A note on Laguerre original ODE and Polynomials (1879)

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(4 April 2020)

Abstract: In the present note a critical discussion of two ODEs and two polynomials that have been wrongly attributed to the French mathematician Edmond Nicolas Laguerre (1834-1886) is provided. It is shown that Laguerre had nothing to do with such a wrong attribution and the actual discoverer was the Russian mathematician Nikolay Yacovlevich Sonine (1849-1915).

Keywords: ODEs, classical orthogonal polynomials, Laguerre ODE, Sonine ODE
Mathematics Subject Classification 33C45 · 34–XX · 42–05

1. Introduction

In addition to Encyclopedia of Mathematics [1], Encyclopedic Dictionary of Mathematics [2], Wikipedia [3], and Wolfram MathWorld [4], several authors of textbooks, for example, **Refs**. [5-12] and peer-reviewed research articles [13-18] relating to the mathematical theory of classical orthogonal polynomials wrongly attributed the second-order linear ODE

$$xy'' + (1-x)y' + ny = 0\,, \tag{1}$$

and its polynomial solution

$$y \equiv L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}\,, \tag{2}$$

to the French mathematician Edmond Nicolas Laguerre (1834-1886), that is to say, Eq.(1) and polynomials (2) are named after Laguerre.

Actually, Eq.(1) and polynomials (2) are, respectively, special cases of another ODE and polynomials, namely

$$xy'' + (\alpha + 1 - x)y' + ny = 0\,, \tag{3}$$

and its polynomial solution

$$y \equiv L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{\alpha + n}{n-k} \frac{x^k}{k!}\,, \tag{4}$$

which are also wrongly attributed to Laguerre since, in the literature, they are named: ‘associated Laguerre differential equation’ and ‘associated Laguerre polynomials’ or ‘generalized Laguerre polynomials’.

2. Laguerre true and authentic ODE and polynomials

The literature relating to the classical orthogonal polynomials refer to two articles written by Laguerre and published in 1878 [19] and 1879 [20], respectively. For instance, in Encyclopedia of Mathematics [1] we can read «*Laguerre polynomials are most frequently used under the condition* $\alpha = 0$*; these were investigated by E. Laguerre* [**Ref**.[19]: E. Laguerre, "Sur le transformations des fonctions elliptiques" Bull. Soc. Math. France **6**,72–78 (1878)], *and are* *denoted in this case by* $L_n(x)$ (*in contrast to them, the* $L_n^{(\alpha)}(x)$ *are sometimes known as generalized Laguerre polynomials*). However, the Laguerre 1878 [19] article has nothing to do with Eqs.(1), (3) and polynomials (2) , (4).

The rest of the literature refer to the Laguerre 1879 [20] article, which also has nothing to do with the above mentioned ODEs and polynomials. But as it is always better to refer to the original articles rather than to second hand account and, I therefore, scrutinized the 1879 article in which Laguerre wrote in French, Page 74, : "... *from where it follows that the polynomial* $f(x)$ *satisfies the second order differential equation*

$$xy''+(1+x)y'-ny=0\,, \qquad (\mathbf{4})$$

of which a second solution is

$$u=\varphi(x)e^{-x}-f(x)\int_x^\infty \frac{e^{-x}dx}{x}\,.$$

Expansion in series easily gives

$$f(x)=x^m+m^2x^{m-1}+\frac{m^2(m-1)^2}{1\cdot 2}x^{m-2}+\frac{m^2(m-1)^2(m-2)^2}{1\cdot 2\cdot 3}x^{m-3}+\cdots+1\cdot 2\cdot 3\cdots m.\text{ "}$$

As we can clearly see, Eq.(**4**) and polynomial $f(x)$ are completely different from Eq.(1) and polynomial (2) and for that reason cannot be deduced from Eq.(3) and polynomials (4) when $\alpha=0$. Consequently, the Laguerre true and authentic ODE and polynomials are, respectively, Eq.(**4**) and $f(x)$.

3. Sonine was the real discoverer

A very small part of the literature [3, 4, 21] referred to the Russian mathematician Nikolay Yacovlevich Sonine (1849-1915) as the discoverer of Eq.(3) and polynomials (4). For instance, Wikipedia wrote: «... *Then they are also named generalized Laguerre polynomials, as will be done here* (*alternatively associated Laguerre polynomials or, rarely, Sonine polynomials, after their inventor* [**Ref.**[22]: Sonine, N. J. "Sur les fonctions cylindriques et le développement des fonctions continues en séries." Math. Ann. **16**,1-80 (1880)] Nikolay Yakovlevich Sonin).»

Effectively, in 1880, Sonine wrote and published the referred article [22] in which he studied in detail Eq.(3) and its polynomial solution (4). He wrote in French, Page 41: "… *But by expanding* $\varphi_m(r,y)$ *according to the powers of* r, *we find*

$$\varphi_m(r,y)=\sum_{n=0}^{n=\infty} T_m^n(y)\cdot r^n\,, \qquad \text{(i)}$$

where

$$T_m^n=\frac{y^n}{\Pi(n)\Pi(m+n)\Pi(0)}-\frac{y^{n-1}}{\Pi(n-1)\Pi(m+n-1)\Pi(1)}+\frac{y^{n-2}}{\Pi(n-2)\Pi(m+n-2)\Pi(2)}-\cdots \qquad \text{(ii)}$$

Thus, by substituting the expansions of $\varphi_m(r,y)$, *we get*

$$\int_0^\infty e^{-y} y^m T_m^n(y) \cdot T_m^{n_1}(y) dy = 0\,, \quad (n_1 \text{ different from } n), \tag{iii}$$

$$\int_0^\infty e^{-y} y^m T_m^n(y) \cdot T_m^n(y) dy = \frac{1}{\Pi(n)\Pi(m+n)}\,. \tag{iv}$$

The polynomials $T_m^n(y)$ *have the following properties*

$$\begin{aligned} (m+n+1)\,\frac{dT_m^{n+1}}{dy} &= T_m^n - \frac{dT_m^n}{dy}, \\ \frac{T_m^{n-1}}{y^{n-1}} &= -\frac{d}{d\frac{1}{y}}\frac{T_m^n}{y^n}, \end{aligned} \tag{v}$$

and satisfy the second order differential equation

$$\frac{d^2 T_m^n}{dy^2} - \left(1 - \frac{m+1}{y}\right)\frac{dT_m^n}{dy} + \frac{n}{y}\,T_m^n = 0. \tag{vi}$$

To expand a function $f(x)$ *into a series of the form*

$$f(x) = \sum_{n=0}^{n=\infty} a_n\, T_m^n(x)\,, \tag{vii}$$

multiply this assumed expansion by $e^{-x} x^m T_m^n(x)$ *and integrate from 0 to ∞. We obtain*

$$a_n = \Pi(n) \cdot \Pi(m+n) \int_0^\infty f(x) e^{-x} x^m\, T_m^n(x)\, dx = 0. \tag{viii}$$

… ”

Remark, Eq.(vi) can be written in the form

$$y\,\frac{d^2 T_m^n}{dy^2} + (m+1-y)\,\frac{dT_m^n}{dy} + n\,T_m^n = 0\,,$$

which is, apart from the symbolic notations, similar in all its details to Eq.(3). Furthermore, for the special case when $m = 0$, this equation reduces to

$$y\,\frac{d^2 T_0^n}{dy^2} + (1-y)\,\frac{dT_0^n}{dy} + n\,T_0^n = 0\,.$$

4. Conclusion

The original Laguerre 1879 article is scrutinized and the so-called Laguerre ODE and polynomials are found to be wrongly attributed to him by the literature relating to the classical orthogonal polynomials. Also the original Sonine 1880 article is studied and found that, unlike

Laguerre Eq.(4), Sonine Eq.(vi) is (apart from the symbolic notations) similar in all its details to Eq.(3) and the same can be said about polynomials (ii). Therefore, Sonine was indisputably the real originator of Eq.(3) and polynomials (4). That is why his name must be logically and fairly attached to his discovery.

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