Riemann Hypothesis

Shekhar Suman

April 2020

Subject Classification code- 11M26

Keywords- Riemann Zeta function; Analytic Continuation; Critical strip; Critical line.

1 Abstract

The Riemann Zeta function is defined as

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s, \ Re(s) > 1$

The Zeta function is holomorphic in the complex plane except for a pole at $s = 1$. *The trivial zeros of* $\zeta(s)$ *are* $-2, -4, -6, \ldots$. *Its non trivial zeros lie in the critical strip* $0 < Re(s) < 1$.

The Riemann Hypothesis states that all the non trivial zeros lie on the critical line $Re(s) = 1/2$.

2 Proof

Analytic continuation of $\zeta(s)$ *is defined as* [*see* 1, *p*.14 , *Eq*. 2.1.4]

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{1}{2} \tag{1}$$

Here [.] *denotes the Greatest Integer Function.*

Let, $s = x + iy$, $0 < x < 1$, $y \in (-\infty, -1/2) \cup (1/2, \infty)$.

For $0 < x < 1$, [*see* 1, *p*.14],

$\frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx = \frac{1}{2}$.

So, using $\frac{1}{2} = \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx$ *in* (1),

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + \frac{s}{2}\int_1^\infty \frac{1}{x^{s+1}}dx + \frac{1}{s-1}$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}dx + s\int_1^\infty \frac{1/2}{x^{s+1}}dx + \frac{1}{s-1}$$

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+1}{x^{s+1}}dx + \frac{1}{s-1}$$

Let, ρ *be a non trivial zero of the Riemann Zeta Function.*

Then, $\zeta(\rho) = 0$; $0 < Re(\rho) < 1$, $Im(\rho) \in (-\infty, -1/2) \cup (1/2, \infty)$.

$$\zeta(\rho) = \rho\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx + \frac{1}{\rho-1} = 0$$

$$\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx = \frac{1}{\rho(1-\rho)};\ 0 < Re(\rho) < 1 \tag{2}$$

The functional equation of the Riemann Zeta function is [*see* [1], *p*.22, 2.6.4],

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

So, by functional equation if ρ *is a zero of the Riemann Zeta function then* $1-\rho$ *is also a zero* [*see* [1], *p*.30].

$\zeta(\rho) = 0 \Rightarrow \zeta(1-\rho) = 0.$

$\zeta(1-\rho) = (1-\rho)\int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx - \frac{1}{\rho} = 0;$

$$\int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx = \frac{1}{\rho(1-\rho)};\ 0 < Re(\rho) < 1 \tag{3}$$

Equating left sides of equation (2) *and* (3),

$\int_1^\infty \frac{[x]-x+1}{x^{\rho+1}}dx = \int_1^\infty \frac{[x]-x+1}{x^{2-\rho}}dx$

$$\int_1^\infty ([x]-x+1)(\frac{1}{x^{\rho+1}} - \frac{1}{x^{2-\rho}})dx = 0 \tag{4}$$

$Let, \rho = \sigma + it\ ; 0 < \sigma < 1,\ t \in (-\infty, -1/2) \cup (1/2, \infty)$

$Since,\ 0 < \sigma < 1\ so\ we\ discuss\ 2\ cases$

$0 < \sigma \leq 1/2\ and\ 1/2 \leq \sigma < 1.$

$Case\ 1 : 0 < \sigma \leq 1/2.$

$Putting,\ \rho = \sigma + it\ in\ equation\ (4),$

$\int_1^\infty ([x]-x+1)(\frac{1}{x^{\sigma+1+it}} - \frac{1}{x^{2-\sigma-it}})dx = 0.$

$\int_1^\infty ([x]-x+1)(\frac{x^{-it}}{x^{\sigma+1}} - \frac{x^{it}}{x^{2-\sigma}})dx = 0.$

$\int_1^\infty ([x]-x+1)(\frac{e^{-it(lnx)}}{x^{\sigma+1}} - \frac{e^{it(lnx)}}{x^{2-\sigma}})dx = 0.$

$\int_1^\infty ([x]-x+1)(\frac{cos(tlnx)}{x^{\sigma+1}} - \frac{cos(t(lnx))}{x^{2-\sigma}})dx +$

$i\int_1^\infty ([x]-x+1)(\frac{-sin(tlnx)}{x^{\sigma+1}} - \frac{sin(t(lnx))}{x^{2-\sigma}})dx = 0$

$Equating\ Real\ part\ to\ zero,$

$\int_1^\infty ([x]-x+1)(\frac{cos(tlnx)}{x^{\sigma+1}} - \frac{cos(t(lnx))}{x^{2-\sigma}})dx = 0$

$$\int_1^\infty ([x]-x+1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx)dx = 0 \tag{5}$$

$$Let, I = \int_1^\infty ([x] - x + 1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}}) cos(tlnx)dx = 0 \quad (6)$$

$$0 \leq x - [x] < 1$$

$$0 < [x] - x + 1 \leq 1.$$

$$([x] - x + 1)(\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx) \leq (\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx).$$

$$I \leq \int_1^\infty (\frac{1}{x^{\sigma+1}} - \frac{1}{x^{2-\sigma}})cos(tlnx)dx \quad (7)$$

$Let,\ lnx = u.$

$\Rightarrow x = e^u.$

$\Rightarrow dx = e^u du$

$when\ x = 1\ then\ u = 0\ and\ when\ x = \infty\ then\ u = \infty.$

$So,\ \int_1^\infty\ becomes\ \int_0^\infty.$

$Since\ by\ equation\ (6),\ I = 0.$

$So,\ equation\ (7)\ gives,$

$$0 \leq \int_0^\infty (\frac{1}{e^{(\sigma+1)u}} - \frac{1}{e^{(2-\sigma)u}})cos(tu)e^u du.$$

$$0 \leq \int_0^\infty (e^{-(\sigma+1)u} - e^{-(2-\sigma)u})cos(tu)e^u du.$$

$$0 \leq \int_0^\infty (e^{-(\sigma+1)u}e^u - e^{-(2-\sigma)u}e^u)cos(tu)du.$$

$$0 \leq \int_0^\infty (e^{-\sigma u}cos(tu) - e^{-(1-\sigma)u}cos(tu))du. \quad 0 < \sigma \leq 1/2$$

$$Now, \int e^{ax}cosbxdx = \frac{e^{ax}[acos(bx) + bsin(bx)]}{a^2 + b^2}.$$

$$0 \leq \left. \frac{e^{-\sigma u}[-\sigma cos(tu) + tsin(tu)]}{\sigma^2 + t^2} \right|_0^\infty - \left. \frac{e^{-(1-\sigma)u}[(\sigma-1)cos(tu) + tsin(tu)]}{(1-\sigma)^2 + t^2} \right|_0^\infty$$

$For,\ 0 < \sigma \leq 1/2,\ -\sigma < 0 \Rightarrow\ e^{-\sigma u} \!\rightarrow 0$ as u $\rightarrow \infty$.

$For,\ 0 < \sigma \leq 1/2,\ -(1-\sigma) < 0 \Rightarrow e^{-(1-\sigma)u} \!\rightarrow 0$ as u $\rightarrow \infty$.

$0 \leq \frac{-(-\sigma)}{\sigma^2+t^2} + \frac{(\sigma-1)}{(1-\sigma)^2+t^2}.$

$0 \leq \frac{\sigma[(1-\sigma)^2+t^2]+(\sigma-1)(\sigma^2+t^2)}{(\sigma^2+t^2)[(1-\sigma)^2+t^2]}$

$0 < \sigma \leq 1/2 \Rightarrow (\sigma^2+t^2)[(1-\sigma)^2+t^2] > 0.$

$0 \leq \sigma(1-\sigma)^2 + \sigma t^2 + \sigma(\sigma^2+t^2) - (\sigma^2+t^2)$

$0 \leq \sigma(1-2\sigma+\sigma^2) + \sigma t^2 + \sigma(\sigma^2+t^2) - (\sigma^2+t^2)$

$0 \leq [\sigma - 2\sigma^2 + \sigma^3] + \sigma t^2 + \sigma^3 + \sigma t^2 - \sigma^2 - t^2.$

$0 \leq 2\sigma^3 - 3\sigma^2 + \sigma + 2\sigma t^2 - t^2.$

$0 \leq \sigma(2\sigma^2 - 3\sigma + 1) + t^2(2\sigma - 1).$

$0 \leq \sigma(2\sigma-1)(\sigma-1) + t^2(2\sigma-1).$

$0 \leq (2\sigma-1)[\sigma(\sigma-1) + t^2].$

$0 \leq (2\sigma-1)[\sigma^2 - \sigma + t^2].$

$$0 \leq (2\sigma-1)[(\sigma-1/2)^2+(t^2-1/4)]. \quad (8)$$

$Since\ t \in (-\infty, -1/2) \cup (1/2, \infty).$

$Thus,\ t^2 - 1/4 > 0.$

$(\sigma - 1/2)^2 + (t^2 - 1/4) > 0$

$So,\ (8)\ gives$

$$0 \leq 2\sigma - 1. \quad (9)$$

$$By\ case\ 1\ we\ had,\ 0 < \sigma \leq 1/2 \Rightarrow 2\sigma - 1 \leq 0. \quad (10)$$

$Combining\ (9)\ and\ (10),\ we\ have$

$0 \leq 2\sigma - 1 \leq 0.$

$2\sigma - 1 = 0$

$\sigma = 1/2.$

$Now\ we\ proceed\ to\ Case\ 2$

$Case\ 2 : 1/2 \leq \sigma < 1.$

$Let, \rho = \sigma + it, 1/2 \leq \sigma < 1,\ t \in (-\infty, -1/2) \cup (1/2, \infty).$

$Let,\ \zeta(\rho) = 0$

$The\ functional\ equation\ of\ the\ Riemann\ Zeta\ function\ is\ [see\ [1],\ p.22, 2.6.4],$

$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \Gamma((1-s)/2)\pi^{-(1-s)/2}\zeta(1-s).$

$So,\ by\ functional\ equation\ if\ \rho = \sigma + it\ is\ a\ zero\ of\ the\ Riemann\ Zeta\ function\ then$
$1 - \rho = 1 - \sigma - it\quad is\ also\ a\ zero\ and\ then$
$1 - \bar{\rho} = 1 - \sigma + it\ \ is\ also\ a\ zero\ [see\ [1], p.30].$

$\zeta(\rho) = 0 \Rightarrow \zeta(1-\rho) = 0.$

$\zeta(1-\rho) = 0 \Rightarrow \zeta(1-\bar{\rho}) = 0$

$Since,\ \rho = \sigma + it.$

$\zeta(1-\bar{\rho}) = 0 \Rightarrow \zeta(1-\sigma+it) = 0.$

$1/2 \leq \sigma < 1 \Rightarrow 0 < 1 - \sigma \leq 1/2.$

$\zeta(1-\sigma+it) = 0,\ 0 < 1 - \sigma \leq 1/2.$

$Let,\ \sigma' = 1 - \sigma.$

$\zeta(\sigma' + it) = 0,\ 0 < \sigma' \le 1/2.$

$So,\ by\ case\ (1),$

$\sigma' = 1/2.$

$(1 - \sigma) = 1/2.$

$\sigma = 1/2.$

$So,\ by\ the\ above\ two\ cases\ we\ get\ that$

$\zeta(\rho) = 0\ ; 0 < Re(\rho) < 1;\ t \in (-\infty, -1/2) \cup (1/2, \infty)$

$\Rightarrow Re(\rho) = 1/2$

$, which\ proves\ the\ R.H.$

3 References

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