Quantum Field Theory as Manifestation of Self-Organized Criticality

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Abstract

Self-organized criticality (SOC) reflects the ability of many complex dynamical systems to self-sustain critical behavior outside equilibrium. Here we provide analytic evidence that quantum field propagators and the probability distribution of SOC share a common foundation. In particular, we find that the formal structure of quantum propagators replicates the *finite scaling ansatz* (FSS) of SOC, which is a generic paradigm for the emergence of complexity in Nature.

Key words: complexity, self-organized criticality, non-equilibrium dynamics, finite size scaling ansatz, quantum field propagator.

1. SOC and the FSS ansatz

Consider a large-scale system of size L undergoing a second-order phase transition. The transition is driven by the control parameter λ as it approaches the critical value λ_c. Near the critical point and for systems of infinite extent ($L \to \infty$), the correlation length ξ diverges as [1-3]

$$\xi \sim (\lambda - \lambda_c)^{-\nu} \; ; \; L \to \infty, \; \lambda \to \lambda_c \tag{1}$$

In the transition region, a relevant variable of the system is also a diverging quantity which scales as

$$A_{\infty}(\lambda) \sim |\lambda - \lambda_c|^{-\zeta} \; ; \; L \to \infty, \; \lambda \to \lambda_c \tag{2}$$

where ζ is a critical exponent. In what follows, we introduce the notation

$$\tau_s = -(\zeta / \nu) \tag{3}$$

There are two distinct cases associated with the power-law (2). If the size of the system greatly exceeds the correlation length, $L >> \xi$, by (1) and (2) we write

$$A_L(\lambda) \sim \xi^{-\tau_s} \; ; \; (L >> \xi, \; \lambda \to \lambda_c) \tag{4}$$

In the opposite case, $L << \xi$, the system size takes over the scaling behavior and (2) turns into

$$A_L(\lambda) \sim L^{-\tau_s} \; ; \; (L << \xi, \; \lambda \to \lambda_c) \tag{5}$$

Taken together, (4) and (5) define the *finite-size scaling* (FSS) ansatz [1-2]

$$A_L(\lambda) = \xi^{-\tau_s} \Phi(L / \xi) \; ; \; (L \to \infty, \; \lambda \to \lambda_c) \tag{6}$$

where the scaling function controls the finite-size effects of critical behavior and is defined as

$$\Phi(x) = \begin{cases} const; & |x| >> 1 \\ x^{-\tau_s} \; ; & x \to 0 \end{cases} \tag{7}$$

To transition from the framework of critical phenomena to SOC, one simply identifies the correlation length with the concept of *avalanche-size*, i.e.,

$$s = \xi \ ; \quad s_{cr} = L \tag{8}$$

The probability distribution defining the FSS ansatz in SOC is a natural extrapolation of (6) and takes the form [2]

$$\boxed{P(s,L) \sim s^{-\tau_s}\Phi(s/s_{cr}) \text{ for } s \gg 1,\ L \gg 1} \tag{9a}$$

$$\boxed{s_{cr}(L) \sim L^{D_0} \text{ for } L \gg 1} \tag{9b}$$

in which τ_s and D_0 are called the *avalanche-size exponent* and the *avalanche dimension*, respectively. Quite generally, (9) shows that, for a system of finite extent and large size avalanches, the avalanche-size probability behaves as a fractal function times a generic cutoff function. To enable all moments of (9) to exist, the cutoff function must decay sufficiently fast. One obtains the following representation of the cutoff function upon power expanding it around zero,

$$\Phi(x) \sim \begin{cases} \Phi(0) + \Phi'(0)x + \frac{1}{2}\Phi''(0)x^2 + \ldots, & x \ll 1 \\ \to 0, & x \gg 1 \end{cases} \tag{10}$$

The avalanche-size probability must be normalized to unity and its average be diverging along with $L \to \infty$, which leads to the following constraints

$$\sum_{s=1}^{\infty} P(s;L) = 1 \qquad \text{for } L < \infty, \tag{11}$$

$$\langle s \rangle = \sum_{s=1}^{\infty} sP(s;L) \to \infty \ \text{ for } L \to \infty \tag{12}$$

Under the assumption that $\Phi(0) \neq 0$, the behavior of (9) for an infinite system size may be approximated as

$$\lim_{L\to\infty} P(s;L) \sim s^{-\tau_s}\Phi(0) \tag{13}$$

Furthermore, to comply with (11) and (12), the avalanche-size exponent must fall in the range

$$1 < \tau_s \leq 2 \tag{14}$$

We proceed next to the analysis of four examples linking the FSS ansatz to the formalism of quantum propagators. Before doing so, it is worth emphasizing that SOC and the FSS ansatz are deeply linked to *multifractals* and the *path integral* approach to quantum theory [14-16].

2. *Case #1*: free scalar relativistic particle

Let us begin by recalling the Klein-Gordon (KG) description of a relativistic particle of mass m in 3+1 spacetime dimensions. It is known that the perturbative treatment of the KG theory may be built by analogy with the Gaussian random walk (RW) model [4]. In the Euclidean version of this theory, ordinary time is analytically continued to $t = i\tau$ and the momentum-space propagator takes the form

$$G(r,p)=\frac{1}{\mathbf{p}^2+p_0^2+m^2}=\int_0^\infty dr\exp(-rm^2)\exp[-r(\mathbf{p}^2+p_0^2)] \tag{15}$$

where $p^\mu=(\mathbf{p},p_0)$, with the energy analytically continued to $p_0=-iE$. Fourier transforming the second integrand in (15) gives the probability distribution of Gaussian random walks of step length r,

$$p(\Delta,r)=(4\pi r)^{-2}\exp[-\Delta(r)] \tag{16}$$

in which

$$\Delta(r)=\frac{\rho^2}{4r}=\frac{|\mathbf{x}|^2+\tau^2}{4r} \tag{17}$$

Since r has units of inverse mass squared, $r=[M]^{-2}=[L]^2$, it is convenient to normalize (16) through the substitution

$$r^0=rm^2 \tag{18}$$

which turns (16) into

$$p^0(\Delta,r^0)=\frac{p(\Delta,r)}{m^4}=(4\pi r^0)^{-2}\exp(-\frac{\rho^2m^2}{4rm^2}) \tag{19}$$

or,

$$p^0(\Delta,r^0)=(4\pi r^0)^{-2}\exp(-\frac{\rho^2m^2}{4r^0})=(4\pi r^0)^{-2}\exp[-\Delta(r)] \tag{20}$$

Comparing (20) and (17) with the FSS ansatz (9) gives

$$\boxed{\tau_s = 2,\ |D_0| = 2} \tag{21}$$

under the following assumptions:

a) the RW step length r^0 is the analogue of the avalanche size s introduced in (9),

b) the scaling dimension of ρ^2 in (17) is $\left[\rho^2\right] = [L]^2$, which implies $|D_0| = 2$.

3. *Case #2*: harmonic oscillator

Consider now the general case of an *anharmonic oscillator* in 1+1 dimensions with Euclidean Lagrangian

$$L_E = \frac{m\dot{x}^2}{2} + \frac{m\omega^2 x^2}{2} + \Delta V(x) = \overline{L_E} + \Delta V(x) \tag{22}$$

where $\overline{L_E}$ denotes the unperturbed Lagrangian and $\Delta V(x)$ a small perturbation to the dynamics of $\overline{L_E}$. Assuming natural units throughout ($\hbar = k = 1$), the path integral for the propagator is given by [5]

$$K_E[\beta, J] = \exp[-\int d\tau \Delta V(\frac{\delta}{\delta J(\tau)})]\overline{K_E}[\beta, J] \tag{23}$$

where $\overline{K_E}(\beta, J)$ represents the path integral of the harmonic oscillator, whose Euclidean Lagrangian $\overline{L_E}$ assumes the form

$$\overline{L_E} = \frac{m\dot{x}^2}{2} + \frac{m\omega^2 x^2}{2} \tag{24}$$

In (23), $t = -i\beta$ is the expression of time as function of the reduced temperature parameter of statistical mechanics ($\beta = T^{-1}$) and $\tau = it$ is the Euclidean time. Furthermore, $\overline{K_E}(\beta, J)$ is the Gaussian integral representing the Euclidean harmonic oscillator driven by the forcing function J. It can be shown that $\overline{K_E}(\beta, J)$ amounts to [5]

$$\overline{K_E}(\beta, J) = [\frac{m\omega}{2\pi\sinh(\omega\beta)}]^{1/2} \exp(J \cdot G_D \cdot J) \tag{25}$$

in which $G_D(\tau_1, \tau_2)$ stands for the Green function of the harmonic oscillator, computed between the initial (τ_1) and final (τ_2) times. For $\beta \to \infty$, $(T = 0)$ the Green function reduces to

$$\lim_{\beta \to \infty} G_D = \frac{1}{2m\omega} \exp(-\omega|\tau_1 - \tau_2|) \tag{26}$$

if one of these two conditions are satisfied

$$|\tau_1 - \tau_2| << \beta \tag{27a}$$

$$\left|\tau_{1,2} \pm \frac{\beta}{2}\right| << \beta \tag{27b}$$

The transition probability between the initial and final times is the square of (26), that is,

$$P_{D(1,2)} = G_D^2 = (\frac{1}{2m\omega})^2 \exp(-\frac{2\omega}{|\tau_1 - \tau_2|^{-1}}) \tag{28}$$

To express (28) in a normalized form, we introduce the dimensionless parameter

$$r^0 = m\omega\beta^2 \tag{29}$$

which turns the propagator (26) and transition probability (28) into, respectively,

$$G_D^0 = \frac{G_D}{\beta^2} = \frac{1}{2r^0}\exp(-\omega|\tau_1 - \tau_2|) = \frac{1}{2r^0}\exp(-\frac{r^0}{m\beta^2}|\beta_1 - \beta_2|) \tag{30}$$

$$P^2_{D(1,2)} = (\frac{1}{2r^0})^2 \exp(-\frac{2r^0}{m\beta^2}|\beta_1 - \beta_2|) \tag{31}$$

Since the scaling dimension of the Euclidean time and of the inverse temperature parameter is $[\tau]=[\beta]=[L]^1=[M]^{-1}$, by (9) and (31) we obtain

$$\boxed{\tau_s = 2,\ |D_0| = 1} \tag{32}$$

4. *Case #3*: oscillator with quartic interaction

Let us look next at the case of a massless scalar field with quartic self-interaction, coupled to an external current J [6]. The generating functional and propagator are respectively given by

$$Z[J] = \int [d\varphi] \exp\{i\int d^4x[\frac{1}{2}(\partial\varphi)^2 - \frac{\lambda^4}{4}\varphi^4 + J\varphi]\} \tag{33}$$

$$G(t_1 - t_2) = \theta(t_1 - t_2)\,\mu(\sqrt[4]{2}\lambda^{-1})\, f(\lambda, \mu t) \tag{34}$$

$$f(\lambda, \mu t) = \mathrm{sn}(\frac{\lambda}{\sqrt[4]{2}}\mu t) \tag{35}$$

where $\theta(t)$ is the time-ordering operator, μ an arbitrary constant with the dimension of a mass (that is, $[M]^1$) and $\mathrm{sn}(\ldots)$ stands for the Jacobi elliptic function. The dimensionless version of (34) reads

$$G^0(t_1 - t_2) = \frac{G(t_1 - t_2)}{\mu} = \theta(t_1 - t_2)(\sqrt[4]{2}\lambda^{-1}) f(\lambda, \mu t) \tag{36}$$

Note that, because λ is a real number, it can be presented as power of another arbitrary real number, which indicates that the value of the avalanche-size exponent is *undetermined*. In line with previous arguments, since time t and energy scale μ entering (36) have scaling dimension ± 1, the oscillator with quartic interaction is characterized by

$$\boxed{\tau_s = undetermined,\ |D_0| = 1} \tag{37}$$

5. *Case #4*: Dirac theory in 1+1 dimensions

In this last example we bring up the fundamental solution of the Dirac equation in 1+1 dimensions, which can be written as ($\hbar = c = 1$) [7]

$$\begin{pmatrix} -m & i\partial/\partial t - i\partial/\partial x \\ i\partial/\partial t + i\partial/\partial x & -m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \tag{38}$$

The corresponding propagator evaluated inside the future light cone $t > |x|$ assumes the form

$$K(x,t)=\frac{m}{2}\begin{pmatrix} -\frac{t+x}{\sqrt{t^2-x^2}}J_1(m\sqrt{t^2-x^2}) & iJ_0(m\sqrt{t^2-x^2}) \\ iJ_0(m\sqrt{t^2-x^2}) & \frac{-t+x}{\sqrt{t^2-x^2}}J_1(m\sqrt{t^2-x^2}) \end{pmatrix} \tag{39}$$

where each matrix entry includes the Bessel functions $J_0(...)$ and $J_1(...)$. Analysis shows that the factor multiplying the Bessel functions in (39) has dimension $|L|^{-1}=[M]^1$, hence normalizing the propagator using a large scale $M_0>>m$ yields

$$K^0(x,t)=\frac{K(x,t)}{M_0} \tag{40}$$

Squaring (40) and repeating the previous line of arguments leads to

$$\boxed{\tau_s=2,\ |D_0|=1} \tag{41}$$

The next section attempts to bridge the gap between SOC and the minimal fractality of spacetime geometry near or above the Fermi scale.

6. From SOC to the minimal fractality of spacetime

We have extensively discussed in [8 - 11] the physical significance of the *minimal fractal manifold* (MFM), a spacetime continuum characterized by arbitrarily small and scale-dependent deviations from four dimensions ($\varepsilon=4-D<<1$). The MFM reflects an evolving setting that starts far-from-equilibrium and gradually reaches the equilibrium conditions mandated by field theory in the limit of four-dimensional spacetime ($\varepsilon=0$). There are well-motivated reasons to believe that dimensional fluctuations driven by ε are asymptotically compatible with the internal structure and dynamics of the Standard Model of particle physics [8-11].

Based on these premises, we introduce the hypothesis that the dimensional deviation ε and the avalanche-size s are interchangeable concepts via

$$\varepsilon = 4 - D = s^{-1} << 1 \tag{42}$$

Furthermore, since ε flows with the energy scale, it likely reaches its uppermost observable value close to the formation of the cosmic microwave background (CMB) [12]. The maximal dimensional deviation is therefore set to

$$\varepsilon_{cr} = \varepsilon_{\max} \approx 10^{-5} \;\; ; \;\; \varepsilon << \varepsilon_{cr} \tag{43}$$

which turns (9) into

$$P(\varepsilon, \varepsilon_{cr}) \sim \varepsilon^{\tau_s} \, \Phi(\varepsilon_{cr}/\varepsilon) \;,\; \varepsilon << 1 \tag{44a}$$

$$\varepsilon_{cr}(\mu) \sim \mu^{D_0}, \; \mu >> 1 \tag{44b}$$

where μ is the dimensionless Renormalization Group scale. Using (44) as a baseline, a planned extension of this work will begin exploring the non-equilibrium regime of vacuum fluctuations, beyond the boundaries of perturbative Quantum Field Theory [13].

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