kappa(a_2)+\kappa(a_2)+\varrho(r_2)\\&=2\kappa(a_2)+\varrho(r_2).\end{aligned}$$

Again it follows that

$$\begin{aligned}\kappa(a_2)+\kappa(a_3)+\kappa(a_4)&=2\kappa(a_2)+\varrho(r_2)+\kappa(a_4)\\&=3\kappa(a_2)+2\varrho(r_2)+\varrho(r_3).\end{aligned}$$

By inducting on this iteration we can now write

$$\begin{aligned}\sum_{j=2}^{\delta(n)}\kappa(a_j)=(\delta(n)-1)\kappa(a_2)+(\delta(n)-2)\varrho(r_2)+(\delta(n)-3)\varrho(r_3)+\cdots+2\varrho(r_{\delta(n)-2})\\+\varrho(r_{\delta(n)-1}))\end{aligned}$$

thereby establishing the identity. □

Remark 2.4. The above result ensures that we can write the partial sums of the determiners $\kappa(a_j)$ of the generators of any additive chain as a linear combination of the regulators $\varrho(r_j)$. We leverage this identity to obtain an identity relating the partial sums of the determiners to the length of the chain and a certain integral of regulators of the chain. We state the result in the following manner as follows.

Theorem 2.5. *Let $1,2,\ldots,s_{k-1},s_k$ be an additive chain producing $n\geq 3$ with associated generators*

$$2,s_3=\kappa(a_3)+\varrho(r_3),\ldots,s_{k-1}=\kappa(a_{k-1})+\varrho(r_{k-1}),s_k=\kappa(a_k)+\varrho(r_k)=n.$$

If we denote by $\delta(n)$ the length of the chain, then we have the idenitity

$$\sum_{j=2}^{\delta(n)+1}\kappa(a_j)=(n-1)+(\delta(n)-1)+\kappa(a_{\delta(n)})-\varrho(r_{\delta(n)+1})+\int\limits_{2}^{\delta(n)-1}\sum_{2\leq j\leq t}\varrho(r_j)dt.$$

Proof. Let $1,2,\ldots,s_{k-1},s_k$ be an additive chain producing $n\geq 3$ with associated generators

$$2,s_3=\kappa(a_3)+\varrho(r_3),\ldots,s_{k-1}=\kappa(a_{k-1})+\varrho(r_{k-1}),s_k=\kappa(a_k)+\varrho(r_k)=n.$$

By letting $\delta(n)$ denote the length of the chain, then we consider the identity

$$\sum_{j=2}^{\delta(n)}\kappa(a_j)=(\delta(n)-1)+\sum_{j=2}^{\delta(n)-1}(\delta(n)-j)\varrho(r_j)$$

in Theorem 2.3. By applying partial summation on the sum on the right-hand side of the expression we have

$$\begin{aligned}\sum_{j=2}^{\delta(n)-1}(\delta(n)-j)\varrho(r_j)&=\sum_{j=2}^{\delta(n)-1}\varrho(r_j)+\int\limits_{2}^{\delta(n)-1}\sum_{2\leq j\leq t}\varrho(r_j)dt\\&=n-1-\varrho(r_{\delta(n)})-\varrho(r_{\delta(n)+1})+\int\limits_{2}^{\delta(n)-1}\sum_{2\leq j\leq t}\varrho(r_j)dt\end{aligned}$$

by appealing to Theorem 2.2, and the result follows by arranging the terms in the expression and appealing to Theorem 2.2. □

Remark 2.6. Next we relate the length of an additive chain function to the corresponding argument $n\geq 3$ in the following inequality. We obtain the following weaker but much less crude bound as follows.

Theorem 2.7. *Let $1,2,\ldots,s_{k-1},s_k$ be the shortest additive chain producing $n\geq 3$ of length $\iota(n)$ with associated generators*

$$2,s_3=\kappa(a_3)+\varrho(r_3),\ldots,s_{k-1}=\kappa(a_{k-1})+\varrho(r_{k-1}),s_k=\kappa(a_k)+\varrho(r_k)=n$$

such that $\varrho(r_j)\geq 2$ for all $j\geq 3$ then we have the inequality

$$\iota(n)\leq\frac{n}{2}.$$

Proof. Let us consider the truncated additive chain $4<s_4,\ldots,s_{k-1},s_k$ producing n with associated generators

$$4\leq s_3=\kappa(a_3)+\varrho(r_3),\ldots,s_{k-1}=\kappa(a_{k-1})+\varrho(b_t),s_k=\kappa(a_k)+\varrho(b_r)=n.$$

By Theorem 2.2, we obtain the corresponding partial sum

$$\sum_{j=3}^{\iota(n)+1}\varrho(r_j)=n-2.$$

It follows from Corollary 2.1

$$\iota(n) \leq \frac{n-2}{\mathrm{Inf}(\varrho(r_j))_{j=3}^{\iota(n)+1}} + 1$$
$$\leq \frac{n-2}{2} + 1.$$

This proves the inequality. □

Proposition 2.1. *Let* $1, 2, \ldots, s_{k-1}, s_k$ *be an additive chain producing* $n \geq 3$ *with associated generators*

$$2 = 1 + 1, \ldots, s_{k-1} = \kappa(a_{k-1}) + \varrho(b_t), s_k = \kappa(a_k) + \varrho(b_r) = n.$$

If we denote the length of the chain by $\delta(n)$*, then the following relation is valid*

$$\sum_{j=2}^{\delta(n)+1} s_j \geq 2n - 2.$$

Proof. We first write

$$\sum_{j=2}^{\delta(n)+1} s_j = \sum_{j=2}^{\delta(n)+1} \kappa(a_j) + \sum_{j=2}^{\delta(n)+1} \varrho(r_j)$$
$$\geq 2 \sum_{j=2}^{\delta(n)+1} \varrho(r_j)$$
$$\geq 2(n-1)$$

by Theorem 2.2. □

Remark 2.8. We improve on the estimate of the partial sums of an additive chain producing $n \geq 3$ in the following result. It has become so clear that we need to a larger extent the argument and the length of an additive chain to determine very roughly the value of their partial sums.

Theorem 2.9. *Let* $1, 2, \ldots, s_{k-1}, s_k$ *be an additive chain producing* $n \geq 3$ *of length* $\delta(n)$ *with associated generators*

$$2, s_3 = \kappa(a_3) + \varrho(r_3), \ldots, s_{k-1} = \kappa(a_{k-1}) + \varrho(r_{k-1}), s_k = \kappa(a_k) + \varrho(r_k) = n.$$

Then we have

$$\sum_{j=2}^{\delta(n)+1} s_j = 2(n-1) + (\delta(n) - 1) + \kappa(a_{\delta(n)}) - \varrho(r_{\delta(n)+1}) + \int_{2}^{\delta(n)-1} \sum_{2 \leq j \leq t} \varrho(r_j) dt.$$

Proof. Under the main assumption we have, by appealing to Theorem 2.2

$$\sum_{j=2}^{\delta(n)+1} \varrho(r_j) = n - 1.$$

By adding the partial sums of the determiners of the same scale to the earlier sum we have by appealing to Theorem 2.3

$$\sum_{j=2}^{\delta(n)+1} s_j = \sum_{j=2}^{\delta(n)} \kappa(a_j) + \kappa(a_{\delta(n)+1}) + \sum_{j=2}^{\delta(n)+1} \varrho(r_j)$$
$$= (n-1)) + (\delta(n)-1) + \kappa(a_{\delta(n)+1}) + \sum_{j=2}^{\delta(n)-1} (\delta(n)-j)\varrho(r_j)$$
$$= 2(n-1) + (\delta(n)-1) + \kappa(a_{\delta(n)+1}) - \varrho(r_{\delta(n)}) - \varrho(r_{\delta(n)+1}) + \int_{2}^{\delta(n)-1} \sum_{2\leq j\leq t} \varrho(r_j)dt$$

by an application of partial summation. □

The inequality established in Theorem 2.7 for additive chains producing $n \geq 3$ confines the scale of further regulators $\varrho(r_j)$ for $j \geq 3$ above one. The following result is generalization of this result by removing this requirement at the compromise of a slightly weaker bound.

Theorem 2.10. *Let $1, 2, \ldots, s_{k-1}, s_k$ be the shortest additive chain producing $n \geq 3$. If we denote by $\iota(n)$ the length of the chain, then we have the inequality*

$$\iota(n) \leq \frac{n+1}{2}.$$

Proof. Let $1, 2, \ldots, s_{k-1}, s_k$ be the shortest additive chain producing $n \geq 3$ and consider the truncated additive chain given $4 \leq s_3, \ldots, s_{k-2}, s_{k-1}$. We break the proof into two special cases; the case $\varrho(r_j) \geq 2$ for all $j \geq 3$ and the case $\varrho(r_3) = 1$. The resulting inequality of the first case has already been given in Theorem 2.7. Thus it suffices to consider the last case. Let us consider the truncated additive chain producing n given by $4 < s_4, \ldots, s_{k-2}, s_{k-1}$, then by Theorem 2.2, It follows that

$$\sum_{j=4}^{\iota(n)+1} \varrho(r_j) = n-3.$$

By adding the terms $1, 2, s_3$ into the truncated chain which also requires removing $s_{\delta(n)+1}$ from the chain as well and appealing to Corollary 2.1, we have

$$\iota(n) \leq \frac{n-3}{\mathrm{Inf}(\varrho(r_j)_{j=4}^{\iota(n)+1}} + 2$$
$$\leq \frac{n-3}{2} + 2$$

and the inequality is established. □

The identity as espoused in Theorem 2.5 offers a rich tool-box in studying Scholz's conjecture concerning an inequality of the shortest additive chain of numbers of the form $2^n - 1$. By rearranging the terms in Theorem 2.5, we have the relation under the same assumption, except that we are now considering the shortest additve chain producing $n \geq 3$ of length $\iota(n)$

$$\iota(n) + n - 1 = \left(\sum_{j=2}^{\iota(n)} \kappa(a_j)\right) + \varrho(r_{\iota(n)}) + \varrho(r_{\iota(n)+1}) + 1 - \int\limits_{2}^{\iota(n)-1} \sum_{2 \leq j \leq t} \varrho(r_j)dt.$$

Then a positive solution to the scholz conjecture reduces to showing that

Conjecture 2.1 (Scholz)**.** Let $1, 2, \ldots, s_{k-1}, s_k$ be the shortest additive chain producing $n \geq 3$ of length $\iota(n)$ with associated generators

$$2, s_3 = \kappa(a_3) + \varrho(r_3), \ldots, s_{k-1} = \kappa(a_{k-1}) + \varrho(r_{k-1}), s_k = \kappa(a_k) + \varrho(r_k) = n$$

then we have the inequality

$$\iota(2^n - 1) \leq \left(\sum_{j=2}^{\iota(n)} \kappa(a_j)\right) + \varrho(r_{\iota(n)}) + \varrho(r_{\iota(n)+1}) + 1 - \int\limits_{2}^{\iota(n)-1} \sum_{2 \leq j \leq t} \varrho(r_j)dt.$$

[1].

References

1. A. Brauer, *On addition chains*, Bulletin of the American mathematical Society, vol. 45:10, 1939, 736–739.
2. M. Clift, *Calculating optimal addition chains*, Computing, vol. 91:3, Springer, 1965, pp 265–284.

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1

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